PLANE AND GEODETIC SURVEYING

FOR ENGINEERS

BY

THE LATE DAVID CLARK M.A. B.Sc.

M.INST.C.E. M.INST.C.E.I. M.AM.SOC.C.E.

PROFESSOR OF CIVIL ENGINEERING, UNIVERSITY OF DUBLIN

VOLUME TWO
HIGHER SURVEYING

FOURTH EDITION REVISED AND ENLARGED

BY

JAMES CLENDINNING O.B.E. B.Sc.(Eng.)

A.M.INST.C.E.

([illegible])

With an Appendix on Mechanical Computation

by

L. J. COMRIE M.A., PH.D. F.R.S.

CONSTABLE & COMPANY LTD

10 ORANGE STREET LONDON W.C.2

LONDON
PUBLISHED BY
Constable and Company Ltd.
10–12 ORANGE STREET, W.C.2.

INDIA AND PAKISTAN
Orient Longmans Ltd.
BOMBAY CALCUTTA MADRAS

CANADA
Longmans, Green and Company
TORONTO

SOUTH and EAST AFRICA
Longmans, Green and Company Ltd.
CAPE TOWN NAIROBI

First Published 1923

PRINTED IN GREAT BRITAIN BY THE WHITEFRIARS PRESS LTD.
LONDON AND TONBRIDGE

PREFACE TO THE FIRST EDITION

The entire field of Plane Surveying having been dealt with in the first volume, the subject-matter of this volume covers the remainder of a degree course, and includes Field Astronomy, Geodetic Surveying and Levelling, Topographical and Reconnaissance Surveying, and Mapping.

As in Volume I, an endeavour has been made to meet the requirements of private students and practising surveyors, particularly with regard to the subject of Field Astronomy. In this branch of surveying it is essential that the student should acquire a sound grasp of the fundamental conceptions before proceeding to a consideration of the actual observations. The author has emphasised this necessity by devoting Chapter I entirely to explanations regarding the quantities dealt with in astronomical determinations. It is hoped that the arrangement of this chapter and the inclusion of typical worked examples on the use of the *Nautical Almanac* will facilitate an understanding of the fundamental principles of astronomical measurements.

Chapter II deals with the instruments and practice of Field Astronomy as applied to the determination of Time, Azimuth, Latitude and Longitude. For each determination the primary method is described as well as the less refined observations by ordinary field instruments.

Chapter III covers the field work of Geodetic Surveying. The catenary method of base measurement by invar wires or tapes, having superseded all others, is treated at length, but a few typical base bars have been described as a matter of interest.

The subjects of Theory of Errors, Survey Adjustment and Geodetic Computations are dealt with in Chapter IV. In a general textbook, considerations of space preclude detailed treatment of these subjects, but it is thought that the matter given will prove amply sufficient for the needs of the majority of readers.

Chapter V deals with Geodetic Levelling, including Precise Spirit and Trigonometrical Levelling.

Topographical, Geographical and Reconnaissance Surveys are treated in Chapter VI. Several of the survey methods have already been given in detail in Volume I, and these are referred to only to the extent of indicating special features applicable to small scale work. The subjects of Barometric Levelling and Photographic Surveying, including Stereo-Photographic and Aerial Surveying, are given at greater length.

Chapter VII deals with Map Construction, and includes a *résumé* of the principal map projections employed.

Sets of illustrative numerical examples and answers are given, and lists of references on the various subjects treated are inserted after the appropriate chapters. As in the case of Volume I, no attempt has been made to prepare exhaustive bibliographies, but the author has selected textbooks and papers which are commonly accessible and many of which

he has himself found useful. He would take this opportunity of expressing his indebtedness to various official publications, particularly those of the Survey of India and the United States Coast and Geodetic Survey.

This volume is illustrated by 114 diagrams, which are original with the exception of Figs, 23, 25, 31 and 70 by Messrs. Bausch and Lomb, 88 by Messrs. C. F. Casella and Co., 29 by M. A. Jobin, 26 by Messrs. C. H. Steward, 22 by Messrs. Troughton and Simms, 45, 57 and 58 by Messrs. E. R. Watts and Son, and 72, 96, 97 and 99 by Messrs. Carl Zeiss. The author gratefully acknowledges the kindness of those firms in placing their illustrations at his disposal, and would also express his thanks to Messrs. The Cambridge and Paul Scientific Instrument Co. for information relative to their invar precise staff.

D. C.

PREFACE TO THE SECOND EDITION

In revising Chapters I and II, we have merely followed closely the author's original text, introducing modifications that were rendered necessary by the recent changes in the mode of reckoning astronomical time and in the *Nautical Almanac*, and revising the paragraphs on the prismatic astrolabe, wireless signals and the determination of longitude, as well as the examples. In addition a number of minor alterations have been made, but there has been no attempt to make a general revision, as our desire was to retain the features of Prof. Clark's presentation.

L. J. Comrie
J. E. E. Craster

PREFACE TO THE THIRD EDITION

In view of the progress that has been made in geodetic methods and instruments since the last edition of this book was published, it seemed to be impossible to carry out the task of preparing a new edition satisfactorily unless a number of additions, now included in this volume, were made. Consequently, while I have tried to avoid altering the original character of the book in any way, or to depart from the original text more than was really necessary, I have found it necessary to make certain additions which I hope will add to the general usefulness of the book, both to the practical man and to the student.

Among other additions may be mentioned short descriptions of some modern light-weight geodetic theodolites and of other field apparatus; a description of modern precise traversing which I believe is more comprehensive than any that can be found in other text-books of the same kind; fairly detailed discussions of the rectangular spherical and Transverse Mercator systems of co-ordinates; lapse-rate formulæ for the graduation of aneroid barometers, etc.

As the 45° prismatic astrolabe has come much into favour with British surveyors during recent years, the paragraphs in Chapter II which deal with the prismatic astrolabe have again been revised in order to provide a fuller description of the 45° model and of the method of using it, and the original illustrations of the 60° prismatic astrolabe of Claude and Driencourt (Figs. 29 and 30 in the last edition) have been omitted to make room for illustrations of the later type of instrument.

The original Chapter IV on Survey Adjustment and Geodetic Computation has been extended to form two separate chapters, one on adjustment and the other on computation, and the matter on both subjects has been considerably expanded. In the original edition, Clark included a discussion on survey adjustment, which, although admirable in its way and sufficient to enable the student to apply the principles of the method of least squares to very simple problems, was not sufficient to enable him to apply them to the more complicated requirements of practice, nor did he make any mention of the exceedingly useful checks that are available during most stages of a least square adjustment. In this edition, the mathematical theory of least squares has been excluded, as it was in the original, but I hope that the matter now added will enable a student to apply the method with confidence to most practical problems.

In the chapters on Field Astronomy and Geodetic Computation I have added a number of proofs of various formulæ which were not given in the original editions as I think these will make the book more complete and add to its general usefulness. At the same time, owing to limitations of space, a proof of every formula is not given and only those which are reasonably simple and which help to illustrate basic principles have been included. Nearly all of these proofs are printed in small type so that

they can be omitted on a first reading or by those who are not interested in the mathematical theory. The book, as it stands, of course, makes no pretence at being a treatise on mathematical geodesy, and some rather difficult proofs, such as that of Colonel A. R. Clarke's formulæ for latitude and longitude from the length and azimuth of long lines, have been omitted on account of their length or mathematical difficulty.

During the last twenty years or so Transverse Mercator co-ordinates have come into very extensive use for official surveys in different parts of the world. These surveys provide fixed points on the ground whose positions are accurately determined, and, in addition to letting them serve as a useful " yard-stick " by which to test the accuracy of his own work, the ordinary private surveyor may often save himself a considerable amount of unnecessary work both in the office and in the field if he makes proper use of these points. Accordingly, I have thought it advisable to include a fairly complete description of the Transverse Mercator system, especially as it is important that those who have to use it should know something of its limitations and of the principles on which it is based. The rectangular spherical system of co-ordinates has also been fairly extensively used both at home and abroad, but the modern tendency is to replace it by the Transverse Mercator. In spite of this, the theory of the rectangular spherical system illustrates important basic principles, and leads up very naturally to the slightly more difficult theory of the Transverse Mercator ; consequently it too has been described at some length. A third system of co-ordinates—the conical orthomorphic—is also used in various places, and, accordingly, the principal formulæ relating to it have been stated but no attempt has been made to prove them.

In most text-books that deal with the subject, the theory of these three main systems of linear geodetic co-ordinates is considered as part of the subject of map projections. In modern practice they tend to be definite co-ordinate systems, rather than ordinary map projections, and accordingly it seems best to treat and consider them as such, and as part of ordinary geodesy, so that the detailed description of them is included in Chapter V rather than in Chapter VIII.

I regret that it has not been possible to spare more space for the important subject of air survey. However, this tends to be a highly specialised subject in itself, which is developing very rapidly, and some good text-books, describing the most recent practice, are now available.

Dr. L. J. Comrie, who originally wrote Appendix I, on Mechanical Computing, for the second edition, has revised this section, with particular reference to the current vogue of using twin calculating machines for rectangular co-ordinate survey.

I owe special thanks to Captain G. T. McCaw, C.M.G., O.B.E., M.A., for much advice and assistance, and particularly for looking over certain parts of the text and for sending me notes on points that appeared to require correction, qualification or amendment. Other acknowledgments are due to Colonel Sir Gerald Lenox-Conyngham, M.A., F.R.S., for various suggestions and for sending me notes of some errors that he had found in the third edition of Vol. I and which are now included in the list given on page xvi ; to Mr. E. J. H. Dale, of Capetown, for

pointing out a rather serious and misleading error in the formation of the normal equations from weighted observation equations which appears in two examples in the previous editions and which is corrected in this one; to the Controller, H.M. Stationery Office, for permission to use information published in Vol. II of the *Admiralty List of Radio Signals*; to Captain S. C. Saward, Director of Surveys, Gold Coast, for permission to include the table given on page 187 for sag correction of a tape used on a slope which is taken from the Gold Coast Survey publication *Tables for Use in the Department*; to the Director of the United States Coast and Geodetic Survey for permission to use the illustrations from U.S.C. & G.S. publications that appear in these pages as Figs. 103, 104, 105, 106 and 107; and to Messrs. Cooke, Troughton & Simms, Messrs. E. R. Watts and Son, and Messrs. H. Wild & Co. for the provision of information or photographs relating to instruments of their manufacture, for permission to include illustrations of these instruments in this book, and in some cases for the loan of the blocks. The description of the adjustment of the transit instrument given on page 55 and the greater part of the description of the adjustment of the zenith telescope given on page 57 have been taken direct from the original author's companion volume, *Field Astronomy*.

J. CLENDINNING.

PREFACE TO THE FOURTH EDITION

In preparing this new edition, besides including descriptions of certain important war-time and post-war developments, I have made various additions which either appeared to myself to be desirable or else were suggested to me by various friends and correspondents.

The main new developments which are described are: the use of a new *Star Almanac for Land Surveyors*, flare triangulation, applications of radar to surveying, the Bergstrand method of measuring distances accurately by means of modulated light waves, and certain astronomical methods which enable determinations of latitude, longitude and azimuth, sufficiently accurate for minor work, to be made with a minimum of observation and computation.

The new *Star Almanac for Land Surveyors*, the first edition of which, containing data for 1951, is being published in 1950, is specially designed for land surveyors engaged on ordinary survey work, as opposed to geodetic work, and it is intended that, for such work, this volume should eventually replace the ordinary edition of the *Nautical Almanac*. Consequently, I have included a description of it and a number of examples of the use of it; also a fuller description of *Apparent Places of Fundamental Stars*, together with some examples based on data taken from it as this publication will now replace the *Nautical Almanac* for reducing all astronomical observations of geodetic or similar order.

Flare triangulation and radar techniques at present involve a considerable amount of skilled staff and special and expensive apparatus, and these are only available to large survey organisations, such as Government Departments, engaged on very extensive projects. Hence, I have felt some doubt about giving so much space to them as I have done. However, as they offer so many opportunities of extending geodetic measurements to cases where such measurements would otherwise be quite impossible, and as some knowledge of them is already being demanded of candidates in certain examinations, it seemed that no up-to-date book on geodesy can now ignore them. The description of flare triangulation is short, and Chapter IX, the new chapter on Radar Ranging and Triangulation, is independent of the remainder of the book, so that it can be omitted by those not interested.

The Bergstrand apparatus for accurate distance measurement, the "geodimeter," which is described in Appendix IV, has not yet come into general use, but, in an electronic age, it is of considerable interest as an example of the application of electronics to surveying, and it promises to have important applications to ordinary triangulation.

In the chapters on Field Astronomy, I have added a short description of position line methods. These methods, although not new in themselves, are being used more and more by surveyors and references to them are now appearing in survey literature. Hence, the book is scarcely

complete without a description of the principles on which they are based. I have also added to these chapters some additional notes on the selection and identification of stars, as this is an important subject in practical work and a fuller treatment appeared to be desirable. In addition, in the chapter on Survey Adjustment I have included a simple example of a least square adjustment by the method of directions and a short description of adjustment by the method of differential displacements.

The chapter on Radar Ranging and Triangulation is based largely on information contained in various articles and publications by Professor C. A. Hart, of University College, London (now Vice-Chancellor of Roorkee University, India), who is chiefly responsible for the early research work on which present-day applications of radar to surveying are founded, and I am much indebted to him for having read through the original draft of the chapter and making many useful suggestions and corrections.

Appendix I on Mechanical Computing was revised by Dr. L. J. Comrie some little time before his death, and contains descriptions of the latest types of computing machines now available for survey work.

Appendix IV, on the New Method of Accurate Distance Measurement by Combined Optical and Electronic Means, is based on articles and information provided by Mr. E. Bergstrand, the designer of the apparatus, and I am very grateful to him, not only for providing me with information that I required, but also for having read and corrected the proof of the Appendix.

Particular thanks are due to the Superintendent of H.M. Nautical Almanac Office for very kindly and courteously providing me with proofs of certain pages of the 1951 editions of the new *Star Almanac for Land Surveyors*, the *Nautical Almanac* and *Apparent Places of Fundamental Stars*, as these editions had not been published when the revision was being prepared. Acknowledgments and thanks are also due to H.M. Astronomer Royal for providing me with information about the radio time signals; to Messrs. Hilger and Watts Ltd. for providing me with information about their Watts "Microptic" theodolite No. 2, and for allowing me to use illustrations of the instrument taken from their catalogue; and to Dr. J. De Graaff-Hunter for reading the drafts of my descriptions of his methods of determining latitude, longitude and azimuth and of his shutter eyepiece. In addition, I have to thank various correspondents for sending me corrections to the third edition and suggestions for the new. If I have not adopted all the suggestions sent me, it is only because limitations of space have prevented me from doing so.

J. CLENDINNING.

"INNISFREE,"
ANGMERING-ON-SEA,
SUSSEX.

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PLANE AND GEODETIC SURVEYING FOR ENGINEERS

HIGHER SURVEYING

CHAPTER I

FIELD ASTRONOMY—INTRODUCTORY

SURVEY methods in general are directed to obtaining only the relative positions of points, but when their absolute positions on the earth's surface are required, recourse must be had to astronomical determinations. The branch of mathematical astronomy with which the surveyor has to deal relates to the determination of time, true meridian, latitude and longitude. Apart from their applications in pure geodesy, observations for these quantities are required for the following purposes.

(1) In the execution of the framework of a large survey it is necessary to obtain the true bearing of at least one of the lines and the latitude and longitude of at least one of the stations, so that the survey may be located as representing a portion of the surface of the earth.

(2) The direction of the true meridian may be determined in order to provide a reference meridian for traverse bearings. Its superiority over the magnetic meridian, by virtue of its permanence, is particularly evident if survey lines have to be retraced at a later date. On other than small surveys, reference to true meridian is essential because of the facility afforded of checking the angular work at intervals by astronomical observations.

(3) Rapid surveys of an exploratory character are sometimes entirely controlled either by means of the latitudes and longitudes of a few points at wide intervals or by the latitudes of points and the directions, or azimuths, of the lines joining them.

(4) The field work of establishing interstate boundaries, involving the setting out of a considerable length of a meridian, a parallel of latitude, or an oblique line, is dependent upon astronomical determinations.

The Celestial Sphere. The celestial bodies with which the surveyor is concerned are the so-called fixed stars and the sun, the moon and planets being of minor importance.

On viewing the stars, one sees a number of bodies of very varied distance from the earth, but all of which are so remote that straight lines from a star to different points on the earth, or even to different points on the earth's orbit round the sun, may for all practical purposes be considered parallel. In other words, the orbit of the earth may be treated as a point in comparison.

Since the surveyor has to deal only with the angular positions of the stars, and not with their linear distances from the earth, it is very convenient to consider them studded upon the surface of an imaginary

sphere, called the celestial sphere, at the centre of which the observer is stationed. This conception is quite legitimate, since the relative angular positions of the stars as they are seen from the earth would remain unchanged if they were projected along the straight lines joining them to the observer until they were situated on the spherical surface. The radius of the celestial sphere is to be imagined indefinitely great, so that the diameter of the earth's orbit may be regarded relatively as a point.

On watching the stars for some time, it is seen that, although they maintain the same situations with respect to each other, their positions relative to the horizon are continually changing, some stars apparently moving more than others in a given time. Indeed, the observed motion resembles that which might be expected if the stars were actually fixed upon a sphere rotating from east to west. This motion is an apparent one, and is due to the real daily rotation of the earth from west to east about the axis joining its north and south poles. The apparent rotation of the celestial sphere is, therefore, equal in rate and opposite in direction to that of the earth, and is about the axis of the earth produced. The points in which the prolongation of the earth's axis meets the celestial sphere are called respectively the north and south celestial poles. About those points the stars appear to travel along concentric circular paths with perfectly uniform angular velocity.

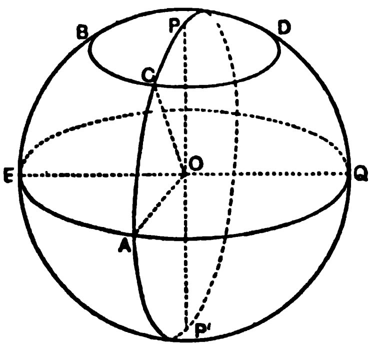


FIG. 1.

It will be found convenient to disregard the fact that the diurnal motion of the celestial sphere is only apparent and to assume that it is real, and that the earth's rotation is annulled.

Geometrical Properties of the Sphere. Many of the quantities involved in astronomical determinations are parts of the celestial sphere, and the surveyor should have some knowledge of spherical geometry and trigonometry. The more important definitions relating to parts of the sphere are here collected for convenience of reference.

(1) A *great circle* of a sphere is the circle in which it is intersected by any plane passing through the centre. The diameters of all great circles are also diameters of the sphere. The circles EAQ, PCAP′, and EBDQ (Fig. 1) are great circles.

A *small circle* is the circle in which the sphere is intersected by any plane not passing through the centre. Thus BCD is a small circle.

(2) The *axis* of any circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle, the ends of the axis being called the *poles* of the circle. Thus P and P′ are the poles of the great circle EAQ and also of the small circle BCD, if the planes of those two circles are parallel.

In the case of great circles, the poles are equidistant from the plane of the circle, but are not so with regard to small circles, the nearer being referred to as *the* pole.

Great circles passing through the poles of a great or small circle are termed *secondaries* to that circle. Thus the great circle PCAP′ is a secondary to both EAQ and BCD.

(3) The distance between two points on the surface of a sphere is measured by :

(*a*) The arc of the great circle on which both points lie, or

(*b*) The angle which this arc subtends at the centre of the sphere.

Thus the distance between the points A and C may be expressed either as the arc AC, since PCAP′ is a great circle, or as the angle AOC.

The distance between a point on the spherical surface and a great or small circle is the distance from the given point to the point of intersection of the circle by that one of its secondaries which passes through the given point. The arc AC, or the angle COA, therefore expresses the distance of C from the circle EAQ, since PCAP′ is the secondary to that circle which passes through C.

(4) The angle between two great circles is measured by :

(*a*) The spherical angle at either of their points of intersection. Thus in Fig. 2, the angle between the great circles PAP′ and PA′P′ is the spherical angle APA′.

(*b*) The angle between the planes in which they lie, as angle AOA′.

(*c*) The plane angle between the tangents to the circles at either intersection, as angle TPT′.

(*d*) The arc intercepted by them on the great circle to which they are both secondaries, as arc AA′ of the great circle EAA′Q.

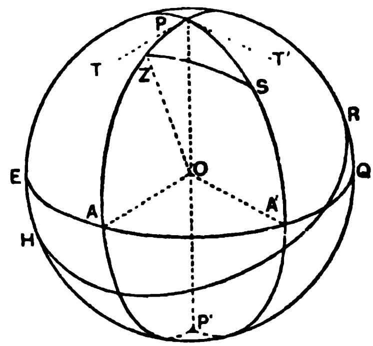


FIG. 2.

(*e*) The distance between their poles. Thus the angle between the great circles EAA′Q and HR is the distance between their poles P and Z, namely the arc PZ of the great circle PAP′ or angle POZ.

(5) A *spherical triangle* is a figure on the surface of the sphere bounded by three arcs of *great* circles. Thus if the points Z and S are joined by an arc of a great circle, PZS is a spherical triangle. Since an arc of a great circle is proportional to the angle at the centre of the sphere subtended by it, the sides of a spherical triangle are measured in terms of the central angles which they subtend. The angles of the triangle are the angles between the great circles which form the sides. Certain relationships then exist among the sides and angles of a spherical triangle and the investigation of these relationships forms part of the subject of spherical trigonometry.

In general, as in plane trigonometry, the formulæ of spherical trigonometry are deduced in a form that is not always suitable for computation, although it may be much the more convenient form for ordinary mathematical investigation or manipulation, or for deducing special formulæ in work, such as spherical astronomy, where a solution depends on the use of spherical trigonometry. Consequently, where necessary, we shall first of all give the formula that is most useful in mathematical work

and then the one that is most useful for computation by logarithms. Sometimes, however, the one used for computing is also suitable for mathematical work, and *vice versa.*

(6) If three of the six parts of a spherical triangle are known, the methods of spherical trigonometry enable the triangle to be solved.

Let A, B and C denote the angles of a spherical triangle, and a, b and c the sides opposite these angles, *i.e.* the angles at the centre of the sphere subtended by the sides. Let $s = \frac{1}{2}(a + b + c)$.

The General Triangle. (1) Given two angles and the side opposite one of them, as A, B and a,

The side b is derived from :—

$$\sin b = \frac{\sin a \sin B}{\sin A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

C is obtained as in case (6) following and then c by substituting c and C for b and B in (i).

As in plane trigonometry, b is ambiguous in this case, since there are two possible values for it, corresponding to the calculated value of one being the supplement of the other. The fact that the greater angle subtends the greater side removes the ambiguity in certain cases, and in practice it is usually possible to select the correct value for the unknown angle.

This formula is suitable either for purely mathematical work or for computing.

(2) Given two sides and the contained angle, as b, c, A,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

B and C can be obtained from case (6).

Formulæ (i) and (ii) are the fundamental formulæ of spherical trigonometry, as all the others can be derived from them.

For logarithmic work a may be computed from

$$\cos a = K \cos (\phi - b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

where $\tan \phi = \dfrac{\sin c \cos A}{\cos c}$, $K = \dfrac{\cos c}{\cos \phi}$. If a is a small angle, or if the complete solution of the triangle is required, compute B and C from

$$\tan \tfrac{1}{2} (B + C) = \frac{\cos \frac{1}{2} (b - c)}{\cos \frac{1}{2} (b + c)} \cot \frac{A}{2} \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

$$\tan \tfrac{1}{2} (B - C) = \frac{\sin \frac{1}{2} (b - c)}{\sin \frac{1}{2} (b + c)} \cot \frac{A}{2} \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

whence $\frac{1}{2}(B + C)$ and $\frac{1}{2}(B - C)$, and $B = \frac{1}{2}(B + C) + \frac{1}{2}(B - C)$ and $C = \frac{1}{2}(B + C) - \frac{1}{2}(B - C)$. Then find a from case (1) or, when a is very small, from one of the tangent formulæ in cases (3) and (5).

(3) Given the three angles, A, B and C,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(vi)}$$

This formula is derived from (ii) by substituting the supplements of the sides and angles in the original triangle to form a new triangle (the "polar triangle") in which these supplements become the corresponding

angles and sides, *i.e.* $a' = (180° - A)$, $A' = (180° - a)$, $b' = (180° - B)$, $B' = (180° - b)$, $c' = (180° - C)$, $C' = (180° - c)$. By using the polar triangle, all formulæ in spherical trigonometry give rise to analagous formulæ in which the sides of the triangle become the angles of the polar triangle, and *vice versa*.

For computing purposes use :—

$$\sin\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}} \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

$$\text{or } \cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}} \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

$$\text{or } \tan\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}} \quad . \quad . \quad . \quad . \quad \text{(ix)}$$

where $S = \frac{1}{2}(A + B + C)$. Similarly, for b and c. Of these three expressions, (ix) is to be preferred for computing purposes.

(4) Given the three sides a, b and c.

For mathematical work the most suitable formula will generally be :—

$$\cos A = \frac{\cos a - \cos b\cos c}{\sin b\sin c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(x)}$$

with similar expressions for $\cos B$ and $\cos C$.

For computing use :—

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \quad . \quad . \quad . \quad . \quad \text{(xi)}$$

$$\text{or } \cos\frac{A}{2} = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xii)}$$

$$\text{or } \tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii)}$$

where $s = \frac{1}{2}(a + b + c)$, with similar expressions for B and C.

(5) Given two angles and the side between them, as A, B and c,

$$\cot b = \frac{\cos c\cos A + \sin A\cot B}{\sin c} \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv)}$$

when a and C can be found from cases (1) and (6).

For computing use :—

$$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c \quad . \quad . \quad . \quad . \quad \text{(xv)}$$

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c \quad . \quad . \quad . \quad . \quad \text{(xvi)}$$

whence $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$, and thence $a = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)$; $b = \frac{1}{2}(a+b) - \frac{1}{2}(a-b)$. C can then be found from case (6).

(6) Given two sides and the angle opposite one of them, as a, b and A.

$$\sin B = \frac{\sin A\sin b}{\sin a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvii)}$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xviii})$$

after which C can be found by substituting C and c for B and b in (xvii). For computing use the same formulæ or

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B) \quad . \quad . \quad . \quad . \quad (\text{xix})$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b) \quad . \quad . \quad . \quad (\text{xx})$$

As in case (1) there may be an ambiguity about B and C as computed from (xvii) and (xviii), but, as before, it is usually not difficult to decide which value to adopt.

(7) The following two formulæ are sometimes useful in mathematical work :—

$$\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A \quad . \quad . \quad . \quad (\text{xxi})$$

$$\sin A \cos b = \sin C \cos B + \cos C \sin B \cos a \quad . \quad . \quad . \quad (\text{xxii})$$

Similar formulæ, involving c and C on the left-hand side, are, of course, obtained by making the appropriate substitutions.

(8) In every spherical triangle the sum of the three angles is greater than 180° and less than 540°. The amount by which this sum exceeds 180° is called the spherical excess and is denoted by E or ϵ. Then

$$E = (A + B + C) - 180° \quad . \quad . \quad . \quad . \quad . \quad (\text{xxiii})$$

$$\tan^2 \tfrac{1}{2}E = \tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c) \quad . \quad . \quad . \quad . \quad (\text{xxiv})$$

In geodetic work the spherical triangles on the earth's surface are comparatively small and the spherical excess seldom exceeds more than a few seconds of arc in numerical value. In this case the spherical excess (in seconds of arc) is usually computed from the approximate formula :—

$$E = \frac{A}{R^2 \sin 1''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxv})$$

where R is the radius of the earth and A is the area of triangle expressed in the same linear units as R (see page 282).

The Right-angled Triangle. If one of the angles of the spherical triangle is a right angle, the formulæ are considerably simplified. Since only two parts, other than the right angle, need be known for solution of the triangle, the following formulæ meet all cases.

Let C be the right angle.

$$\begin{aligned}
\sin a &= \sin A \sin c = \cot B \tan b \\
\sin b &= \sin B \sin c = \cot A \tan a \\
\cos c &= \cos a \cos b = \cot A \cot B \\
\cos A &= \cos a \sin B = \tan b \cot c \\
\cos B &= \cos b \sin A = \tan a \cot c
\end{aligned}$$

Some Useful Series

The following series, here tabulated for reference purposes only, are often useful in astronomical and geodetic work.

(1) **Binomial Expansion.**

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}a^3 + \dots$$
$$\frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}x^{n-r}a^r + \dots\dots$$

(2) **Taylor's Theorem for a single variable (with remainder term),**

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\dots\dots$$
$$+ \frac{h^n}{n!}f^n(x) + \frac{h^{n+1}}{(n+1)!}f^{n+1}(a) \text{ where } x < a < (x+h).$$

(3) Maclaurin's Theorem (with remainder).

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots\dots$$
$$+ \frac{x^n}{n!}f^n(0) + \frac{x^{n+1}}{(n+1)!}f^n(a) \text{ where } 0<a<x$$

(4) Exponential Series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\dots \quad [x^2 < \infty]$$

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \dots \quad [x^2 < \infty]$$

$e = 2{\cdot}7182818285$ value of e^x when $x = 1$.

(5) Logarithmic Series.

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots \quad [-1 < x \leqq 1]$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots\dots \quad [-1 \leqq x < 1]$$

$$\log_e(x+1) = \log_e x + 2\left[\frac{1}{(2x+1)} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \right] [x > 0]$$

$\log_{10} N = \log_e N \,.\, \log_{10}e$ where $\log_{10}e = 0{\cdot}4342944819$

$\log_e N = \log_{10}N \,.\, \log_e 10$ where $\log_e 10 = 2{\cdot}3025850930$

(6) Trigonometric Series. The following series hold when θ is expressed in circular measure :—

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\dots\dots\dots \quad [\theta^2 < \infty]$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\dots\dots\dots \quad [\theta^2 < \infty]$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \frac{62\theta^9}{2835} + \dots\dots \quad \left[\theta^2 < \frac{\pi^2}{4}\right]$$

$$\cot\theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \frac{2\theta^5}{945} - \frac{\theta^7}{4725} + \dots\dots\dots \quad [\theta^2 < \pi^2]$$

$$\sec\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24} + \frac{61\theta^6}{720} + \ldots\ldots\ldots \qquad \left[\theta^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{cosec}\theta = \frac{1}{\theta} + \frac{\theta}{6} + \frac{7\theta^3}{360} + \frac{31\theta^5}{15120} + \ldots\ldots\ldots \qquad [\theta^2 < \pi^2]$$

$$\sin^{-1}\theta = \theta + \frac{1}{2}\cdot\frac{\theta^3}{3} + \frac{1\times3}{2\times4}\cdot\frac{\theta^5}{5} + \frac{1\times3\times5}{2\times4\times6}\cdot\frac{\theta^7}{7} + \ldots \qquad [\theta^2 < 1]$$

$$\cos^{-1}\theta = \frac{\pi}{2} - \sin^{-1}\theta$$

$$\tan^{-1}\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \ldots\ldots\ldots \qquad [\theta^2 < 1]$$

$$\tan^{-1}\theta = \frac{\pi}{2} - \frac{1}{\theta} + \frac{1}{3\theta^3} - \frac{1}{5\theta^5} + \frac{1}{7\theta^7} - \ldots\ldots\ldots \qquad [\theta^2 > 1]$$

$$\cot^{-1}\theta = \frac{\pi}{2} - \tan^{-1}\theta$$

$$\sec^{-1}\theta = \frac{\pi}{2} - \frac{1}{\theta} - \frac{1}{2}\cdot\frac{1}{3\theta^3} - \frac{1\times3}{2\times4}\cdot\frac{1}{5\theta^5} - \frac{1\times3\times5}{2\times4\times6}\cdot\frac{1}{7\theta^7} + \ldots \qquad [\theta^2 > 1]$$

$$\operatorname{cosec}^{-1}\theta = \frac{\pi}{2} - \sec^{-1}\theta.$$

(7) *Reversion of Series.*—If a quantity y can be expressed as a series in ascending powers of another quantity x, in which each succeeding term in the series is numerically less than the term that precedes it, it is often required to express x as a converging series in ascending powers of y. This may be done by the use of the following formulæ :—

If $\quad y = ax + bx^2 + cx^3 + dx^4 + ex^5 +$

where $\quad ex^5 < dx^4 < cx^3 < bx^2 < ax.$ Then

$$x = \frac{y}{a} - \frac{b}{a^3}y^2 + \frac{(2b^2 - ac)}{a^5}y^3 - \frac{(5b^3 - 5abc + a^2d)}{a^7}y^4$$
$$+ \frac{(14b^4 - 21ab^2c + 6a^2bd + 3a^2c^2 - ea^3)}{a^9}y^5 -$$

If $\quad y = ax + bx^3 + cx^5 + dx^7 + ex^9 +$

where $\quad ex^9 < dx^7 < cx^5 < bx^3 < ax.$ Then

$$x = \frac{y}{a} - \frac{by^3}{a^4} + \frac{(3b^2 - ac)}{a^7}y^5 - \frac{(12b^3 - 8abc + a^2d)}{a^{10}}y^7$$
$$+ \frac{(55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - ea^3)}{a^{13}}y^9 -$$

DEFINITIONS OF ASTRONOMICAL TERMS

The quantities dealt with in field astronomy may be classified as relating to : (1) the earth, the celestial sphere, and the observer ; (2) the position of a celestial body ; (3) the classification of stars ; (4) the sun ; (5) corrections ; (6) time.

The Earth, the Celestial Sphere, and the Observer (Fig. 3)

The Terrestrial Equator, $E_1A_1Q_1$, is the great circle of the earth, the plane of which is at right angles to the axis of rotation. The north and south terrestrial poles, P_1 and P_1', are its geometrical poles, and are consequently equidistant from it.

The Celestial Equator, EAQ, is the great circle of the celestial sphere in which it is intersected by the plane of the terrestrial equator. It is therefore midway between the celestial poles P and P′.

The Horizon. (*a*) The *Sensible Horizon* is a plane passing through the observer at right angles to the direction of gravity at the place of observation. It is the plane in which the line of sight of a level telescope lies. (*b*) The *True, Rational* or *Geocentric Horizon* is a parallel plane through the centre of the earth.

The distance between these planes being negligible in comparison with the radius of the celestial sphere, they are to be regarded as coincident and intersecting the celestial sphere in a great circle called the *Celestial Horizon,* or simply the *Horizon.* In Fig. 3, HR represents the horizon of the place X. In the case of the relatively near bodies forming the solar system, the radius of the earth subtends an appreciable angle at the celestial body, and results of observations from the sensible horizon must be corrected to the values they would have if taken at the centre of the earth from the true horizon (see Parallax, page 19).

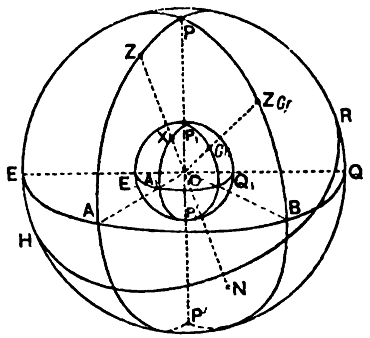


FIG. 3.

The Zenith and Nadir are the poles of the celestial horizon, the zenith Z being the point overhead in which the direction of a plumb line at the observer would meet the celestial sphere, the nadir N being the corresponding point vertically below him.

Terrestrial Meridians are lines of intersection of the surface of the earth by planes passing through its axis, as $P_1A_1P_1'$. If the earth were perfectly spherical, they would be great circles through the poles and secondaries to the equator.

Celestial Meridians are corresponding great circles of the celestial sphere passing through the celestial poles. The celestial meridian of a place or an observer, referred to at the place as *the* meridian, is that great circle which passes through the poles and the zenith and, necessarily, through the nadir also. It is therefore a secondary of the celestial horizon as well as of the equator, and cuts both the horizon and the equator at right angles. PZP′ is the meridian of the observer X.

Vertical Circles are great circles of the celestial sphere through the zenith and nadir. They are all secondaries to the celestial horizon. The observer's meridian is one such circle.

The Prime Vertical is that particular vertical circle which is at right

angles to the meridian, and which therefore passes through the east and west points of the horizon.

The Latitude (ϕ) of a place is the angle between the direction of a plumb line at the place and the plane of the equator. It is marked + or − (or N. or S.) according as the place is north or south of the equator. It will be seen from the definitions of the zenith and the celestial equator that latitude may also be defined as the angle between the zenith and the celestial equator. The latitude of X is therefore represented by ZOA.

The Co-latitude of a place is the complement of the latitude; and is therefore the angle ZOP between the zenith and the celestial pole.

The Longitude (L) of a place is the angle between a fixed reference meridian, called the prime or first meridian, and the meridian of the place. Longitude is measured from 0° to 180° eastwards or westwards from the prime meridian, and must therefore be marked E. or W. The prime meridian universally adopted for astronomical and geodetic work is that of Greenwich. $PZ_{Gr}BP'$ being Greenwich meridian, the longitude of X is angle BOA west, measured on the plane of the equator.

The Position of a Celestial Body

The fact that the situation of a point on the surface of the earth is completely specified by the co-ordinates latitude and longitude suggests the employment of similar means to designate the position of a body on the celestial sphere. In any system of co-ordinates for this purpose, it is necessary to adopt a plane of reference, one of the two angular co-ordinates being measured at right angles to this plane, and the other upon it. This plane should pass through the centre of the sphere, and the first co-ordinate is consequently measured upon a secondary to the great circle formed by the intersection of the reference plane with the sphere. The second is expressed as the angular distance between a certain specified point on this great circle and the secondary through the body. The radius to the specified point may be regarded as a fixed direction of reference. Thus in Fig. 4, let ABCD be the great circle of reference, or its plane, C being the fixed point thereon. The position of the heavenly body S is designated by (1) the angle SOB, or arc SB, of ESBF, the secondary to ABCD through S, (2) the angle COB, on the plane of reference, or the arc CB of the reference circle, between the fixed direction OC and the plane of the secondary through S. To avoid all ambiguity, it is necessary in designating the first co-ordinate to state on which side of the reference plane the body is situated, and to have an established method of measuring the second angle as regards direction.

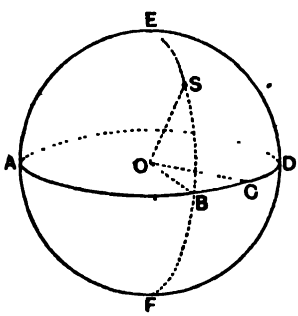


FIG. 4.

Co-ordinate Systems. Several systems of co ordinates are available according to the particular reference plane and reference point adopted. Three systems are of service in field astronomy : (1) altitude and azimuth ;

(2) declination and right ascension; (3) declination and hour angle. The horizon is the reference plane in (1), and the equator in (2) and (3).

The Altitude (h) of a celestial body is its angular distance above the horizon, and is measured on the vertical circle passing through the body. In Fig. 5 angle AOS, or arc AS, represents the altitude of the star S to an observer whose horizon is HAR.

Note. It follows from the definition of latitude that the altitude of the pole P equals the latitude of the place of observation.

Co-altitude or Zenith Distance (z) is the complement of the altitude, and is therefore the angular distance between the body and the zenith, *i.e.* angle SOZ, or arc SZ.

The Azimuth (A) of a celestial body is the angle between the observer's meridian and the vertical circle through the body. There is no universally accepted manner of reckoning azimuth, both quadrantal and whole circle methods being used, and in the latter the north point of the horizon is most often taken as the origin. ZPR being the meridian, and R the north point of the horizon in Fig. 5, the azimuth of S may be expressed as angle ROA, arc RA, or the spherical angle PZS, measured from north towards east. Occasionally azimuths are reckoned clockwise from the south point of the horizon. In this case, the azimuth of S would be the exterior angle, HOA = 180° + ROA, where H is the south point of the horizon.

The Altitude and Azimuth System. The co-ordinates altitude and azimuth are much used in field astronomy. The observation of altitude is that most frequently required, and, since it is measured, subject to correction, by the angle of elevation as registered on the vertical circle of a theodolite, the observation is a simple one.

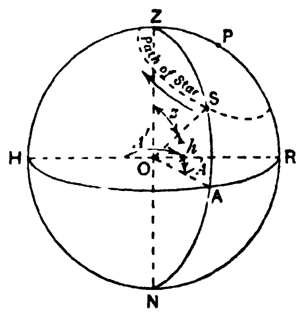


FIG. 5.

Since these co-ordinates not only depend upon the position of the observer, but are constantly changing owing to the diurnal apparent motion of the celestial body about the pole (Fig. 5), they are useless as a means of permanently recording the positions of heavenly bodies. For this purpose it is necessary to employ a system of invariable co-ordinates, and the equator must be the reference plane, since a star in its daily motion maintains a practically constant angular distance from the equator. The point of reference on the equator must partake of the apparent movement for constancy of the second co-ordinate. This is secured in the declination and right ascension system.

The Declination (δ) of a celestial body is its angular distance from the plane of the equator, and is measured on the great circle—generally called the declination circle—through the body and the celestial pole. Declination is measured from 0° to 90°, and is marked + or − according as the body is north or south of the equator. Thus in Fig. 6 the declination of the star S is measured by the angle BOS or the arc BS.

Co-declination or Polar Distance (p) is the angular distance between the body and the pole. It is termed north polar distance when measured from

the north pole, its value being 90° — the declination or 90° + the declination according as the declination is + or —. When reference is made to the south pole, the quantity is called south polar distance, and equals 90° — the declination or 90° + the declination according as the latter is — or +. The angle SOP, or arc SP, is the north polar distance of S, and angle SOP′, or arc SP′, the south polar distance.

The Right Ascension (*R.A.*) of a celestial body is the angle between the declination circle through a fixed point on the equator called the *First Point of Aries*, ♈ (page 18), and the declination circle of the body. It is measured eastwards from ♈, the right ascension of S in Fig. 6 being angle ♈OB or arc ♈B of the equator. Right ascension is intimately related to time measurement, and it is found convenient to reckon it in time units from 0 to 24 hours instead of from 0° to 360° (see page 22)

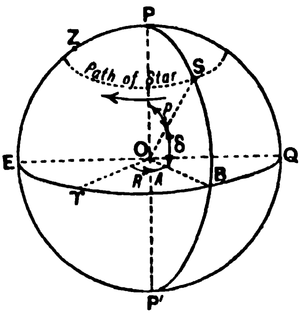


Fig. 6.

The Declination and Right Ascension System. The observations of field astronomy do not involve the measurement of declination and right ascension, but these co-ordinates are constantly required in the reduction of observations and are employed for recording the positions of celestial bodies. Their values for a fixed point in the heavens, although nearly constant, are not absolutely so. Owing to the fact that the earth is not quite spherical, the attractions of the sun, moon and planets exert a disturbing couple which causes a slow secular movement of the plane of the equator, called precession. In addition, due to variation in the value of this couple, a small periodic disturbance, called nutation, is produced. These phenomena cause a variation in the position not only of the plane of the equator but also of the first point of *Aries*, and in consequence, although the positions of the stars relatively to each other are unaffected, both their declinations and right ascensions undergo slow alteration, increasing in some cases, decreasing in others. The fact that the earth's velocity in its orbit is an appreciable fraction of the velocity of light causes a displacement, known as aberration, of star positions from their mean position. A further source of variation in the magnitude of the co-ordinates arises from the fact that the stars do not occupy quite fixed situations on the celestial sphere. This motion, called proper motion, is partly real, due to an actual change of position of the body in the heavens, and partly apparent, due to the movement of the solar system in space. The correction of star positions by the use of *Apparent Places of Fundamental Stars* (page 33) is described and illustrated in pages 41–44. The variation of the declination and right ascension of the sun is very much greater than for the stars, and is treated on page 18.

The Declination and Hour Angle System. The plane of the equator is also used as the reference plane in this system of co-ordinates, the first co-ordinate being declination as defined above. The reference direction is that of the observer's meridian, the reference point being that in which the celestial equator is intersected by the meridian.

The Hour Angle (t) of a celestial body is the angle between the meridian of the observer and the meridian or declination circle of the body. When it is considered in relation to the length of the day, it is measured westwards (or in the same direction as the apparent motion of the body) from the part of the observer's meridian situated above the pole, that is from the position of upper transit (see below), and is reckoned in arc from 0° to 360°, or in time from 0^h to 24^h. Thus, in Fig. 6, the hour angle of S is about 240° or 16^h and is represented by the large angle EOB = 180° + QOB, arc EQB, or the corresponding large spherical angle ZPS. Sometimes, however, and more particularly in theoretical work, it is more convenient to reckon it from 0° to 180°, or from 0^h to 12^h, in terms of the smallest angle west (positive) or east (negative) of the meridian above the pole. In this case the hour angle of S in Fig. 6 would be taken as arc E♈B, small angle EOB, or the small spherical angle ZPS, and would be negative.

Owing to the apparent motion of a celestial body, its hour angle is constantly changing, but at a uniform rate in the case of a star, or at a nearly uniform rate in the case of the sun.

Circumpolar Stars are sometimes defined as those having polar distances less than the latitude of the place of observation. Such stars are, at any point in their diurnal path round the pole, always above the observer's horizon, and therefore do not set.

In Fig. 7, the latitude of the observer whose horizon is HOR is angle ZOE = POR. Any star such as S, the polar distance of which is less than POR, and which therefore lies within the segment PTR of the celestial sphere, is continually above the horizon, and is non-setting to the observer. Stars in the segment P'HK are never visible to him, while all the others appear to rise and set. No stars are non-setting to a person on the equator, his horizon being POP', while to an observer at the north pole all the stars in the northern celestial hemisphere are non-setting, since his horizon is EOQ. For purposes of field astronomy, however, the term circumpolar is usually confined to stars near to the pole, *i.e.* to those whose polar distance is less than 10°.

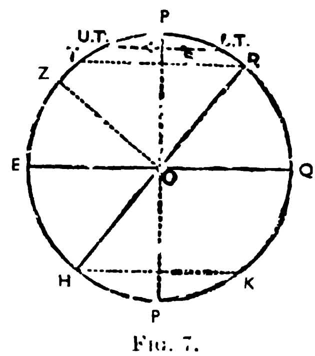


FIG. 7.

Transit or Culmination. When a celestial body crosses the observer's meridian, it is said to transit or culminate. In one revolution round the pole every body crosses the meridian twice, the two transits being distinguished as the upper and lower respectively (U.T. and L.T. in Figs. 7 and 8), the former occurring on the same side of the pole as the observer's zenith and the latter on the opposite side. Fig. 8, which represents the polar region from the point of view of the observer, shows that lower transits can be observed only in the case of non-setting stars, and that a star attains its greatest altitude at upper transit and its least altitude (or greatest distance below the horizon, if a setting star) at lower transit.

Elongation. A celestial body is said to elongate or to be at elongation

when it appears to attain its maximum distance from the meridian. The two elongations, E.E. and W.E. in Fig. 8, are distinguished as the eastern and western elongations respectively.

The Astronomical Triangle. Many of the observations of field astronomy involve the evaluation of such parts of the spherical triangle formed

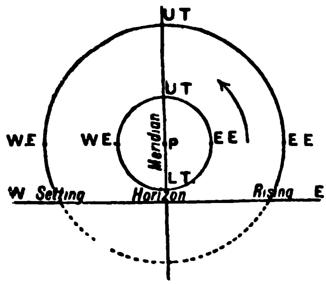


Fig. 8.

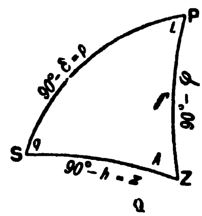


Fig. 9.

by the pole, the zenith and a celestial body as will enable it to be solved for the quantity under determination. This triangle PZS (Fig. 9) is called the astronomical triangle. Its three sides are :

PZ (an arc of the observer's meridian) = the co-latitude of the observer

PS (an arc of the declination circle of the body) = the co-declination or polar distance of the body

ZS (an arc of the vertical circle through the body) = the co-altitude or zenith distance of the body.

The angles are :

at P = the hour angle of the body
at Z = its azimuth
at S = the parallactic angle.

Solution of the triangle requires that three of the parts should be known The side PS is obtained from the declination of the body tabulated in the *Nautical Almanac* or other similar almanac Two of the quantities PZ, ZS and t commonly make up the data Of these PZ can be obtained by independent observations, ZS may be observed, and t is obtained from the readings of a chronometer The unknowns, usually t, A or ZS, are then calculated by means of the formulæ given on pages 4 to 6 or their equivalents.

The Right-Angled Astronomical Triangle. Two important cases occur according as the parallactic angle or the azimuth is a right angle.

In the former case (Fig 10) the celestial body appears to the observer

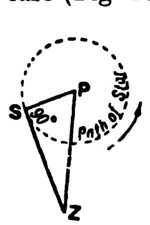


Fig. 10.

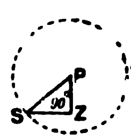


Fig. 11.

to attain its maximum distance east or west of the meridian, *i.e.* the body is at elongation. Solution of the triangle for the altitude, azimuth and hour angle of the body in terms of its declination and the latitude of the place gives :

$$\sin h = \frac{\sin \phi}{\sin \delta}$$

$$\sin A = \frac{\cos \delta}{\cos \phi}$$

$$\cos t = \frac{\tan \phi}{\tan \delta}$$

When the azimuth is a right angle, the body is on the observer's prime vertical (Fig. 11), and

$$\sin h = \frac{\sin \delta}{\sin \phi}$$

$$\cos t = \frac{\tan \delta}{\tan \phi}$$

The Classification of the Stars

Nomenclature. For convenience in distinguishing particular stars, they have been arranged in named groups called constellations. This system of classification originated with the ancients and the number of constellations has been somewhat extended in modern times, but several of the groupings which have been proposed are not in common use.

The brighter stars of each constellation are named by assigning to each a letter of the Greek alphabet followed by the name of the constellation in the genitive case. The letters are assigned roughly in order of brightness. Thus α *Tauri* is the most brilliant star belonging to the constellation *Taurus*, the Bull ; β *Leonis* is the second brightest in the constellation *Leo*, the Lion, and so on. Letters indicating the relative brightness of stars refer only to the constellation to which they belong and do not serve to compare the brightness of stars in different constellations. Thus η *Ursæ Majoris* is a much brighter star than α *Draconis*. The fainter stars in a constellation are distinguished by numerals increasing in the order of their right ascensions.

A second way of referring to a particular star is to quote its index number in one of the various star catalogues that have been published. Thus Boss 1234 is the star so numbered in the Preliminary General Catalogue of Lewis Boss.

Several of the more important stars have been given special names, *e.g.* α *Ursæ Minoris* or *Polaris*, the Pole Star ; α *Lyræ* or *Vega* ; β *Orionis* or *Rigel*, etc. A full list of these names will be found in the *Nautical Almanac*, or in Norton's *Star Atlas* and in the Nautical Almanac office annual publication entitled *Apparent Places of Fundamental Stars*, which gives the apparent positions of 1,535 stars.

Magnitude. The quantity termed the magnitude of a star is a measure of its brightness. Magnitudes are noted in the *Nautical Almanac* for all the stars of which particulars are given, and are indicated by numbers,

the smaller the number the brighter being the star. The brightness of a star of magnitude m is about $2\frac{1}{2}$ times that of a star of magnitude $m + 1$. There are 10 stars brighter than magnitude 1·0, and their magnitudes are given as fractions. Thus the magnitude of *Vega* is 0·1, indicating a brightness 0·9 of a magnitude greater than the unit. Of those stars, *Sirius* and *Canopus* have their magnitudes represented by the negative quantities $-1·6$ and $-0·9$ showing that their brilliancy is 2·6 and 1·9 magnitudes greater than a star of unit magnitude.

Since a bright star is easily discovered in the field of view of a telescope, and observations of it can consequently be made with greater convenience than in the case of a faint star, a table of magnitudes is of service in enabling the surveyor to select suitable stars for observation. Under favourable conditions, stars down to about the sixth magnitude are visible to the naked eye, but, if possible, observations with a small theodolite should be restricted to those that are brighter than the sixth magnitude.

The Sun

The preceding explanations of the apparent positions of celestial bodies have been given more particularly with reference to the stars, which are so remote that the values of their astronomical co-ordinates are unaffected by the orbital motion of the earth. In considering the apparent motion of the sun, however, it must be kept in view that (*a*) its mean distance from the earth is only about 1/250,000 that of the nearest star, (*b*) the earth performs a circuit of the sun yearly. The apparent path of the sun in the heavens is therefore the result of both the diurnal and annual real motions of the earth.

Effect of Earth's Annual Motion. The principal facts relating to the earth's motion about the sun and their effects upon the apparent position of the sun are as follows.

(1) The earth's orbit lies in a plane. The apparent path of the sun is necessarily in the same plane, and, since this plane passes through the centre of the celestial sphere, it intersects the latter in a great circle, called the *Ecliptic*. The apparent motion of the sun is along this great circle.

(2) The earth's orbit is an ellipse having the sun in one of the foci. The earth is thus at varying distances from the sun, being nearest about January 2, when it is said to be in perihelion, the sun being then in perigee, and farthest about July 4, when the earth is said to be in aphelion and the sun in apogee. The points of perihelion and aphelion in the earth's orbit, being situated at the ends of the major axis of the ellipse, are termed the apses of the orbit, the line joining them being called the apse line.

(3) The plane of the ecliptic is not coincident with that of the equator. The angle between them is called the *Obliquity of the Ecliptic*, the value of which at 1950·0 is 23° 26′ 45″ with a mean annual diminution of 0″·47. The axis of the earth is therefore inclined to the plane of the ecliptic at an angle of 90° minus the obliquity of the ecliptic, or about $66\frac{1}{2}$°, and remains practically parallel to itself throughout the year.

The effect of the obliquity is shown by the diagrammatic plan and

sections of the earth's orbit, Figs. 12, 13 and 14. On or about March 21 the axis of the earth is perpendicular to the line joining the earth and the

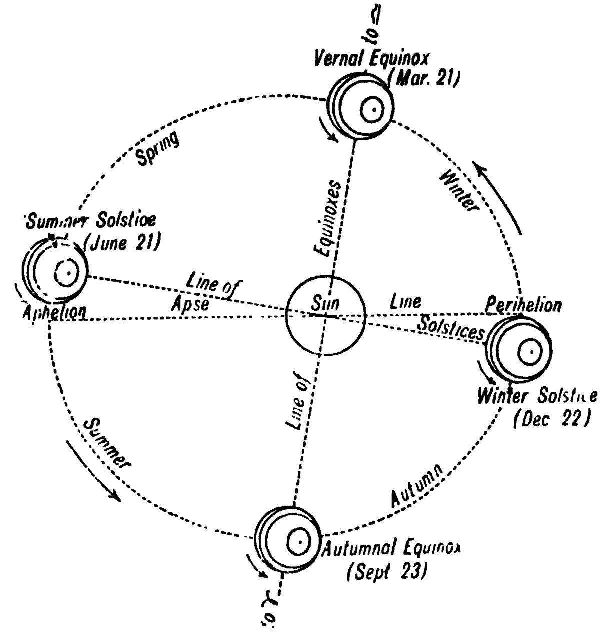


FIG. 12. PLAN OF THE EARTH'S ORBIT.

sun, and the sun is in the plane of the equator. The instant at which this occurs is called the *Vernal Equinox*, day and night being of equal duration throughout the earth. About June 21 the earth is so situated

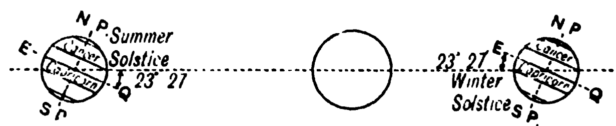


FIG. 13. SECTION ON LINE OF SOLSTICES.

that its axis is in the plane perpendicular to the ecliptic which contains the line joining the earth and the sun, this line now making an angle with the plane of the equator equal to the obliquity of the ecliptic. The sun is there-

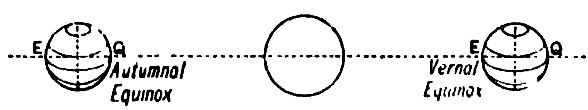


FIG. 14. SECTION ON LINE OF EQUINOXES.

fore vertically over a point on the parallel of latitude 23° 27′ N., called the Tropic of Cancer, this instant being termed the *Summer Solstice.* The *Autumnal Equinox* occurs about September 23, and the *Winter Solstice* about December 22, the sun being then over the Tropic of Capricorn, or 23°

27′ S. The line of equinoxes is the line of intersection of the planes of the ecliptic and the equator, and is at right angles to the line of solstices.

Apparent Motion of the Sun. The apparent motion of the sun produced by the annual circuit of the earth may now be traced. At the vernal equinox the sun is seen from the earth as if projected along the line of equinoxes upon a certain point on the celestial equator. This equinoctial point was at one time in the constellation *Aries*, but, owing to the precession of the equinoxes (page 12) towards the west at the rate of about 50″·3 a year, it is now situated in the constellation *Pisces*. It is called the *First Point of Aries* (♈), and has already been referred to as the point from which right ascensions are reckoned. The sun's right ascension at the vernal equinox is therefore zero. The changing position of the earth in its orbit causes the sun to travel apparently eastward along the ecliptic, *i.e.* in the opposite direction to the apparent diurnal rotation of the celestial sphere. The latter motion being superimposed, the sun appears to move westward across the heavens daily at a slightly slower rate than the stars. Since right ascensions are measured eastwards from ♈, the right ascension of the sun increases, at a variable rate (page 22), by about 1° or 4^m daily, having the values 6^h at the summer solstice, 12^h at the autumnal equinox, and 18^h at the winter solstice, the return to the vernal equinox completing the annual circuit of the ecliptic.

Since the sun crosses the equator at the vernal equinox, its declination is then zero. From March 21 until June 21 the sun travels northwards from the equator, attaining its maximum north declination of about 23° 27′ at the summer solstice. From June 22 to September 23 the north declination decreases until it is again zero at the autumnal equinox. The sun then crosses the equator at the other equinoctial point, called the *First Point of Libra* (♎), and proceeds southwards, its south declination increasing to a maximum of 23° 27′ at the winter solstice on December 22. The south declination thereafter decreases until the equator is again crossed at ♈.

Corrections

In general, the immediate results of astronomical measurements require correction before they can be utilised. Excluding corrections which depend entirely upon the instrument and method of observation employed, the more important corrections required in ordinary field astronomy are here described.

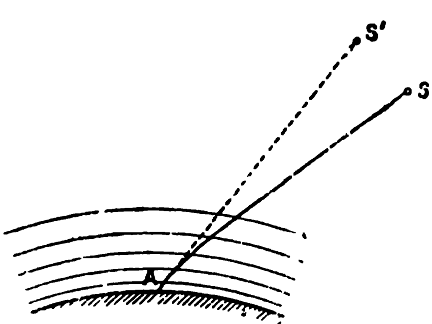


FIG. 15.

Refraction. Rays of light proceeding from a celestial body to an observer, on entering and passing through the earth's atmosphere, undergo a gradual bending due to the increasing density of the atmosphere as the surface of the earth is approached. In consequence of the observer's being conscious only of the direction in which the rays reach him, the body

is seen displaced from its actual position in altitude. Thus in Fig. 15, the celestial body S appears to the observer A to be situated at S′, and a refraction correction, $R = S'AS$, must be applied to reduce the observed or apparent altitude h' to the true altitude h. The correction is always negative to observed altitudes.

Except for low altitudes, the magnitude of the correction is nearly proportional to the cotangent of the apparent altitude, being zero when the body is in the zenith. It also depends upon the density of the air through which the rays pass, and this varies with changing barometric pressure and temperature. To obtain the value of the correction for a particular observed altitude, barometric pressure and temperature, reference is made to refraction tables, those of Bessel * being most commonly used.

In the absence of tables, and neglecting pressure and temperature corrections, refraction is approximately given by

$$R = 58'' \cot h'$$

provided h' is large. If the body is very near the zenith, R is approximately 1″ for each degree of zenith distance.

The value of the refraction correction is always somewhat uncertain, since the assumptions made in the calculation of tabulated values are not applicable to all states of the atmosphere. With a view to minimising the uncertainty, observations should not be made of stars of low altitude, and should be duplicated in such a way that errors of refraction are more or less balanced out by taking the mean of the resulting values.

When accurate values of the altitudes of celestial bodies are required the barometric pressure and atmospheric temperature should be read at the time of observation. For all purposes required in field astronomy, it is sufficient to observe the barometric pressure by means of an ordinary aneroid which has been carefully standardised against a good mercury barometer.

Parallax. Parallax is the apparent change in the direction of a body when viewed from different points. The particular case with which we are concerned may be distinguished as parallax in altitude, or diurnal parallax, and relates to the difference between the altitude of a celestial body as observed from the sensible horizon at the surface of the earth and its altitude from the true horizon at the earth's centre. Thus in Fig. 16, if A is an observer, and O the centre of the earth, the altitudes of a body S at A and O being respectively h' and h, the parallax of S is $h - h' = ASO = P$.

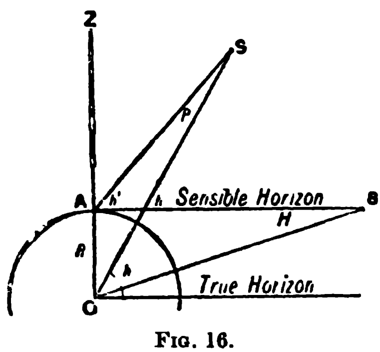


FIG. 16.

The co-ordinates of celestial bodies are given in the *Nautical Almanac* with reference to the centre of the earth, and altitudes to be used in

* Given in some mathematical tables, such as Chambers's *Seven-Figure Mathematical Tables*. The new *Star Almanac for Land Surveyors* (page 33) contains a useful refraction table which gives the refraction correction to the nearest second of arc, which is good enough for work of a minor order of accuracy.

reductions must also be referred to the centre. Observed altitudes are converted to geocentric altitudes by application of the parallax correction, which is represented by the angle P subtended at the body by the earth's radius, and which is always positive. The value of the correction becomes negligible in the case of the stars because of their remoteness and the comparative insignificance of the length of the earth's radius.* In the case of observations of the sun or other bodies of the solar system, however, parallax becomes appreciable, and an appropriate value for the correction must be applied.

Parallax vanishes when the body is in the zenith. It attains its greatest value when the body is on the horizon, as at S_1, and the quantity is then termed the horizontal parallax, H, of the body. On the assumption that the earth is spherical, the amount of the correction may be investigated as follows.

Let D = distance of the body from the earth's centre
R = mean radius of the earth :

then $\sin H = \frac{R}{D}$, whence H.

From triangle OAS, $\frac{\sin P}{\sin (90° + h')} = \frac{R}{D}$

$\therefore \sin P = \sin H \cos h'$.

Except in the case of the moon, P and H are sufficiently small to allow of the approximation

$$P = H \cos h'.$$

The mean value of the sun's horizontal parallax is $8''\cdot80$. It varies from $8''\cdot95$ early in January to $8''\cdot66$ early in July on account of the sun's varying distance from the earth, and is given in the *Nautical Almanac* for every tenth day of the year.

Note that parallax has the effect of moving the observed body away from the zenith whereas refraction has the effect of moving it towards the zenith. The correction for parallax is therefore additional to the observed altitude, the correction for refraction being subtractive.

Semi-diameter. In taking a single measurement of the sun's altitude, the observation is made upon either the lower or the upper edge or limb. To reduce the measured altitude to that of the sun's centre, a correction for semi-diameter is applied, positively in the one case and negatively in the other. The amount of the correction is the value of the angle subtended by the sun's radius. It varies throughout the year between limits of about $16' 18''$ and $15' 45''$ owing to the varying distance of the earth from the sun. Its value is tabulated in the *Nautical Almanac* for every day of the year.

* In the case of the stars, there is another type of parallax called "annual parallax" which depends on the angle subtended at the star by the line joining the earth to the sun. Owing to the vast distance of the stars, annual parallax for most of them is very small, and is ordinarily negligible, but a correction for it has already been applied to the R.A. and declination of 35 of the nearer stars which have annual parallaxes exceeding $0''\cdot100$ and which are listed on page xxxii of *Apparent Places of Fundamental Stars*.

The necessity for applying the correction is obviated if, as is frequently the case, observations are made on both limbs. The mean of the two measured altitudes then represents the apparent altitude of the centre at the instant corresponding to the average of the times of the two observations, provided that the interval of time between the observations is small.

Time

A knowledge of some of the aspects of time measurement is required by the surveyor because of its intimate relationship with hour angle, right ascension and longitude.

The rotation of the earth on its axis and its complete revolution in its orbit, being each performed with absolute regularity, afford suitable standards by which to divide up what is known as time or duration, the former interval being the day, the latter the year. It will be sufficient to consider the division of time into days, as the measurement of long periods does not enter into field astronomy.

The Day. The day may be defined as the interval between successive transits of a celestial body in the same direction across the meridian: according as the body is a star, the sun or the moon, we have a sidereal, solar or lunar day.

Although the rate of rotation of the earth is perfectly uniform, yet the day as measured by transits of the sun or moon is of variable length, since the apparent diurnal motion of those bodies is not simply due to the earth's axial rotation. The sidereal day may, however, be regarded as constant. It is not strictly so, owing to the existence of precession and nutation, which, by affecting the position in space of the earth's axis, have a minute effect upon the times of transit of the stars. By adopting the interval between the transits of ♈ as the day, we have a unit which for every practical surveying purpose is invariable.

Sidereal Time. The sidereal day is the interval of time between two successive upper transits of the first point of *Aries*.

The sidereal time at upper transit of ♈ is 0^h. A clock rated to keep sidereal time is called a sidereal or astronomical clock. It is set to register 0^h at upper transit of ♈ and 24^h at the succeeding upper transit, and at any instant the reading of the clock is the sidereal time which has elapsed since ♈ was on the meridian. It therefore follows from the definition of hour angle (page 13), and with the convention that hour angle is measured westwards from the meridian of the observer, that

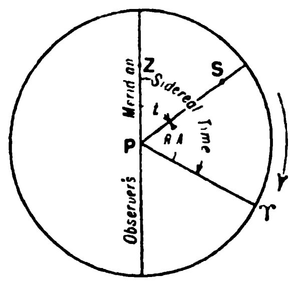


FIG. 17.

The sidereal time at any instant is the hour angle of the first point of Aries, and further, *The hour angle of a star is the sidereal time that has elapsed since its transit.*

Thus in Fig. 17, which represents a plan of the celestial sphere, the hour angle of ♈ is ZP♈ = the sidereal time at the instant represented. The hour angle of the star S is ZPS, but the right ascension of this star is

♈PS, right ascension being measured eastwards, and we therefore have the further relationship;

Star's Hour Angle + Star's Right Ascension = the Sidereal Time,

24^h being subtracted when the sum exceeds 24^h.

Note. The reader will find it instructive to make several sketches similar to Fig. 17 with ♈ and S in various positions, and to verify the relationship in each case.

At the instant when the star makes its upper transit its hour angle is zero, and the above equation reduces to

Star's Right Ascension = the Sidereal Time of its Transit.

A sidereal clock therefore records the right ascensions of stars as they make their upper transits.

It will now be evident why hour angle and right ascension are measured in time in preference to angular units. In the case of the stars, those angles are reckoned in sidereal time units. Since a rotation of the celestial sphere through 360° occupies 24^h, one hour is equivalent to 15°, and hours, minutes and seconds are converted to degrees, minutes and seconds by multiplying throughout by 15. Thus, $5^h\ 12^m\ 43^s$ is equivalent to 78° 10′ 45″. Conversely, division by 15 reduces angular measure to time. Conversion tables are given in the *Nautical Almanac.*

The sidereal division of time, although of great importance in astronomy, is not suited to the needs of everyday life, for the purposes of which the sun is the most convenient time measurer. On account of the increase in the sun's right ascension from 0^h to 24^h in the course of the year, the noon transit of the sun is recorded by the sidereal clock as occurring at 0^h on March 21, 6^h on June 21, 12^h on September 23, and 18^h on December 22. The adoption of sidereal time for ordinary purposes would thus lead to endless inconvenience, which is obviated by the use of solar time.

Solar Time. The interval between successive noon transits of the sun is not constant, two causes operating to produce a variation in the length of the solar day.

(1) The apparent diurnal path of the sun differs from those of the stars because it lies in the ecliptic. In consequence, even although the eastward progress of the sun in the ecliptic were uniform, the time elapsing between the departure of a meridian from the sun and its return thereto would vary because of the obliquity of the ecliptic.

(2) The sun does not advance at a constant rate along the ecliptic, because the motion of the earth in its orbit is not uniform on account of the ellipticity of the orbit.

The variation in the solar day may be examined with reference to the sun's right ascension. The obliquity of the ecliptic and the sun's unequal motion both cause a variable rate of increase of the sun's right ascension. If the yearly change in right ascension from 0^h to 24^h were at a uniform rate, then, since right ascensions are measured on the plane of the equator, the solar day, although different from the sidereal day, would be of constant length throughout the year.

Time as reckoned from the transits of the sun is called *Apparent Solar Time.* It is the time given by a sun-dial, but, owing to the variable length of the day, cannot be recorded by a clock having a constant rate.

To obviate this disadvantage, there has been adopted for common use a system of measurement called *Mean Solar Time*, or simply mean time, in which the day is of invariable length, and which is at the same time free from the disadvantages of sidereal time. The mean solar day is the average of all the apparent solar days of the year. Mean solar time may be conceived as being determined by an imaginary point, called the mean sun, which moves at a uniform rate along the *equator*, its constant rate of increase of right ascension being the average rate of increase of the sun's right ascension. The mean solar day may therefore be defined as the interval between two successive transits of the mean sun.

Civil and Astronomical Reckoning. In the ordinary reckoning of time the day commences at midnight, and is divided into two periods, *a.m.* (*antemeridiem*) and *p.m.* (*postmeridiem*). Astronomers, however, with a view to avoiding ambiguity, count the mean solar day as beginning at midnight, but divide it continuously from 0^h to 24^h. The former system is called civil time, and the latter astronomical time.

To Convert Civil Time into Astronomical.—If the civil time is *a.m.*, make no change; if *p.m.*, add 12^h.

Thus, June 30, 2 a.m. is expressed astronomically as June 30^d 02^h.
June 30, 2 p.m. „ „ June 30^d 14^h.

To Convert Astronomical Time into Civil. If the astronomical time is less than 12^h, make no change and mark *a.m.*; if greater than 12^h, subtract 12^h, marking *p.m.*

Thus, June 30^d 10^h is equivalent to June 30, 10 a.m.
June 30^d 20^h „ „ June 30, 8 p.m.

It is to be observed that what are called civil and astronomical times are simply systems of reckoning mean solar time.*

The Hour Angle and Right Ascension of the Sun. The relationships between hour angle, right ascension and time which have been given on pages 21 and 22 for sidereal time are similar in the case of solar time. We have, therefore,

At any instant the apparent solar time is the hour angle of the sun + 12^h, *and the mean solar time is the hour angle of the mean sun* + 12^h.

Sun's Hour Angle + Sun's Right Ascension = the Sidereal Time, and similarly,

Mean Sun's Hour Angle + Mean Sun's Right Ascension = the Sidereal Time.

As before, since the hour angle is zero at upper transit,

Sun's Right Ascension = the Sidereal Time of Apparent Noon, and

Mean Sun's Right Ascension = the Sidereal Time of Mean Noon.

Interconversion of Time Systems. The aim of observations for time is usually to determine mean time at any instant, but since the mean sun is merely a convenient hypothesis, mean time cannot be found directly. Apparent solar time is obtained from sun observations, while star observations give sidereal time, and it is therefore necessary to be able to deduce the mean time equivalent of the computed result.

Relationship between Mean and Apparent Solar Time. The difference at any instant between apparent and mean time is known as the *Equation*

* Before 1925 January 1, the astronomical day was reckoned from noon, instead of from midnight. For further explanation see the *Nautical Almanac*.

of Time (E). This quantity is usually defined as the amount which must be added to or subtracted from apparent time to give mean time. It may be expressed as the angle SPM (Fig. 18) between the meridian of the true sun and that of the mean sun, and is therefore the difference at any instant between the respective hour angles or right ascensions of the true and mean suns. In particular, at the instant of apparent noon, the sun's hour angle being then zero, the mean time equals the equation of time $+ 12^h$. Thus if the value of the equation of time at apparent noon on a certain day is $+ 8^m$, apparent noon occurs at 12·8 p.m., the mean sun being ahead of the true sun. If the value is $- 8^m$, the sun transits at 11.52 a.m.

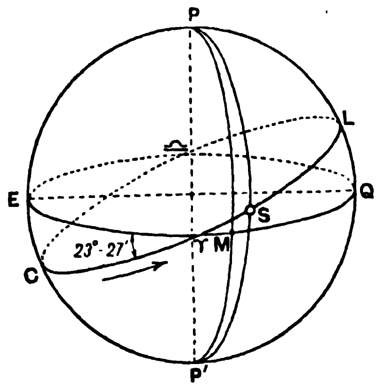


Fig. 18.

The amount of the equation of time and its variation are due to the two sources of variation in the length of the apparent solar day, namely, the obliquity of the ecliptic and the sun's unequal motion therein. The difference due to obliquity has a maximum value of about $\pm 10^m$, that due to unequal motion about $\pm 7^m$. The period of these two components are different, and on combining them we have the equation of time as represented in Fig. 19. The true sun and the mean sun are on the same meridian four times during the year, the equation of time then vanishing.

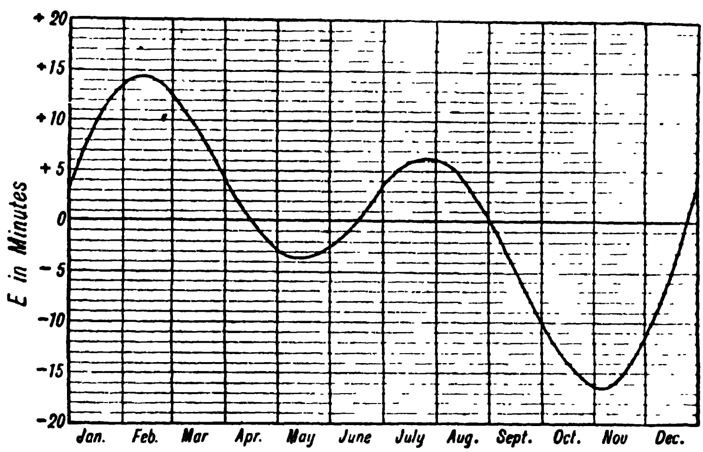


Fig. 19.

This occurs about April 16, June 14, September 1 and December 25. The maximum values attained are roughly $+ 14^m$ about February 11, $- 4^m$ about May 14, $+ 6^m$ about July 27, and $- 16^m$ about November 3.

The equation of time is tabulated in the *Nautical Almanac* for every midnight, but in the sen e apparent — mean, so that it must be added

algebraically to mean time to give apparent time, and *vice versa*. The G.M.T. of apparent noon, *i.e.* the moment when the sun transits at Greenwich, is also given. In the new *Star Almanac for Land Surveyors* (page 33) 12 hours have been added to the equation of time and the tabulated quantity E is given at 6-hour intervals. Consequently, the tabulated value of E for 12^h is the G.A.T. of mean noon, and, for other tabulated values, E is the equation of time plus 12 hours, or G.A.T. minus G.M.T. plus 12 hours.

Relationship between Mean and Sidereal Time Intervals. The difference between the lengths of the mean and sidereal days is a constant amount, since both are defined by points (the mean sun and ♈) which move at constant rates along the equator. The motion of the mean sun being in an eastward direction, the mean solar day is evidently longer than the sidereal by the daily increase in the right ascension of the mean sun. Now the total increase of 24^h in this quantity is accomplished in the interval from one vernal equinox to the next. This period is called the tropical year, and is equivalent to about 365·2422 mean solar days. The daily increase in right ascension of the mean sun, or the amount by which the mean solar day exceeds the sidereal, is therefore

$$\frac{24^h}{365{\cdot}2422} = 3^m\ 56^s{\cdot}56 \text{ sidereal.}$$

The matter may be considered in another way. In a tropical year the earth actually makes 366·2422 revolutions on its axis, but since it has in that period travelled round the sun, the latter has made one upper transit less than ♈, so that we have the relationships

$$365{\cdot}2422 \text{ mean solar days} = 366{\cdot}2422 \text{ sidereal days}$$

$$\frac{\text{Mean time units in any interval}}{\text{Sidereal time units in the same interval}} = \frac{365{\cdot}2422}{366{\cdot}2422} = \frac{1}{1{\cdot}002738}$$
$$= 0{\cdot}997270$$

$$1 \text{ mean solar day} = 24^h\ 03^m\ 56^s{\cdot}56 \text{ sidereal time}$$

$$1 \text{ sidereal day} = 23^h\ 56^m\ 04^s{\cdot}09 \text{ mean time.}$$

The conversion of mean time intervals to sidereal units, or *vice versa*, is facilitated by the use of the tables of equivalents published in the *Nautical Almanac*, and in *Apparent Places of Fundamental Stars* as Tables III and IV.

It is to be understood that the above relationships refer only to the conversion of time *intervals* from one unit to another, and do not enable one to deduce the mean time at a particular instant from the sidereal time of that instant, or *vice versa* (for which see pages 39 *et seq.*).

Connection between Time and Longitude. Let PA and PB (Fig. 20) be the meridians of two places separated by δL of longitude,

t_A and t_B = the respective hour angles at those places of a time measurer T (♈, the apparent sun, or the mean sun) at any instant

Then $t_A - t_B = \text{APB} = \delta L$.

Difference of longitude, expressed in time units, is therefore equal to difference of local time, whether the time system is sidereal, apparent solar or mean solar.

In particular, in Fig. 21, let PEP′Q be the meridian of Greenwich, S the sun, and M the mean sun. It is evidently afternoon at a place X west of Greenwich, the sun having crossed the meridian PAP′ some

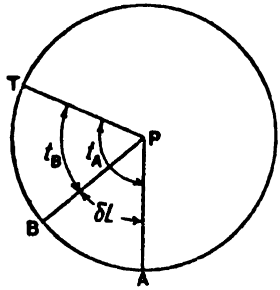


FIG. 20.

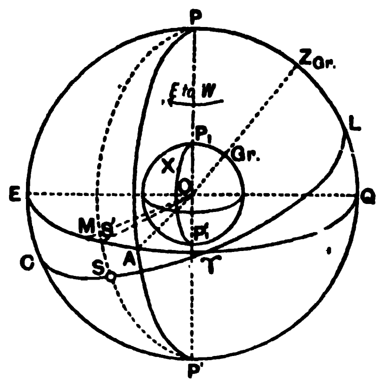


FIG. 21.

little time before. The local apparent time is AOS′ (the hour angle of the sun) $+ 12^h$ and AOM $+ 12^h$ represents local mean time. Now at Greenwich the sun transited some hours before, Greenwich apparent time being represented by QOS′ $+ 12^h$, and Greenwich mean time by QOM $+ 12^h$. Therefore, with the convention that west longitudes are positive and east longitudes negative, we have

Greenwich Time — Local Time = Longitude in time.

At the instant of local midnight, local time is 0^h, so that

Greenwich Time of Local Midnight = Longitude in time.

These relationships are applicable to sidereal, apparent solar or mean solar time. The local time referred to, however, is that at the meridian of the place and is not necessarily that adopted for every day purposes. To obviate the inconvenience arising through each place using its own local time, the various countries of the world have adopted *Standard Times*, which are referred to the meridian of Greenwich and in most cases differ from Greenwich time by an integral number of hours.* Thus Greenwich time itself is used in Great Britain, Ireland, France, Belgium, Spain and Portugal. Mid-European time is that of the meridian 1^h E. Canada is divided into five time belts, the Atlantic, Eastern, Central, Mountain and Pacific, which are respectively 4^h, 5^h, 6^h, 7^h and 8^h W the last four of these are adopted in the United States. A complete list of the standard times adopted in various countries is given in the *Nautical Almanac*.

* Because of its almost universal use as a standard time, Greenwich Mean Time (G.M.T.) is now often called Universal Time (U.T.).

Interpolation, by Successive Differences, from the Nautical Almanac and other similar Tables

To Interpolate the Value of a Varying Tabulated Quantity by Means of Second and Third Differences. The required value may be obtained (*a*) by simple or linear interpolation between the successive tabulated values on the assumption that the rate of change is uniform and equal to its value at the middle of the interval, (*b*) by interpolating strictly, taking higher order differences into account.

Let $f_{-2}, f_{-1}, f_0, f_1, f_2$, etc., be successive values of a function that it is desired to interpolate, and let the first differences between these values, be denoted by Δ', with appropriate suffixes, *e.g.* $f_{-1} - f_{-2} = \Delta'_{-\frac{3}{2}}$, $f_{-1} - f_0 = \Delta'_{-\frac{1}{2}}, f_1 - f_0 = \Delta'_{\frac{1}{2}}, f_2 - f_1 = \Delta'_{\frac{3}{2}}$. Similarly, the differences known as the second differences, are denoted by Δ'', with suffixes, *e.g.* $\Delta'_{-\frac{1}{2}} - \Delta'_{-\frac{3}{2}} = \Delta''_{-1}$, $\Delta'_{\frac{1}{2}} - \Delta'_{-\frac{1}{2}} = \Delta''_0$, $\Delta'_{\frac{3}{2}} - \Delta'_{\frac{1}{2}} = \Delta''_1$, and so on. These differences may be represented thus :—

Number or Argument	Function or Entry	Differences				
		First	Second	Third	Fourth	Fifth
F_{-2}	f_{-2}					
		$\Delta'_{-\frac{3}{2}}$				
F_{-1}	f_{-1}		Δ''_{-1}			
		$\Delta'_{-\frac{1}{2}}$		$\Delta'''_{-\frac{1}{2}}$		
F_0	f_0		Δ''_0		Δ^{iv}_0	
		$\Delta'_{\frac{1}{2}}$		$\Delta'''_{\frac{1}{2}}$		$\Delta^{v}_{\frac{1}{2}}$
F_1	f_1		Δ''_1		Δ^{iv}_1	
		$\Delta'_{\frac{3}{2}}$		$\Delta'''_{\frac{3}{2}}$		
F_2	f_2		Δ''_2			
		$\Delta'_{\frac{5}{2}}$				
F_3	f_3					

Note that $\Delta''_0 + \Delta''_1 = \Delta'_{\frac{3}{2}} - \Delta'_{-\frac{1}{2}}$, $\Delta^{iv}_0 + \Delta^{iv}_1 = \Delta'''_{\frac{3}{2}} - \Delta'''_{-\frac{1}{2}}$, etc.

Suppose now that we wish to find f_n, *i.e.* the value of the function at a fraction n of the way between f_0 and f_1. There are a number of formulæ available, but, generally speaking, the most suitable one, especially if third and higher order differences may be neglected (as they may in most cases which arise in practical astronomy and geodesy), is Bessel's :—

$$f_n = f_0 + n\Delta'_{\frac{1}{2}} + B''(\Delta''_0 + \Delta''_1) + B'''\Delta'''_{\frac{1}{2}} + B^{iv}(\Delta^{iv}_0 + \Delta^{iv}_1) + B^{v}\Delta^{v}_{\frac{1}{2}} + \dots$$

where n is the fractional value of the interval between two tabular values and

$$B'' = \frac{n(n-1)}{2.2!}$$

$$B''' = \frac{n(n-1)(n-\frac{1}{2})}{3!}$$

$$B^{iv} = \frac{(n+1)n(n-1)(n-2)}{2.4!}$$

$$B^{v} = \frac{(n+1)n(n-1)(n-2)(n-\frac{1}{2})}{5!}$$

Neglect of the last four terms in the equation is equivalent to simple linear interpolation. The third term is a maximum when $n = \frac{1}{2}$ and its value is then one-eighth of the average second difference. Hence, as successive differences in general decrease very rapidly in value, the criterion for deciding if second differences must be taken into account is whether one-eighth of the second difference is negligible in value or not. The coefficient, B''' of the fourth term has a maximum positive value when n is approximately 1/5, its value then being $+ 0{\cdot}008$, and a maximum negative value when n is approximately 4/5, its value then being $- 0{\cdot}008$.

The quantities B'', B''', B^{iv}, etc. are called the Besselian Interpolation Coefficients. Values for B'' are tabulated, for different values of n, in Tables XVIII and XIX and for B''' and B^{iv} in Table XVIII of the *Nautical Almanac*. B'' is always negative, since n is always less than unity; B''' is plus for values of n between 0 and 0·5 and negative for values of n between 0·5 and 1·0; and B^{iv} is always positive. Table XX, giving $B''(\Delta''_0 + \Delta''_1)$, can be used when $\Delta''_0 + \Delta''_1$ is less than 300.

Example. Find log 7531 given the following logarithms :—

Number	Logarithm	Δ′	Δ″	Δ‴
7300	863 3229			
		+ 59088		
7400	869 2317		− 792	
		+ 58296		+ 19
7500	875 0613		− 773	
		+ 57523		+ 21
7600	880 8136		− 752	
		+ 56771		+ 20
7700	886 4907		− 732	
		+ 56039		+ 18
7800	892 0946		− 714	
		+ 55325		
7900	897 6271			

Here, the third differences remain practically constant, so there is no point in proceeding beyond them. In this example :—

$$f_0 = 8750613,\ n = 0{\cdot}31,\ \Delta'_{\frac{1}{2}} = + 57523$$

$$(\Delta''_0 + \Delta''_1) = - 773 - 752 = - 1525,\ \Delta'''_{\frac{1}{2}} = + 21$$

From Table XVIII, $B'' = - 0{\cdot}0535$, $B''' = + 0{\cdot}0066$

$$\begin{aligned}
f_0 &= 8750613 \\
n \times \Delta'_{\frac{1}{2}} = 0{\cdot}31 \times + 57523 &= + 17832{\cdot}13 \\
B''(\Delta''_0 + \Delta''_1) = - 0{\cdot}0535 \times - 1525 &= + \quad 81.59 \\
B'''\Delta'''_{\frac{1}{2}} = + 0{\cdot}0066 \times 21 &= + \quad 0{\cdot}14 \\
&= 8768526{\cdot}86 \\
\therefore \log 7531 &= 8768527
\end{aligned}$$

Another common interpolation formula, the Gregory-Newton formula, makes use of the binomial coefficients and is as follows :—

$$f_n = f_0 + n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{2!}\Delta''_1 + \frac{n(n-1)(n-2)}{3!}\Delta'''_{\frac{3}{2}} + \ldots$$

A table of the second and third binomial coefficients, to three decimal places in terms of n to two decimal places, is given on page 437 of *Chambers's Seven-Figure Mathematical Tables.* Accordingly, it may be more convenient to use this formula, instead of the one given, when the differences are small or the *Nautical Almanac* is not available. In general, however, the Gregory-Newton formula is inferior to the Newton-Bessel formula.

Interpolation may also be performed without differences by using the Lagrange formula. This formula, for differences up to the third, may be written

$$f_n = L_{-1}f_{-1} + L_0 f_0 + L_1 f_1 + L_2 f_2$$

where

$$L_{-1} = -\frac{n(n-1)(n-2)}{6}$$

$$L_0 = \frac{(n+1)(n-1)(n-2)}{2}$$

$$L_1 = -\frac{(n+1)n(n-2)}{2}$$

$$L_2 = +\frac{(n+1)n(n-1)}{6}$$

and

$$L_{-1} + L_0 + L_1 + L_2 = 1.$$

In ordinary practice, the Lagrange formula is not nearly so convenient to use as the Bessel formula, but cases sometimes do arise when it is better to employ it, especially when a calculating machine is being used. One case where it may be useful is when the interval of a table is being halved, so that $n = \frac{1}{2}$, in which case the formula reduces to the very simple form

$$f_{\frac{1}{2}} = 0{\cdot}0625\,(-f_{-1} + 9f_0 + 9f_1 - f_2).$$

In the above, it has been assumed that the argument advances by equal intervals, as nearly always occurs in practice. Sometimes, however, values of a function are given at intervals which are not equal and here it becomes necessary to use the method of "divided differences." As this case occurs very seldom in ordinary field astronomy and geodesy it is not given here but will be found described in Whittaker and Robinson's *The Calculus of Observations*, or in the same authors' shorter work *A Short Course in Interpolation.*

In using the *Nautical Almanac*, or other similar tables, the interval of tabulation is so close that third and higher order differences are seldom or never required, so that the Bessel formula may be written :—

$$\begin{aligned} f_n &= f_0 + n\Delta'_{\frac{1}{2}} + B''(\Delta''_0 + \Delta''_1) \\ &= f_0 + n\Delta'_{\frac{1}{2}} + B''(\Delta'_{\frac{3}{2}} - \Delta'_{-\frac{1}{2}}). \end{aligned}$$

Inverse Interpolation. Suppose that, in the last example, we are given the logarithm 876·4975 and we want to find the anti-logarithm of this from the table. This is the reverse case to the one we had before. Symbolically, it means that the numerical value of f_n is given and the argument, F_n, corresponding to this value is required. Obviously, we have the solution if we can determine n.

One method is to proceed by successive approximations. First, find an approximate value of n from the equation

$$n\Delta'_{\frac{1}{2}} = f_n - f_0$$

With this value of n, enter Table XVIII and obtain approximate values for B'', B''' and B^{iv}. Substitute these in the equation.

$$n\Delta'_{\frac{1}{2}} = f_n - f_0 - B''(\Delta''_0 + \Delta''_1) - B'''\Delta'''_{\frac{1}{2}} - B^{iv}(\Delta^{iv}_0 + \Delta^{iv}_1)$$

and solve for n. If third and subsequent differences are negligible, this solution will generally give a value of n sufficiently accurate for all practical purposes. If not, use this value of n to obtain new values of B'', B''' and B^{iv} from Table XVIII or XIX in the *Almanac* and substitute these values in the last equation.

Example. From the table of differences used in the previous example find the number whose logarithm is 876 4975.

$$\text{Here } f_n = 876\ 4975,\ f_0 = 875\ 0613,\ \Delta'_{\frac{1}{2}} = 57523.$$

$$\therefore\ f_n - f_0 = 14362$$

$$\therefore\ n = \frac{14362}{57523} = 0{\cdot}2497 \text{ approximately.}$$

From Table XVIII :—

$$B'' = -0{\cdot}047,\ B''' = +0{\cdot}008,\ B^{iv} = +0{\cdot}009$$

$$\Delta''_0 + \Delta''_1, = -1525;\ \Delta'''_{\frac{1}{2}} = +21$$

Then the equation gives

$$\begin{aligned} 57523n &= 14362 - 0{\cdot}047 \times 1525 - 0{\cdot}008 \times 21 \\ &= 14362 - 71{\cdot}675 - 0{\cdot}168 \\ &= 14290{\cdot}11 \end{aligned}$$

giving $n = 0{\cdot}24842$ approximately.

Using this value of n and Table XIX we get

$$B'' = -0{\cdot}0467. \quad \text{Also, } B''' = +0{\cdot}008$$

and

$$\begin{aligned} 57523n &= 14362 - 71{\cdot}22 - 0{\cdot}17 \\ &= 14290{\cdot}62 \end{aligned}$$

which gives $n = 0{\cdot}24843$

Hence, the anti-log of 876 4975 is 752 4843, as may be verified from a table of seven-figure logarithms.

The method of successive approximations may be avoided, and the interpolation may be made directly, by the use of the following formula which, with different symbols, is the one given in Winterbotham's *Survey Computations* for inverse interpolation up to fourth differences.

$$n = \frac{\delta}{A} - \frac{B}{A} \cdot \frac{\delta^2}{A^2} + \left(\frac{2B^2}{A^2} - \frac{C}{A}\right)\frac{\delta^3}{A^3} - \left(\frac{5B^3}{A^3} - \frac{5BC}{A^2} + \frac{D}{A}\right)\frac{\delta^4}{A^4}$$

where $f_n - f_0 = \delta$ and

$$\begin{aligned} A &= \Delta'_{\frac{1}{2}} - \tfrac{1}{2}\Delta''_m + \tfrac{1}{12}\Delta'''_{\frac{1}{2}} + \tfrac{1}{12}\Delta^{iv}_m \\ B &= \tfrac{1}{2}\Delta''_m - \tfrac{1}{4}\Delta'''_{\frac{1}{2}} - \tfrac{1}{24}\Delta^{iv}_m \\ C &= \tfrac{1}{6}\Delta'''_{\frac{1}{2}} - \tfrac{1}{6}\Delta^{iv}_m \\ D &= \tfrac{1}{24}\Delta^{iv}_m \end{aligned}$$

in which $\Delta''_m = \frac{1}{2}(\Delta''_0 + \Delta''_1)$ and $\Delta^{iv}_m = \frac{1}{2}(\Delta^{iv}_0 + \Delta^{iv}_1)$.

In nearly all cases that occur in practice the higher order terms in this equation are negligible.

A very useful pamphlet on interpolation, in the form of a reprint from the *Nautical Almanac* for 1937 and entitled "Interpolation and Allied Tables," may be obtained from H.M. Stationery Office for 1*s*. This pamphlet, among other things, contains useful hints on interpolation when a calculating machine is used. The more essential tables, so far as the surveyor is concerned, and an explanation of the methods of using them, are, however, contained at the back of each year's issue of the *Almanac*.

Double Interpolation. In the cases so far considered the quantity to be interpolated is a function of a single variable only. Sometimes, however, a quantity is tabulated which is a function of two variables. For example, the Table on page 187 gives a factor k, to be used in determining the catenary correction for a tape or wire used on a slope. This factor depends on the slope and on a quantity B, which is a function of the weight of the tape per unit length, the span involved and the pull used.

Let Z, with suitable suffixes, represent the tabulated function or entry, X_0, X_1, X_2 . . ., Y_0, Y_1 Y_2 . . . successive values of the arguments in the columns and rows and let it be required to interpolate the value Z_{mn} for an mth part of the interval $X_1 - X_0$ and an nth part of the interval $Y_1 - Y_0$. If the table has not already successive differences tabulated, form one as follows :—

X . .	Y				
	Y_0 Δc	Δr Δ''	Y_1 Δc	Δr Δ''	Y_2 Δc
X_0 . .	Z_{00} Δc_{00}	Δr_{00} Δ''_{00}	Z_{01} Δc_{01}	Δr_{01} Δ''_{01}	Z_{02} Δc_{02}
X_1 . .	Z_{10} Δc_{10}	Δr_{10} Δ''_{10}	Z_{11} Δc_{11}	Δr_{11} Δ''_{11}	Z_{12} Δc_{12}
X_2 . .	Z_{20}	Δr_{20}	Z_{21}	Δr_{21}	Z_{22}

Here $\Delta c_{00} = Z_{10} - Z_{00}$; $\Delta c_{01} = Z_{11} - Z_{01}$; $\Delta c_{10} = Z_{20} - Z_{10}$, etc.

$\Delta r_{00} = Z_{01} - Z_{00}$; $\Delta r_{10} = Z_{11} - Z_{10}$; $\Delta r_{01} = Z_{02} - Z_{01}$, etc.

$\Delta''_{00} = \Delta c_{01} - \Delta c_{00} = \Delta r_{10} - \Delta r_{00}$, etc.

Then

$$Z_{mn} = Z_{00} + m\Delta c_{00} + n\Delta r_{00} + mn\Delta''_{00} + \tfrac{1}{2}m(m-1)(\Delta c_{10} - \Delta c_{00}) + \tfrac{1}{2}n(n-1)(\Delta r_{01} - \Delta r_{00})$$

Usually the "mesh" of the table is chosen so that only the first four erms in this formula are required.

Example. Take part of the table on page 187 which is used for computing sag correction on a slope when the tension is applied at the lower end of the tape, and find the value of k for $B = 0{\cdot}125$, $\theta = 21° 15'$. Then $m = 0{\cdot}5$, $n = 0{\cdot}25$.

B	$\theta°$				
	20°		25°		30°
0·12	0 000508 + 86	− 40 − 8	0·000468 + 78	− 45 − 7	0·000423 + 71
0·13	594 + 93	− 48 − 8	546 + 85	− 52 − 9	494 + 76
0·14	687	− 56	631	61	570

Here, $Z_{00} = 0{\cdot}000508$; $\Delta r_{00} = -40$; $\Delta c_{00} = +86$; $\Delta''_{00} = -8$;
$(\Delta c_{10} - \Delta c_{00}) = +7$; $(\Delta r_{01} - \Delta r_{00}) = -5$.

$$\therefore\ k = 0{\cdot}000508 + 0{\cdot}000001\,(0{\cdot}5 \times 86 - 0{\cdot}25 \times 40 - 0{\cdot}5 \times 0{\cdot}25 \times 8 - \tfrac{1}{2} \times 0{\cdot}5 \times 0{\cdot}5 \times 7 + \tfrac{1}{2} \times 0{\cdot}25 \times 0{\cdot}75 \times 5)$$
$$= 0{\cdot}000508 + 0{\cdot}000001\,(43 - 10 - 1 - 0{\cdot}875 + 0{\cdot}469)$$
$$= 0{\cdot}000540.$$

In surveying, the occasions on which "Tables of Double Entry," such as the one just given, are most frequently used are when rectangular co-ordinates or the convergence of the meridians are tabulated in terms of latitude and longitude.

It is obvious that interpolation by successive differences is of great value when constructing special tables, either of single or double entry, because the function can be computed for a comparatively few values of the argument only and intermediate values interpolated by the use of the successive differences. Here it is well, when computing the preliminary table, to compute it to at least two places more than that ultimately required in the completed table.

The Use of the Nautical Almanac and of the Star Almanac for Land Surveyors

The Nautical and Star Almanacs. *The Nautical Almanac*,* in common with similar foreign publications, contains such particulars of the varying positions of celestial bodies as are required for the reduction of observations.

Commencing with the issue for 1931, the *Nautical Almanac* was redesigned, and furnished with a very complete explanation. It will be assumed, in the remarks that follow, that the reader has a copy of it and/or of the *Star Almanac for Land Surveyors* before him. Certain further slight changes were introduced in 1935 and subsequent years, so that the examples in this chapter will be taken mainly from 1951.

The values of the published elements are continually changing, and in applying any of the quantities to the reduction of an observation its value at the time of the observation is required. It is therefore generally necessary to interpolate the value corresponding to a given instant from

* Published annually by the Admiralty two years in advance.

the tabulated values. Further, the value of the elements are given for certain times on the meridian of Greenwich, and it is usually required to obtain their values for a particular instant of local time at a place east or west of Greenwich. It is then necessary to express the given time in terms of Greenwich time before the interpolation can be performed.

The ordinary standard edition of the *Nautical Almanac* contains many tables, such as tables giving the positions of the moon and planets, which the surveyor never needs. Accordingly, as a result of a proposal made at the 1947 Conference of Commonwealth Officers, the *Nautical Almanac* Office is preparing a special *Star Almanac for Land Surveyors*, and it is expected that the first issue, giving tables for 1951, will be published some time during the summer of 1950. The new almanac will only contain tables that are of immediate use to ordinary surveyors, whose requirements are an ephemeris of the sun and the positions, to an accuracy of about a second of arc, of a large number of stars. Hence, the volume will contain an ephemeris of the sun, apparent places of about 650 stars (including all stars brighter than magnitude 4·0), a star index, circumpolar star tables, times of sunrise and sunset, a refraction table and interpolation tables, together with a minimum of explanatory matter. Besides being much less bulky and expensive, the new almanac, in many respects, will be more convenient to use than the ordinary edition of the *Nautical Almanac*, as the interval of tabulation will normally be much closer. Thus, the position of the sun will be given for 6-hour intervals for every day of the year instead of at daily intervals as in the *Nautical Almanac*, so avoiding the necessity for interpolation by second differences. Against this, the right ascension of the sun and the equation of time will only be given to a tenth of a second of time and the declination to a tenth of a minute of arc, while apparent places of stars will be given to a tenth of a second of time in right ascension and to a second of arc in declination. This is accurate enough for ordinary work of a minor order. If greater accuracy is needed, the standard edition of the *Nautical Almanac* should be used, as it gives the places of the sun and stars to a hundredth of a second of time in right ascension and to a tenth of a second of arc in declination, but for first-order geodetic or similar work it is necessary to use the Nautical Almanac Office publication *Apparent Places of Fundamental Stars*, which is described on page 41, or a similar star catalogue. Commencing with the issue for 1952, however, the *Nautical Almanac* will no longer give the positions of stars, so that these will then have to be obtained from the *Star Almanac for Land Surveyors* for minor work or from *Apparent Places of Fundamental Stars* when greater accuracy is needed. In fact, after 1950 the surveyor will only need the *Nautical Almanac* when data for the sun are required to a greater degree of accuracy than that to which thay are given in the *Star Almanac*.

The manner in which the *Nautical Almanac* and the *Star Almanac for Land Surveyors* are used for working out common problems is illustrated below. In most cases, two solutions of each problem are given, the data used in the first being taken from the standard edition of the *Nautical Almanac* and the data used in the second being taken from

the *Star Almanac.** The procedures in the two cases are somewhat different, although the basic principles remain the same. When the ephemeris of the sun is used, it should be remembered that the data given on pages 6 to 21 of the *Nautical Almanac* are for the *apparent* sun at G.M.T. 0^h, and those given on pages 22 to 29 are for the apparent sun at transit at Greenwich, that is for the apparent sun at G.A.T. 12^h. In the case of the *Star Almanac*, the tabulated values of R and E_1 are for the *mean* sun at 6-hour intervals of G.M.T. In addition, in the *Star Almanac*, the quantity tabulated as R on pages 2–25 is R.A. of the *mean* sun $\pm 12^h$ and the quantity tabulated as E, for which we shall use the symbol E_1, is Equation of time $+ 12^h$. Hence the value of R for 0^h in the tables is also the sidereal time at G.M T. 0^h and the value of E_1 for 12^h is also the G.A.T. of Greenwich mean noon. From this it will be seen that R A. of apparent sun $= R - E_1$.

An index to places of stars, arranged alphabetically by the names of the constellations, is given in the *Nautical Almanac* (pp. 615 616), in the *Star Almanac* (pp. 53–55) and in *Apparent Places of Fundamental Stars* (pp. 529-538). Hence, to find the data relating to any star, it is only necessary to look up the star in the index and find the relevant page in the tables of apparent places.

Notation. The following notation † is adopted :—

Altitude	h	Right ascension of mean sun $\pm$ 12^h quantity tabulated as R in the S.A.	R
Apparent places of fundamental stars	$A.P.$	Greenwich apparent time	G.A.T
Azimuth	A	Greenwich mean time	G.M.T.
Co-altitude or zenith distance	z	Greenwich sidereal time	G.S.T.
Co-declination or polar „	p	Greenwich apparent noon	G.A.N.
Declination	δ	Greenwich mean noon	G.M.N.
Equation of time	E	Greenwich sidereal noon	G.S.N.
Equation of time $+ 12^h$, or quantity tabulated as E in the S.A.	E_1	Local apparent time	L.A.T.
		Local mean time	L.M.T.
Hour angle	t	Local sidereal time	L.S.T.
Latitude	ϕ	*Nautical Almanac*	*N.A.*
Longitude	L	*Star Almanac for Land Surveyors*	*S.A.*
Right ascension	R.A.		

The astronomical method of writing dates is year, month, day, hour, etc.

1. To obtain the Sun's Declination at any given Date and Greenwich Mean Time.

Rule. Express the hours, minutes and seconds as a fraction of a day. This can be done by ordinary arithmetic or, more conveniently, by means of Table V (pp. 542–3) of the *Nautical Almanac.* So determine n, the fraction to be used in the interpolation. Then obtain from the Tables

* The data used in the examples on pages 34 to 41 are actually taken from some proofs of the *Nautical* and *Star Almanacs* giving data for 1951 which were kindly provided by the Superintendent of the Nautical Almanac Office, as at the time of writing the issues of these *Almanacs* for that year had not been published.

† The notation here is that commonly used in surveying, and differs in some instances from that used in the *Nautical Almanac*, and in astronomy generally.

for the Sun given on pages 6–21 the sun's apparent declination at Greenwich for zero hour on the given date, and interpolate for the hours, minutes and seconds by the methods already described for interpolation by successive differences.

Example. To find the value of the sun's declination at 1951 Jan. $4^d\ 11^h\ 50^m\ 30^s$ G.M.T. from the following data :—

	Date	Declination	Δ'	Δ''
From *N.A.*, p. 6,	1951 Jan. 3	$-22°\ 55'\ 29''{\cdot}3$		
			$+337''{\cdot}7$	
	4	22 49 51·6		$+27''{\cdot}2$
			$+364{\cdot}9$	
	5	22 43 46·7		$+27{\cdot}1$
			$+392{\cdot}0$	
	6	-22 37 14·7		

Here it is seen that the third differences are negligible.

From Table V of the *N.A.* (pp. 542–3), $11^h\ 50^m\ 30^s = 0{\cdot}49341$. Hence, $n = 0{\cdot}49341$ and B'' Table XVIII $= -0{\cdot}062$.

$$f_0 = \text{Declination at Jan. } 4^d\ 0^h = -22°\ 49'\ 51''{\cdot}6$$
$$n\Delta'_1 = 0{\cdot}49341 \times +364''{\cdot}9 = +\ 3\ 00{\cdot}0$$
$$B''(\Delta''_0 + \Delta''_1) = B''(\Delta'_1 - \Delta') = -0{\cdot}062 \times +54''{\cdot}3 = -\ 3{\cdot}4$$
$$\text{Sum} = f_n = \text{Dec. at } 4^d\ 11^h\ 50^m\ 30^s = -\ 22\ 46\ 55{\cdot}0$$

In this case the neglect of second differences would have caused an error of 3″, which is about the maximum error that could occur with the sun's declination. The maximum error with the right ascension or equation of time is $0^s{\cdot}1$.

The sun's right ascension and the equation of time can be found in a similar manner using the data given on pages 6 to 21 of the *N.A.* In these cases the second differences are very small and can usually be neglected.

The *Star Almanac* gives the declination of the sun at 6-hour intervals of G.M.T. and hence the problem becomes one of simple interpolation from the tables. Thus, with the following tabulated values from p. 2 of the *S.A.* :—

G.M.T.	Sun's Declination		
0^h	$-22°\ 49'{\cdot}9$		
			$+1'{\cdot}5$
6	22 48·4		
			$+1{\cdot}5$
12	22 46·9		
			$+1{\cdot}6$
18	22 45·3		

we have :—

Given time $= 11^h\ 50^m\ 30^s$
$= 6^h + 5^h\ 50^m\ 30^s$
$= 6^h + 0{\cdot}974$ of 6^h

From *S.A.*, p. 2, Declination of sun at 6^h G.M.T. $= -22°\ 48'{\cdot}4$
$0{\cdot}974 \times +1'{\cdot}5$ (or from the Interpolation Table for the sun on p. 63 of the *S.A.*) $= +1{\cdot}5$
Sum = Declination of sun at 1951, Jan. $4^d\ 11^h\ 50^m\ 30^s$ $= -22\ 46{\cdot}9$

Here, for the sake of illustration, we have interpolated from the value for 6^h, but, as the given time is almost 12^h and a tabulated value for 12^h is available, it would have been simpler to interpolate backwards from this value. Thus :—

Given time $= 11^h\ 50^m\ 30^s$
$= 12^h - 0^h\ 09^m\ 30^s$
$= 12^h - 0{\cdot}026$ of 6^h

From *S.A.*, p. 2. Declination of sun at 12^h G.M.T. $= -22°\ 46'{\cdot}9$
$-0{\cdot}026 \times +1'{\cdot}5$ $= 0{\cdot}0$
Sum = Declination of sun at 1951, Jan. $4^d\ 11^h\ 50^m\ 30^s$ $= -22\ 46{\cdot}9$

If the R.A. and E are needed, R and E_1 may be obtained by simple linear interpolation in the same way, interpolation for R being facilitated if desired by the use of the special Interpolation Table for R on pages 64–65. E is then equal to $E_1 - 12^h$, but, as the R.A. of the true or apparent sun is required, and the tabulated values of R are for the mean sun, we use the relation :—

$$\text{R.A. of true or apparent sun} = R - E_1.$$

Interpolations for the sun's declination and for E_1 can be simplified by means of the special Interpolation Table for the Sun on page 63 of the *S.A.*, as this is in terms of actual interpolation intervals and avoids the necessity for working out the interpolation coefficient obtained by dividing the interpolation interval by 6^h.

2. To find the Greenwich Time corresponding to a Given Instant of Local Time.

Rule. To the given local time, expressed in astronomical reckoning, apply the longitude in time, adding if west and subtracting if east. The resulting Greenwich time is on the same system, apparent, mean or sidereal, as the local time.

Example. Find G.A.T. corresponding to L.A.T. April 24 (*a*) 12.15 p.m., (*b*) 5.23 p.m. at a place in longitude 32° 15′ E.

			h	m
(*a*) Given L.A.T., expressed astronomically	=	Apr. 24	12	15
East longitude in time	=	—	2	09
Corresponding G.A.T.	=	Apr. 24	10	06
(*b*) Given L.A.T., expressed astronomically	=	Apr. 24	17	23
East longitude in time	=	—	2	09
Corresponding G.A.T.	=	Apr. 24	15	14

3. To find the Local Mean Time corresponding to a Given Instant of Local Apparent Time.

In this case it is required to find the value of E, the equation of time, at the given instant. As the time given is apparent, we utilise the G.M.T. of transit of the sun at Greenwich (*N.A.* pp. 22–29), *i.e.* the G.M.T. of apparent noon, as the difference between this time and 12^h is the equation of time at apparent noon.

Rule. Find G.A.T. corresponding to the given L.A.T. Interpolate the value of G.M.T. of transit at Greenwich to this G.A.T. Then

$$\text{L.M.T.} = \text{L.A.T.} + \text{G.M.T. of transit at Greenwich} - 12^h$$

Example. A time observation on 1951 Jan. 3, at a place in longitude 64° 45′ W., gave L.A.T as $17^h\ 34^m$. Find the corresponding L.M.T.

			d	h	m
Given L.A.T.	=	Jan.	3	17	34
West longitude	=			+ 4	19
G.A.T.	=		3	21	53
	=	$3^d\ 12^h + 9^h\ 53^m$			
	=	$3^d\ 12^h + 0^d{\cdot}4118$			

		h	m	s
From *N.A.*, p. 22, G.M.T. at G.A.T. $3^d\ 12^h$	=	12	04	16″·64
$0{\cdot}4118 \times +\ 27^s{\cdot}72$	=		+	11·42
$-\ 0{\cdot}061 \times -\ 0^s{\cdot}72$	=		+	0·04
Sum = G.M.T. of transit as at G.A.T. $3^d\ 21^h\ 53^m$	=	12	04	28·10
L.A.T.	=	17	34	00·00
Sum − 12^h = L.M.T.	=	17	38	28·10

If the *Star Almanac* is used, we must determine the proper interpolation interval since the given time is apparent time and the values given in

the *S.A.* are for G.M.T. Hence we must convert the nearest G.M.T. from which we are going to interpolate into apparent time and subtract the result from the G.A.T. equivalent of the given L.A.T. Thus, after finding the G.A.T. as above :—

		d	h	m	s
From *S.A.*, p. 2, E_1 at 3^d 18^h G.M.T.	=	0	11	55	36·5
3^d 18^h G.M.T.	=	3	18	00	00·0
Sum = 12^h = G.A.T. at 3^d 18^h G.M.T.	=	3	17	55	36·5
G.A.T. corresponding to given L.A.T. from above	=	3	21	53	00·0
Difference = interpolation interval	=		3	57	23·5
	=	0·66 of 6^h			

		d	h	m	s
E_1 at 3^d 18^h G.M.T. from above	=		11	55	36·5
0·66 × $- 7^s·0$ (or from the Interpolation Table on p. 63 of the *S.A.*)	=				4·6
Sum = E_1 at 3^d 21^h 53^m G.A.T.	=		11	55	31·9
L.A.T.	=	3	17	34	00·0
Difference + 12^h = L.M.T.	=	3	17	38	28·1

It should be noted here that, although the interpolation interval of 3^h 27^m $23^s·5$ is in terms of apparent time, the 6-hour interval between the times of successive tabulated values is in terms of mean time. Consequently, from the strictly theoretical point of view, this 6-hour interval of mean time should be converted into the equivalent interval of apparent time when working out the interpolation coefficient 0·66. To do this, we could use the rule :—

Apparent time interval = mean time interval + change in value of E_1 during the interval.

The maximum value of the change in the value of E_1 during a 6-hour interval never exceeds $7^s·5$, which is only 0·002 of an hour. Hence, for all practical purposes, so far as working out an interpolation coefficient is concerned, we can neglect, as we have neglected above, the difference between 6 hours of mean time and the interval of apparent time which corresponds to a mean time interval of 6 hours.

4. To find the Local Apparent Time corresponding to a Given Instant of Local Mean Time.

Rule. Interpolate the value of *E* from those given for mean midnight, and add the value so obtained algebraically to the L.M.T.

Example. Find L.A.T. corresponding to L.M.T. 1951 Jan. 15^d 21^h 06^m at a place in longitude 64° 45′ W.

			d	h	m
Given L.M.T.	=	Jan.	15	21	06
West longitude	=			+ 4	19
G.M.T.	=	Jan.	16	01	25
	=	Jan.	$16^d·0590$		

		m	s
From *N.A.*, p. 6, *E* for Jan. 16^d 0^h	=	− 9	27·86
0·0590 × $- 20^s·83$	=	−	1·23
− 0·014 × $+ 1^s·40$	=	−	0·02
Sum = *E* for Jan. 16^d 01^h 25^m	=	− 9	29 11

			d	h	m	s
Given L.M.T.	=	Jan.	15	21	06	00·00
E (Apparent − Mean)	=			−	9	29·11
Sum = Required L.A.T.	=	Jan.	15	20	56	30·89

The computation by means of the *S.A.* is equally simple :—

		d	h	m	s
Given L.M.T.	= Jan.	15	21	06	00·0
West longitude	=		+ 4	19	00·0
G.M.T.	=	16	01	25	00·0
	= Jan.	$16^d\ 00^h + 0{\cdot}24$ of 6^h			
From *S.A.*, p. 2, E_1 at $16^d\ 00^h$ G.M.T.			11^h	50^m	$32^s{\cdot}1$
$0{\cdot}24 \times -5^s{\cdot}2$ (or from Interpolation Table for Sun)	=				$-1{\cdot}2$
Sum $= E_1$ at Jan. $16^d\ 01^h\ 25^m$ G.M.T.	=		11	50	30·9
Given L.M.T.	= Jan.	15	21	06	00·0
Sum $- 12^h$ = Required L.A.T.	= Jan.	15	20	56	30·9

5. To find the Sun's Right Ascension and Declination at any Instant of Local Time.

Rule. First obtain the corresponding G.T. If this is G.M.T. interpolate the values given for every midnight. If the time obtained is G.A.T., interpolate the values at transit at Greenwich (*N.A.* pp. 22–29), which are, of course, for G.A.T. $= 12^h$.

Example. Find the sun's R.A. on 1951 Jan. $4^d\ 16^h$ L.A.T. at a place in longitude 87° 15′ E.

		d	h	m
Given L.A.T.	= Jan.	4	16	00
East longitude	=		− 5	49
G.A.T.	= Jan.	4	10	11
	= Jan.	$3^d\ 12^h + 22^h\ 11^m$		
	= Jan.	$3^d\ 12^h + 0^d{\cdot}92431$		

		h	m	s
From *N.A.*, p. 22, R.A. at G.A.T. $3^d\ 12^h$	=	18	53	29·73
$0{\cdot}92431 \times +264^s{\cdot}36$	=		+ 4	04·35
$-0{\cdot}017 \times -0^s{\cdot}73$	=		+	0·01
Sum = R.A. at G.A.T. Jan. $4^d\ 10^h\ 11^m$	=	18	57	34·09

In the case of the *S.A.*, the tabulated values of R are for the mean sun, not for the true or apparent sun, and it is the value for the apparent sun which is required. Hence we must interpolate for E_1 and R and use the expression :

$$\text{R.A. of apparent sun} = R - E_1.$$

Here also, as the given time is apparent time, we must begin, as we began in *Example* 3, by reducing the nearest G.M.T., $4^d\ 6^h$, from which we are going to interpolate to G.A.T. so as to obtain a real interpolation interval. This interval will then be in terms of apparent time whereas the interval for which the values are tabulated is 6 hours of mean time. In practice, however, when calculating the interpolation coefficient, it is sufficient, within the limits of accuracy involved, to assume that 6 hours apparent time are the equivalent of 6 hours mean time and hence the interpolation coefficient can be obtained by dividing by 6 the interpolation interval found by subtracting the apparent time at $4^d\ 6^h$ G.M.T. from the G.A.T. corresponding to the given L.A.T., or else we can use this interval as the argument in entering the Interpolation Table on page 63 and the Interpolation Table for R on page 65 of the *S.A.*

		d	h	m	s
From *S.A.*, p. 2, E_1 at Jan. $4^d\ 6^h$ G.M.T.	=	0	11	55	22·6
Jan. $4^d\ 6^h$ G.M.T.	=	4	6	00	00·0
Sum $- 12^h$ = G.A.T. at Jan. $4^d\ 6^h$	=	4	5	55	22·6
G.A.T. at given L.A.T. as above	=	4	10	11	00·0
Difference = interpolation interval	=		4	15	37·4

	h	m	s
From above, E_1 at Jan. 4^d 6^h G.M.T. =	11	55	22·6
Increase for interval of 4^h 15^m $37^s{\cdot}4$ (from Interpolation Table for Sun on p. 63 of the *S.A.*) =			− 4·9
Sum E_1 at Jan. 4^d 10^h 11^m G.A.T =	11	55	17·7
From *S.A.*, p. 2, R at Jan. 4^d 6^h G.M.T. =	6	52	09·8
Increase for interval of 4^h 15^m $37^s{\cdot}4$ (from Interpolation Table for R on p. 65 of the *S.A.*) =			+ 42·0
Sum R at Jan. 4^d 10^h 11^m G.A.T. =	6	52	51·8
E_1 from above =	11	55	17·7
$R - E_1$ =	− 5	02	25·9
R.A. of sun at Jan. 4^d 16^h L.A.T. =	18	57	34·1

It may be useful to note here that, in calculations which involve finding the R.A. of the sun, an approximate estimate of the number of hours in the R.A., which is good enough to use as a check in seeing that 12 hours have not been missed or wrongly applied, can be obtained by remembering that the R.A. of the sun is zero on or about March 22 of each year and increases from then until the same date in the following year at the rate of about 2^h for each succeeding month. Using this rule, we see that the hours in the sun's R.A. on Jan. 4 are approximately $9{\cdot}4 \times 2^h = 19^h$.

In finding the sun's declination, the differences for 6-hour intervals are comparatively small when the work is only being taken to a tenth of a minute of arc and hence we can use the same interpolation interval as before.

From *S.A.*, p. 2, Declination at Jan. 4^d 6^h G.M.T. =	− 22°	48′·4
Increase for interval of 4^h 15^m $37^s{\cdot}4$ (from Interpolation Table for Sun on p. 63 of the *S.A.*) =		+ 1·1
Sum = Declination at Jan. 4^d 16^h L.A.T. =	− 22	47·3

6. To find the Local Mean Time of Local Apparent Noon.

Rule. Interpolate the G.M.T. of transit at Greenwich to the given longitude.

Example. Find the approximate L.M.T. of L.A.N. in longitude 5^h 42^m East on 1951 Jan. 4.

	h	m	s
From *N.A.*, p. 22, G.M.T. of transit at Greenwich =	12	04	44
− 0·24 × + $27^s{\cdot}4$ =		−	7
Sum = L.M.T. of L.A.N. =	12	04	37

When using the *S.A.*, correct E_1 for the time equivalent of the longitude and subtract the result from 24^h.

	h	m	s
From *S.A.*, p. 2, E_1 for 1951 Jan. 4^d 12^h G.M.T.	11	55	16
Increase for interval of 5^h 42^m (from Interpolation Table for Sun on p. 63 of the *S.A.*)			+ 7
Sum E_1 at 12^h L.M.T.	11	55	23
	24	00	00
Difference = L.M.T. of L.A.N. =	12	04	37

7. To find the Local Mean Time of Local Sidereal 0^h.

Rule. Correct the G.M.T. of transit of first point of Aries for the date

at the rate of $9^s{\cdot}8296$ for every hour of longitude, adding if east and subtracting if west. This is most conveniently done with the aid of Table IV.

Example. Find the L.M.T. of local sidereal 0^h on 1951 Jan. 4 at a place in longitude $4^h\ 23^m$ E.

		h	m	s
From *N.A.*, p. 7, Transit of Aries, Jan. 4	=	17	06	00·79
Correction for 4^h East (Table IV)	=		+	39·32
Correction for 23^m East (Table IV)	=		+	3·77
Sum = L.M.T. of local sidereal 0^h	=	17	06	43·88

When the computation is made with the aid of the *S.A.*, we must derive the mean time of transit from the tabulated value of R of the mean sun at mean noon.

		h	m	s
From *S.A.*, p. 2, R.A. of mean sun at 12^h G.M.T. $- R + 12^h$	=	18	53	08·9
Sidereal interval until transit of Aries at Greenwich		5	06	51·1
Mean time equivalent of $5^h\ 06^m\ 51^s{\cdot}1$ sidereal time	=	5	06	00·8
Mean time of transit of Aries at Greenwich	=	17	06	00·8
Correction for 4^h East	=		+	39·3
,, ,, 23^m East	=		+	3·8
Sum = L.M.T. of local sidereal 0^h	=	17	06	43·9

8. To find the Local Mean Time corresponding to a Given Instant of Local Sidereal Time.

Rule. Find the L.M.T. of the preceding local sidereal 0^h, as in the previous example, and to it add the mean time equivalent (from Table IV) of the given sidereal time.

Example. Find the L.M.T. corresponding to 1951 Jan. $5^d\ 13^h\ 08^m\ 54^s{\cdot}2$ L.S.T at a place in longitude $4^h\ 23^m$ East.

		d	h	m	s
From *Example* 7, L.M.T. of preceding L.S. 0^h	=	4	17	06	43·88
Mean time equivalent (Table IV) of 13^h	=		12	57	52·22
08^m	=			7	58·69
54^s	=				53·85
$0^s{\cdot}2$	=				·20
Sum = L.M.T.	=	5	06	13	28·84

9. To find the Local Sidereal Time Corresponding to a Given Instant of Local Mean Time.

Rule. Correct the G.S.T. at 0^h at the rate of $9^s{\cdot}8565$ for every hour of longitude, adding if west and subtracting if east, and to this add the sidereal equivalent of the given L.M.T.

Example. Find the L.S.T. corresponding to 1951 Jan. $5^d\ 06^h\ 12^m\ 32{\cdot}3^s$ L.M.T., at a place in longitude $4^h\ 23^m$ East.

		h	m	s
From *N.A.*, p. 6, G.S.T. at 0^h	=	6	55	07·22
Correction for 4^h East	=		—	39·43
Correction for 23^m East	=		—	3·78
Sidereal equivalent of 6^h	=	6	00	59·14
12^m	=		12	01·97
$32^s{\cdot}3$	=			32·39
Sum = L.S.T.	=	13	07	57·51

Since the R.A. of the mean sun is equal to the sidereal time of mean noon, the sidereal time at 0^h M.T. will be the mean sun's R.A. at that

time $\pm$ 12^h, which is the value of R at 0^h M.T. Hence, if we are using the *S.A.*, we have :—

Tabulated value of R for Jan. 5^d 0^h = G.S.T. at 0^h G.M.T. = 6^h 55^m 07^s·2 and the remainder of the calculation proceeds as before.

10. To find the Local Mean Time of Transit of a Star.

Rule. Since the R.A. of the star equals the L.S.T. at the instant of its upper transit, convert this L.S.T. to L.M.T., as in *Example* 8. The lower transit is separated by 12 sidereal hours from the upper.

Note. For the purpose of selecting stars for observation, a rough estimate of the time of transit is usually all that is required. It is then sufficient to subtract from the R.A. of the star, increased if necessary by 24^h, the G.S.T. at 0^h for the date.

Example. Find the approximate L.M.T. at which β *Ursæ Minoris* will be on the meridian on 1951 Jan. 7.

		h	m
From *S.A.*, p. 40, R.A. of β *Ursæ Minoris*	=	14	51
From *S.A.*, p. 2, R for Jan. 7^d 0^h = G.S.T. at 0^h	=	7	03
Difference = L.M.T. of transit	=	7	48

11. To find the Local Sidereal Time of Elongation of a Star.

Rule. To the R.A. of the star apply its hour angle at elongation (page 15), in time, adding for west and subtracting for east elongation. The result, increased or diminished by 24^h if necessary, represents the L.S.T. of elongation.

Example. Find the L.S.T. at which β *Ursæ Minoris* will elongate on the evening of 1951 July 27, at a place in latitude 55° 52′ N.

			h	m	s
From *S.A.*, pp. 40 41, R.A. of β *Ursæ Minoris*	=		14	50	51·7
δ	=	+ 74° 21′ 25″			
At elongation $\cos t$	=	$\dfrac{\tan\phi}{\tan\delta}$			
log tan 55° 52′	=	0·168835			
log tan 74° 21′ 25″	=	0·552818			
log cos t	=	9·616017			
t	=	65° 36′ 09″ =	4	22	24·6
L.S.T. of W. elongation	=		19	13	16·3

As the sidereal time at 0^h is approximately 20^h 15^m (*S.A.*, p. 15), it is evident that the western elongation is the one that occurs in the evening.

Use of Apparent Places of Fundamental Stars. In 1941 the number of stars whose mean and apparent places are given in the *Nautical Almanac* was reduced to 208, and the places of these stars are now (1950) only tabulated to 0^s·01 in right ascension and 0″·1 in declination. In the same year, a separate volume, giving the apparent places of 1,535 stars, was issued under the title *Apparent Places of Fundamental Stars* and, after a break caused by the war, it is now being published annually. This publication, which should be used in preference to the *Nautical* or *Star Almanacs* when work of the highest order of precision is being undertaken, or when a greater choice of stars than those now listed in these *Almanacs* is desired, gives the apparent places of stars whose declination is less than $\pm$ 60° to 0^s·001 in R.A. and 0″·01 in declination, and of stars whose declination exceeds $\pm$ 60° to 0^s·01 in R.A. and 0″·01 in declination. The interval of tabulation is every tenth upper transit at Greenwich for all of the 1,483 stars whose declination is less than $\pm$ 81°, and for the 52 circumpolar stars (*i.e.* those within 9° of either pole) the interval is

for each upper transit at Greenwich throughout the year. The data required for survey work are the apparent places and these may normally be obtained with sufficient accuracy by interpolation, using differences up to the second order and the Bessel interpolation formula, from the list of apparent places given in pages 36–511. The tabulated values have had all necessary corrections, such as the corrections for proper motion, annual aberration, annual parallax, etc., applied to them with the exception in many cases of certain corrections for short-period terms of nutation.* When the period of tabulation is the interval between 10 upper transits, these short-period terms complete their oscillations within this period and hence they cannot conveniently be included in the tabulated values, since to do so would cause difficulties in interpolation. They are, however, very small in magnitude and can be neglected in all except work of high precision. If it is desired to take them into account, let $\Delta\alpha$ be the correction to right ascension and $\Delta\delta$ the correction to the declination. Then

$$\begin{aligned} \Delta\alpha \text{ in seconds of time} &= d\alpha(\psi)\,.\,d\psi + d\alpha(\epsilon)\,.\,d\epsilon, \\ \Delta\delta \text{ in seconds of arc} &= d\delta(\psi)\,.\,d\psi + d\delta(\epsilon)\,.\,d\epsilon, \end{aligned}$$

in which the quantities $d\psi$ and $d\epsilon$, the short-period terms of nutation in celestial longitude ψ † and obliquity ϵ, are given for every day of the year in Table I (pp. 512–513), and the quantities

$$\begin{aligned} d\alpha(\psi) &= \tfrac{1}{15}\,(\cos\epsilon + \sin\alpha\tan\delta\sin\epsilon)\,; & d\delta(\psi) &= \cos\alpha\sin\epsilon \\ d\alpha(\epsilon) &= -\tfrac{1}{15}\cos\alpha\tan\delta\,; & d\delta(\epsilon) &= \sin\alpha \end{aligned}$$

are tabulated on pages 36–407 under the entries for each star.

In the case of the circumpolar stars with polar distances less than 9° whose positions are tabulated on pages 408–511, these corrections have already been applied, as here the interval between entries is only that between successive upper transits, and this covers the periods of the short-period terms.

An example, illustrating the application of these corrections is given below.

A table, Table II, in *Apparent Places* also gives the sidereal time at 0^h G.M.T. for every day of the year to $0^s{\cdot}001$, whereas the corresponding table in the *N.A.* only goes to $0^s{\cdot}01$ and to $0^s{\cdot}1$ in the *S.A.* Consequently, Table II in *Apparent Places* should be used when time is computed to a thousandth of a second. When apparent sidereal time is taken from the table to pass from mean solar time or from G.M.T. to apparent sidereal time and *vice versa*, a very small correction for change in nutation in

* The corrections for annual parallax have only been applied to thirty-five stars for which parallaxes equal to or greater than $0''{\cdot}100$ are available, and a list of these stars is given on page xxxii of the Introduction to the Tables. In 1948, however, the International Astronomical Union decided that these corrections should be included for a greater number of stars as soon as the necessary arrangements could be made. At present, therefore, the corrections for annual parallax that have been omitted from the tabulated values may exceed those for the short-period terms of nutation.

† Celestial longitude is the angular distance measured eastwards along the ecliptic from the First Points of Aries to the point where a great circle drawn through the star perpendicular to the ecliptic passes through the latter.

right ascension between 0^h and a given G.M.T. has to be applied. This can be done by adding together the long- and short-period terms of nutation given in the table for the given date and the day following and taking the difference. The amount of the correction is this difference multiplied by the solar interval since 0^h and divided by 24^h. Examples of these computations are given below.

As noted on page 33, the positions of stars will no longer be given in the 1952 and subsequent issues of the *Nautical Almanac* so that, for all computations relating to stellar observations, the surveyor will then have the choice of the *Star Almanac for Land Surveyors* for minor work or *Apparent Places of Fundamental Stars* for work of greater accuracy.

Example 1. Find the apparent place of α *Aquilæ* on 1951 June 13·6.

From page 341 of *Apparent Places* on 1951 June 10·1 R.A. $= 19^h\ 48^m\ 26^s{\cdot}086$ $\delta = +8°\ 44'\ 09''{\cdot}38$.

Interval of tabulation is 10 transits and hence $n = 0{\cdot}35$.

For R.A., $\Delta'_{\frac{1}{2}} = +0^s{\cdot}228$; $\Delta'_{\frac{3}{2}} = +0^s{\cdot}195$; $\Delta'_{-\frac{1}{2}} = +0^s{\cdot}256$
$\Delta''_0 + \Delta''_1 = \Delta'_{\frac{3}{2}} - \Delta'_{-\frac{1}{2}} = -0^s{\cdot}061$

	h	m	s
R.A. on June 10·1	= 19	48	26·086
$n\Delta'_{\frac{1}{2}} = 0{\cdot}35 \times +0^s{\cdot}228$			+ 0·080
$B''(\Delta''_0 + \Delta''_1)$ (Table VI)	=		+ 0·003
Sum = R.A. on June 13·6	= 19	48	26·169

For δ, $\Delta'_{\frac{1}{2}} = +2''{\cdot}14$, $\Delta'_{\frac{3}{2}} = +2''{\cdot}13$; $\Delta'_{-\frac{1}{2}} = +2''{\cdot}08$
$\Delta''_0 + \Delta''_1 = \Delta'_{\frac{3}{2}} - \Delta'_{-\frac{1}{2}} = +0''{\cdot}05$

	°	′	″
δ on June 10·1	= + 8	44	09·38
$n\Delta'_{\frac{1}{2}} = 0{\cdot}35 \times +2''{\cdot}14$	=		+ 0·75
$B''(\Delta''_0 + \Delta''_1)$ (Table VI)	=		− 0·00
Sum = δ on June 13·6	= + 8	44	10·13

These values are accurate for most ordinary work, but in observations of the highest precision it is necessary to add the corrections for the short-period terms in nutation.

From page 341 of the tables we find for α *Aquilæ*

$$d\alpha(\psi) = +0^s{\cdot}057;\quad d\delta(\psi) = +0''{\cdot}18$$
$$d\alpha(\epsilon) = 0{\cdot}005;\quad d\delta(\epsilon) = -0{\cdot}89$$

and from Table I for the date June 13·6 we get by a straightforward linear interpolation

$$d\psi = -0{\cdot}14;\quad d\epsilon = +0{\cdot}10$$

Hence

$$\Delta\alpha = +0{\cdot}057 \times -0{\cdot}14 + 0{\cdot}005 \times +0{\cdot}10 = -0^s{\cdot}008$$
$$\Delta\delta = +0{\cdot}18 \times -0{\cdot}14 + -0{\cdot}89 \times +0{\cdot}10 = -0''{\cdot}11$$

Consequently, the corrected values are

$$\text{R.A.} = 19^h\ 48^m\ 26^s{\cdot}169 - 0^s{\cdot}008 = 19^h\ 48^m\ 26^s{\cdot}161$$
$$\delta = +8°\ 44'\ 10''{\cdot}13 - 0''{\cdot}11 = +8°\ 44'\ 10''{\cdot}02.$$

Example 2. Find the local apparent sidereal time at a place in longitude $8^h\ 32^m\ 41^s{\cdot}016$ W. at G.M.T. $13^h\ 42^m\ 15^s{\cdot}914$ on 1951 June 22.

		h	m	s
Mean solar interval from 0^h	=	13	42	15·914
Corrections to mean solar time to give sidereal time (Table III)	=		2	15·034
				0·044
Apparent sidereal time at 0^h (Table II)	=	17	57	28·668
Change in nutation from 0^h to 14^h (Table II)				+ 0·003
Sum = apparent Greenwich time	=	31	41	59·663
Difference in longitude	=	− 8	32	41·016
Sum = local apparent sidereal time	=	23	09	18·647

Here the sum of the long- and short-period terms of nutation for June 22 given

in Table II is $+ 0^s{\cdot}308$ and for June 23 it is $+ 0^s{\cdot}314$. Hence the correction for change in nutation is $(+ 0{\cdot}314 - 0{\cdot}308) \times 14/24 = + 0{\cdot}006 \times 14/24 = + 0{\cdot}003$.*

Example 3. To find G.M.T. on 1951 April 23 corresponding to a local apparent sidereal time of $18^h\ 26^m\ 19^s{\cdot}804$ in longitude $8^h\ 32^m\ 41^s{\cdot}016$ W.

		h	m	s
Local apparent sidereal time	=	18	26	19·804
Difference in longitude	=	+ 8	32	41·016
Sum = Greenwich apparent sidereal time	=	26	59	00·820
Apparent sidereal time at 0^h (Table II)	=	14	00	55·194
Sidereal interval	=	12	58	05·626
Corrections to sidereal time to give mean solar time (Table IV)			− 2	07·457
			−	0·016
Change in nutation from 13^h to 0^h (Table II)	=		−	0·002
Sum = G.M.T. at local apparent sidereal time $18^h\ 26^m\ 19^s{\cdot}804$	=	12	55	58·151

The sums of the long- and short-period terms of nutation for April 23 and April 24 are $+ 0^s{\cdot}156$ and $+ 0^s{\cdot}160$ respectively and so the correction for the change between 13^h and 0^h is $- 0{\cdot}004 \times 13/24 = - 0^s{\cdot}002$.

Modifications when Standard Time is Used. It is, generally speaking, more convenient to keep the mean time clock on the standard time used in the country of observation, although the apparent and sidereal times required will always be local. Conversion from one system of time to another is then done by converting the given time to the corresponding Greenwich time, as in paragraph 2, then converting to the required system of time at Greenwich, and finally reducing the time so obtained to the desired local or standard time. The principal conversions dealt with may be summarised, in the new form, as follows.

3a. To find the Standard Mean Time corresponding to a Given Instant of Local Apparent Time.

G.A.T. = L.A.T. + west longitude
= L.A.T. − east longitude
G.M.T. = G.A.T. − 12^h + G.M.T. of transit at Greenwich, interpolated from apparent noon to the given G.A.T.
Standard mean time = G.M.T. − west longitude of standard meridian
= G.M.T. + east longitude of standard meridian.

4a. To find the Local Apparent Time corresponding to a Given Instant of Standard Mean Time.

G.M.T. = Standard mean time + west longitude of standard meridian
= Standard mean time − east longitude of standard meridian
G.A.T. = G.M.T. + Equation of time (Apparent − Mean), interpolated from 0^h to the given G.M.T.
L.A.T. = G.A.T. − west longitude
= G.A.T. + east longitude.

8a. To find the Standard Mean Time corresponding to a Given Instant of Local Sidereal Time.

G.S.T. = L.S.T. + west longitude
= L.S.T. − east longitude

* The tables in *Apparent Places* give values for both apparent and mean sidereal times and in most cases the sum of the two terms on nutation in R.A. is equal to the difference between the apparent and the mean sidereal times, but this is not always so as end-figure discrepancies sometimes occur.

G.M.T. = Mean time of the preceding transit of the first point of Aries + mean time equivalent (Table IV in *N.A.* and *A.P.*) of given G.S.T.

Standard mean time = G.M.T. − west longitude of standard meridian
= G.M.T. + east longitude of standard meridian.

9a. To find the Local Sidereal Time Corresponding to a Given Instant of Standard Mean Time.

G.M.T. = Standard mean time + west longitude of standard meridian
= Standard mean time − east longitude of standard meridian

G.S.T. = Sidereal time at 0^h + sidereal equivalent (Table III in *N.A.* and *A.P.*) of given G.M.T.

L.S.T. = G.S.T. − west longitude
= G.S.T. + east longitude.

Examples

1. Compute the value of the sun's declination for the instant on the morning of 1941 Oct. 28 at which its hour angle is 50° at a place in longitude 50° E. The sun's declination at transit at Greenwich is

	°	′	″	″
1941 Oct. 26	− 12	24	06·6	
				− 1226·0
27	− 12	44	32·6	
				− 1214·0
28	− 13	04	46·6	
				− 1201·7
29	− 13	24	48·3	

2. Find the true altitude of the sun's centre from an observation which gave an apparent altitude of 55° 34′ 23″ to the sun's upper limb. Take the sun's horizontal parallax as 9″ and semi-diameter as 15′ 53″.

3. Find the standard mean time of L.A.N. on 1941 July 21, at a place in New South Wales in longitude 142° 30′ E. Standard time in New South Wales is 10^h fast on Greenwich.

Date	G.M.T. of Transit at Greenwich			
	h	m	s	s
1941 July 19	12	06	06·89	
				+ 3·93
20	12	06	10·82	
				+ 3·40
21	12	06	14·22	
				+ 2·84
22	12	06	17·06	

4. The Greenwich sidereal time at Greenwich mean midnight on a particular day is found from the *N.A.* to be 7^h 20^m 35^s. An observation of a star is taken in longitude 2° west at local sidereal time 17^h 30^m 50^s. The correction of S.T. for longitude is $9^s{\cdot}86$ per hour. Find the local mean time at instant of observation.
366·2422 sidereal days = 365·2422 mean solar days. (Univ. of Lond., 1918).

5. At what standard time does ♈ make its upper transit on 1941 Nov. 4, at a place in longitude 122° 15′ W. in the Pacific time belt (8^h W.) ?
G.M.T. of G.S.N., Nov. 4 = 21^h 05^m $09^s{\cdot}3$.

6. Calculate to the nearest second the Indian standard mean time (corresponding to the meridian 5^h 30^m E.) of transit of *β Draconis* (R.A. = 17^h 29^m $08^s{\cdot}5$) on 1941 July 2, at a place in longitude 84° 30′ E., given that G.S.T. at 0^h on July 2 = 18^h 38^m $33^s{\cdot}7$.

7. From the following data calculate the Greenwich mean time of transit of the star A at the place B.

	h	m
Right ascension of star A	10	00
Sidereal time of mean midnight at Greenwich	22	00
Longitude of place B		8 W

8. Find, to the nearest second, at what Eastern standard time (corresponding to the meridian 5^h W of Greenwich) *γ Cassiopeiæ* (δ = + 60° 23′ 56″, R.A. = 0^h 53^m $13^s{\cdot}1$)

elongates on the evening of 1941 Aug. 31, at a place in latitude 50° N. and longitude 70° W., and state whether the elongation is eastern or western. The transit of the first point of Aries at Greenwich on Aug. 31 is at $1^h\ 24^m\ 39^s{\cdot}1$.

9. At what L.S.T. was *β Ceti* ($\delta = -18°\ 24'\ 47''$, R.A. $= 0^h\ 39^m\ 40^s{\cdot}5$) on the prime vertical at a place in latitude 22° 32′ S., and what was then its altitude ?

10. At what L.S.T. did *α Boötis* ($\delta = +19°\ 35'\ 35''$, R.A. $= 14^h\ 12^m\ 05^s{\cdot}7$) attain an altitude of 60° on the east side of the meridian at a place in latitude 32° 17′ N. ?

CHAPTER II

FIELD ASTRONOMY—OBSERVATIONS

THE quantities to be obtained by the observations of field astronomy are time, azimuth, latitude and longitude. Each can be determined in several ways, and the selection of a suitable method is based chiefly upon the instrumental means available and the degree of precision required.

The observations to be described do not exhaust those available for the different determinations. The principal methods are given, including those employed for the most refined field determinations as required in geodetic survey. It is impossible to state definitely the probable accuracy to be expected from the different methods of determination as it depends very largely upon the capability of the instrument used.

The data required in the reduction of observations include, in many of the methods, quantities which necessitate astronomical observation for their evaluation. Thus, observations for time are made in connection with determinations of azimuth, latitude and longitude. In some cases the same observation yields more than one quantity. For determinations of low grade it is satisfactory to make one observation serve a double purpose, but in deliberate work each unknown is generally observed for independently, though accurate values for time and latitude may be derived simultaneously by observations on three stars at equal altitudes and time and azimuth by two stars on the same vertical.

ASTRONOMICAL AND GEODETIC POSITIONS

From the definitions of time, azimuth, latitude and longitude, it is evident that the values of those quantities are influenced by the direction of the vertical or plumb line at the place. The results of astronomical determinations therefore include the effect of local deviations caused by the irregular distribution of mass in the earth's crust. The amount of local deflection cannot be measured directly. It is deduced by comparison of the astronomical position obtained by observation with the geodetic position computed with reference to the spheroid which best represents the form of the whole earth or a particular part of it (page 321). Hence, in order to obtain an estimate of the local deflection at any single point it is necessary to have a number of astronomical observations at various survey stations covering a fairly considerable area surrounding the given point. The discrepancies at different points are of great interest and importance in geodetic investigations. Their numerical values are usually relatively small but they are seldom or never zero. In many cases they exceed the probable errors of the instrumental observations in themselves, and sometimes they have been found to amount to as much as 30″ of arc. Consequently, although the determination of their values, and the elimination of their effects, belong to the province of the professional geodesist rather than to that of the professional engineer, it is important for the latter to recognise the limitations they impose on the value of astronomical determinations, by themselves, as a check on

positions established by triangulation or traverse, or as a framework to be substituted in place of either. For these reasons, therefore, it appears advisable to give some further consideration to this subject.

Astronomical and Geodetic Latitudes and Longitudes and the Deviation of the Plumb Line. The difference between the geodetic and the astronomical latitude is illustrated in Fig. 22. The curve EPQP′ represents a section through a meridian on the earth's surface, PP′ being the polar axis and EQ the equatorial axis. Since the earth is an oblate spheroid in shape, this curve will be an ellipse. At any point A draw the line bAB normal to the curve at A. Join A to C, the centre of the ellipse. Then the angle ACE is the geocentric latitude ψ and the angle ABE is the geodetic latitude ϕ.

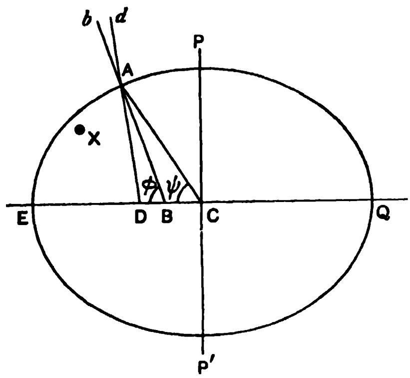


Fig. 22.

Now suppose an abnormally heavy mass to be concentrated at the point X. A plumb bob suspended at A will then be drawn by gravitational attraction towards X, so that the plumb line will no longer lie along the normal bAB but along some other line dAD. All astronomical observations for latitude ultimately depend on the indications of a spirit level, and, as the level is at the centre of its run when its tangent plane there is perpendicular to the plumb line, it will be seen that the latitude measured by the instrument will be the angle ADE and not the geodetic latitude ABE. The measured latitude ADE is called the astronomical latitude, and the difference between this and the geodetic latitude—the angle BAD in the figure—is called the deviation of the vertical, or of the plumb line, in latitude. If the mass X lies to one side of the meridian plane, and not exactly on it, there will be a similar deviation in the local time, so that the astronomical longitude L_A will not coincide with the geodetic longitude L_G. When the point of observation is not on the equator this difference between astronomical and geodetic longitudes will lead to a difference between the astronomical and geodetic azimuths, since the effect will be slightly to tilt the plane in which horizontal angles are measured. Let A_A and A_G be the astronomical and geodetic azimuths. Then it can be proved that :—

$$A_A - A_G = -(L_A - L_G) \, . \sin \phi_G$$

where ϕ_G is the geodetic latitude. This equation is known as the "Laplace equation," and points at which both geodetic and astronomical longitudes and azimuths have been observed are known as "Laplace points." In important and extensive geodetic surveys it is now customary to correct the observed astronomical azimuths by means of this equation after values have been obtained for the deviations in longitude. This is not necessary when all the points lie on, or very close to, the equator, as $\sin \phi_G$ is then zero or very small.

Effect of the Deviations of the Plumb Line on the Survey of Position. In order to appreciate the effect which the deviations of the plumb line may have on the relative positions of points established by survey or geodetic methods, consider Fig. 23 which represents a short arc AB of a meridian. To simplify matters let this arc be short enough to be considered an arc of a circle of radius R and centre C. Then, a direct measurement, by triangulation or traverse, between the points A and B will give the length of the arc AB, which is equal to R × $\Delta\phi_G$, where $\Delta\phi_G$ is the difference in the geodetic latitudes of A and B. Owing to deviations of the verticals at the two points, astronomical measures will give a difference of astronomical latitude of $\Delta\phi_A$. Hence, assuming that, as the deviations are very small, R is not appreciably altered in value, the length of the arc AB by computation from the astronomical measures will be $R \times \Delta\phi_A$, so that the difference in lengths as obtained from geodetic and astronomical measurements will be $R(\Delta\phi_G - \Delta\phi_A)$. If $(\Delta\phi_G - \Delta\phi_A) = 10''$, this difference amounts to about 1,000 ft. on the earth's surface. Deviations in longitude will produce a somewhat similar effect, the linear displacement in this case being given by $R(\Delta L_A - \Delta L_G) \cos \phi$.*

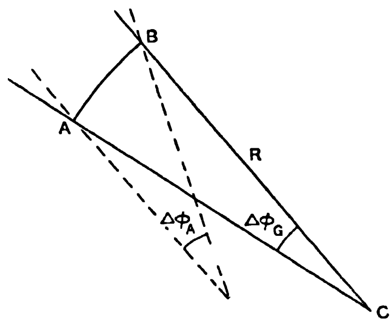


FIG. 23.

From the above explanation it will readily be realised that closely spaced astronomical determinations of position are not of much value as a check on such operations as theodolite traversing or on triangulation (*see* Vol. I, page 232). On the other hand, astronomically determined azimuths are invaluable for checking the propagation of angular error in traversing, especially in low latitudes where the difference between the geodetic and astronomical azimuths is always small although, unless corrections are made, astronomical azimuths also may have their limitations when applied to very refined work.

Effect of the Deviation of the Vertical on Azimuth Control.† It has already been pointed out (page 48) that slight deviations of the plumb line will cause discrepancies between the astronomical and geodetic azimuths. Unless they are allowed for, these small deviations in azimuth will naturally cause local "swings" in traversing or triangulation, but,

* It is usual to express the deviations of the vertical in terms of seconds of arc measured along the meridian and along the prime vertical, and these components are called the meridional and prime vertical components respectively. The meridional component is $\phi_A - \phi_G$ and the prime vertical component is $(L_A - L_G) \cdot \cos \phi_G$. One advantage of expressing the deviation in longitude in this way, instead of directly in terms of $(L_A - L_G)$, is that it affords a convenient method of forming a rough mental estimate of the amount of each component in feet. For this purpose, 1″ of arc on the earth's surface, measured either along the meridian or along the prime vertical, may be taken at 100 ft. approximately.

† For further information on this subject see a very useful and interesting paper by Captain G. T. McCaw, C.M.G., O.B.E., M.A., on "Astronomical Azimuths and their Introduction into Triangulation" in *Empire Conference of Survey Officers. Report of Proceedings. 1928.*

so far as fixing the relative positions of points by either of these methods is concerned, the effects will not be nearly so serious as those of the displacements in astronomically determined positions caused by the deviations in latitude and longitude. The reason for this, of course, is that the length of any arc subtending a given angle depends on the radius of the circle of which the arc forms a part, and the length of any survey line, or series of lines, is almost infinitesimally small when compared with the radius of the earth.

The disturbance in azimuth in latitude 55°, corresponding to a deviation of 10″ in longitude, which is rather a large figure, is 8″·19 of arc, and the lateral displacement caused by this at the end of a line one mile in length is 0·21 ft., or one part in twenty-five thousand of the length of the line. If the latitude is 10° the corresponding deviation in azimuth is 1″·74 and the linear displacement in position 0·044 ft., or one part in a hundred and twenty thousand of the length of the line. It follows, therefore, that, when great accuracy is sought, some corrections to the observed azimuths may be necessary when working in high latitudes, and, if such corrections are not made, work carried out in high latitudes may not appear to give such good results as work carried out by exactly similar methods in places near the equator. It should, however, be noted that small discrepancies between the geodetic and astronomical longitudes and azimuths, due to deviations of the vertical, must not be confused with discrepancies caused by normal accidental errors of observation. If the observed values of $(L_{,} - L_{G})$ are no more than those that can be expected from the quality of the observations themselves, it would be wrong to suppose that the whole effect is due to the displacement of the zenith, or to apply the Laplace equation. Even at the equator, differences between the computed geodetic and the observed astronomical azimuths are usually found. The Laplace equation can be written :—

$$(L_{,} - L_{,,}) = - (A_{,} - A_{,,}) \cdot \operatorname{cosec}\phi$$

so that, with $\phi = 0°$ and $(A_{,} - A_{,,})$ finite, this would lead to an infinite value of $(L_{,} - L_{,,})$. This, of course, is absurd, and here it must be assumed that the apparent $(A_{,} - A_{,,})$ is not a true value, in the sense that it represents a real deviation to which the Laplace equation is applicable, but is due entirely to ordinary errors of observation.

Owing to the application of wireless time signals to determining longitude, there is now no great difficulty in taking fairly frequent observations for longitude. If this is done, and $(L_{,} - L_{,,})$ is determined at a number of points, the corrections to the astronomical azimuths at these points can be worked out. The geodetic longitudes will normally be based on the astronomical longitude at the initial station of the survey and, at this point, the true geodetic longitude will not usually coincide with the astronomical longitude. Hence, all the computed geodetic longitudes will be affected by the unknown $(L_{,} - L_{,,})$ at the beginning of the survey. So far as any work likely to be carried by engineers is concerned, however, it will suffice if the mean $(L_{,} - L_{G})$ is taken, and the result applied, with its proper sign, to each longitude determination, including that at the initial station, to give a corrected $(L_{,} - L_{G})$ from which to work out the azimuth correction. Even this, or for that matter

the application of a Laplace correction at all, is only necessary for the very highest class of work, and for most purely engineering surveys the disturbances to azimuth caused by local deviations of the vertical can be considered to be negligible and can thus be ignored.

A further point to be considered is the frequency of check azimuth observations on such work as theodolite traversing and triangulation. If no correction is to be applied to the observed azimuths it follows that there is not much point in taking observations at closer intervals than those in which the probable error in bearing, due to the propagation of angular error, is less than the probable error of the observed azimuths, the latter p.e. including any that may arise from the unknown deviations of the vertical.

The numerical values of $(L_{\prime} - L_{\prime\prime})$ vary a good deal, but, when they are unknown, they may, for purposes of estimating probable error but not, of course, for purposes of applying corrections, be taken at about $\pm 3''$, though this value is often greatly exceeded in practice.

Astronomically Determined Latitudes and Longitudes as a Control for Mapping. It might at first be thought that, owing to the ease and accuracy with which observations may now be made with modern instruments, astronomical observations for latitude and longitude might serve admirably as a framework for mapping on a fairly small scale. Provided there are no unusual climatic difficulties, a given area can be covered with a network of astronomically determined points much more quickly than it can be covered with a network of points fixed by triangulation or traverse. Moreover, any errors of observation that may be made at one point are not carried forward when fixing other points, whereas in triangulation or traversing errors of observation tend to be propagated and magnified. In addition, if it is desired to survey a particular district before the framework in other districts is complete, or nearly complete, astronomical latitudes and longitudes can be observed at once in that district without having to wait for the framework to be completed elsewhere. This is not the case with triangulation or traversing, which, as a rule, have to be pushed forward as a whole and not surveyed in bits and pieces.

In spite of all these advantages, the possibility of the existence of deviations of the vertical, of unknown and varying amounts, sets a limit to the usefulness of astronomically determined latitudes and longitudes as a framework for mapping. If the scale of the map is very small, or if the map is to be a rough geographical one only, where errors in position of, say, a quarter of a mile or more do not greatly matter, the use of an astronomically determined framework can be justified, especially if speed of production is a main consideration; but, when the scale is reasonably large, and reasonable accuracy is desired, it is inadvisable to rely solely on astronomically fixed points for systematic mapping purposes, and it is far better to lay down a proper network of points fixed by triangulation or by accurate theodolite traverse. It must be stated, however, that there has been a tendency in recent years to rely on astronomical fixings as a basis for mapping on medium scales, more particularly when, in order to produce the map quickly, the detail survey has been done by air survey methods.

INSTRUMENTS

The theodolite is the most generally useful instrument for the observations of field astronomy, and all sizes from 3 in. to 12 in. are employed for the purpose. For determinations in connection with geographical mapping, the 5-in. or 6-in. micrometer instrument gives sufficiently accurate results, and is that most commonly used. The astronomical or nautical sextant may be employed in certain cases for the measurement of altitude. For primary determinations, as required in geodetic surveying, the principal instruments used are the portable transit instrument for time, the geodetic theodolite for azimuth, and the zenith telescope, or the theodolite with eyepiece micrometer, for latitude observations. In recent years, however, owing to the general standard of excellence and the adaptability of modern geodetic theodolites, the tendency, even in the most precise work, has been to take all astronomical observations, whether of time, azimuth or latitude, with an instrument of this kind. The theodolite is an all-round instrument, capable of measuring ordinary horizontal and vertical angles, as well as of being used for astronomical observations. Consequently, when it can be used for astronomical work, there is no point in carrying other instruments as well.

The Theodolite. The modern large theodolite, described on pages 191 *et seq*, is usually entirely suitable for astronomical work, but certain types of engineer's transit theodolite are imperfectly adapted for the measurement of altitude. The provision of micrometer reading of the circles instead of verniers is very desirable, but of even greater importance is the necessity for the mounting of a sensitive spirit level on the frame carrying the vertical circle micrometers or verniers. This altitude level, being independent of the inclination of the telescope, defines the horizon throughout the observation, and serves to show whether the instrument remains stable. Its sensitiveness should be about 5″ per 1·5 or 2 mm. division for a 5-in. or 6-in. theodolite, and must be carefully determined so that altitudes observed with the bubble off centre may be corrected. The use of a striding level to ensure horizontality of the horizontal axis is also necessary.

Accessory parts required are: (*a*) means for illuminating the field of view; (*b*) a diagonal eyepiece; (*c*) sight vanes fitted on the upper and lower sides of the telescope tube to give a line of sight parallel to that of the telescope to facilitate pointing to a star; (*d*) a dark glass to fit on the eyepiece when observing the sun.

For star observations it is necessary to illuminate the field sufficiently to enable the cross-hairs to be seen. In older instruments this is usually effected by having the trunnion axis hollow and attaching a small lamp to one of the standards. The light is projected along the axis, and is reflected by a very small mirror in the telescope. Alternatively, a paper reflector may be attached in front of the objective. It should be bent over the lens and be provided with an opening to enable the light from the star to enter the telescope. A lamp is held or placed so that just sufficient light is reflected down the tube. In the more modern type of instrument, such as the Wild or Tavistock, the illumination of the cross-hairs is effected by means of a special low-voltage system, built into the

instrument itself. This system may be operated either from dry batteries or from accumulators, and the intensity of the light may be controlled by means of a variable resistance provided for the purpose. The same system is also now used extensively to provide illumination for the micrometers used in reading the circles. When a theodolite is provided

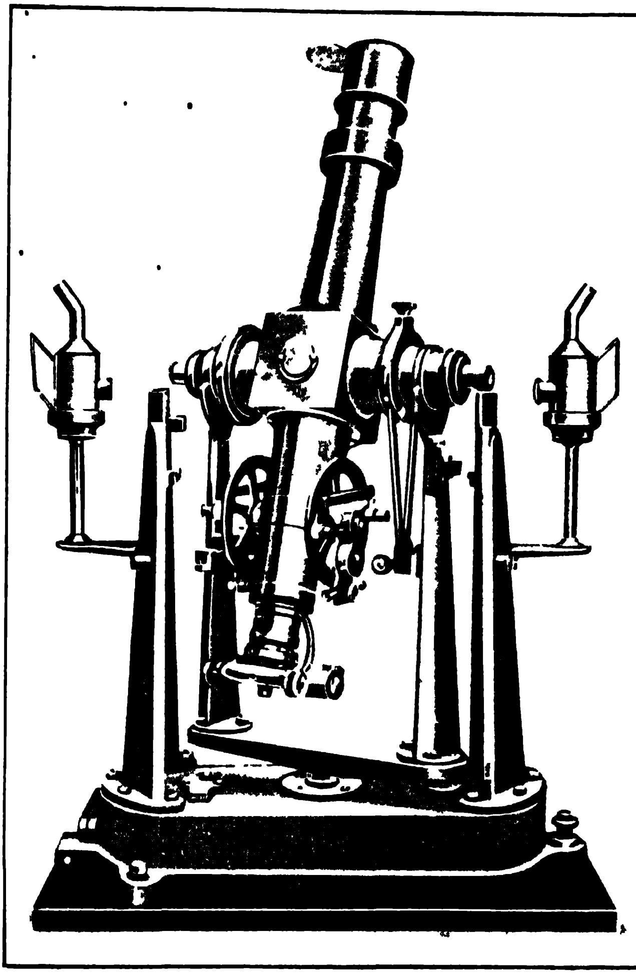


FIG. 24. PORTABLE TRANSIT INSTRUMENT.

(By permission of Messrs. Troughton & Simms.)

with a system of complete internal electrical illumination it is well, when precise angular observations are being taken, to use the electrical illumination of the micrometers for daylight observations as well as for night ones. The advantage of doing so is that the illumination of the circle graduations is more even and constant than it would be if ordinary

daylight illumination were used, and small errors in reading are thereby reduced.

The diagonal eyepiece (Vol. I, page 28) is used when the altitude exceeds about 45°. The proportions of the theodolite must be such that the telescope can be transited without having to disturb the focus of the eyepiece. In observing the sun with the diagonal eyepiece, it should be remembered that the prism inverts the image, so that with the usual Ramsden type the sun appears right side up but inverted in azimuth.

The Portable Transit Instrument. This instrument (Fig. 24) is an adaptation of the transit circle used in observatories, but is of much smaller size. It is set in the plane of the meridian for observing star transits for time and longitude determinations, and is occasionally utilised for the observation of azimuth, the referring object (page 98) being placed sufficiently near the plane of the meridian to be within the range of the eyepiece micrometer. The instrument may also be fitted for latitude observations by the Talcott method (page 113)

Various sizes are used, that illustrated being suitable for primary determinations. In the larger instruments, the telescope objective has a focal length of 36 to 45 in. and an aperture of about 3 in. The powers of the various eyepieces provided range up to over 100 diameters. The telescope is mounted on a rigid axis resting on wye supports as in the theodolite, and provision is made for reversing the axis without lifting the telescope by hand. A sensitive striding level is essential. The vertical circles are of small diameter, and are used only for setting the telescope approximately at any required altitude. The instrument must have a very stable support, the best form being a masonry pier well founded and insulated from vibrations by having a narrow surrounding air space below the ground.

In a design of portable transit instrument, known as the broken-telescope transit, which is used on the Continent and in America, the light which passes through the object glass is reflected through a right angle by means of a prism placed in the trunnion axis.* This hollow axis is continued beyond the supports, and the eyepiece is fitted at one end. A small electric lamp for the illumination of the field is carried at the other end of the axis. With this optical arrangement the standards can be made much lower than in the ordinary pattern, so that the instrument is very compact.

A transit is observed by taking the time at which the star crosses each of several vertical hairs forming the reticule. When the times are taken by the eye and ear method (page 79), no more than five hairs can be used in order that the intervals between successive passages may permit of the times being booked. When the instants are registered by chronograph, eleven or more passages are recorded, and the influence of accidental errors of observation on the mean result is consequently reduced.

Still better results are obtained by the use of the transit or impersonal micrometer, fitted with its movable vertical hair in the focal plane of the telescope. The micrometer drum carries five contact points, which make an electric circuit as they pass a fixed contact spring, and the instants

* For an example of an instrument in which the broken-telescope principle is used, see the description of the Wild Astronomical Theodolite on page 207.

of the contacts are registered on the chronograph sheet. In observing a transit it is only necessary, as the star crosses the field of view, to keep it continuously bisected by the movable hair. Two milled heads are provided for actuating the micrometer screw so that, by using both hands, a steady motion may be imparted to the hair. An automatic cut-out is fitted, and no record is transmitted except while the star is traversing the middle part of the field defined by two fixed vertical hairs. Four complete revolutions of the micrometer screw are required to carry the hair across this space, so that twenty contacts are made and registered on the chronograph sheet.

The principal adjustments of the transit instrument are those of the striding level and horizontal axis and of the collimation line. The former adjustments are made in the manner described for the theodolite (page 195). The collimation adjustment is performed by sighting a well-defined distant point with the middle hair of the eyepiece micrometer. The exact bisection of the object sighted is effected by movement of the azimuth tangent screws. On reversing the telescope in its supports, the line of sight should still bisect the object. If it does not, the error may be eliminated by moving the hair half-way towards the image of the object by the adjusting screws controlling the diaphragm. The test and adjustment are repeated if necessary, but the *complete* elimination of collimation error is not really essential (see page 219).

The portable transit instrument is seldom used nowadays in survey work except possibly for primary determinations of longitude where the highest possible standard of accuracy is required. For all ordinary geodetic purposes a good geodetic type of theodolite, fitted with an eyepiece micrometer, is used instead. In some theodolites (*e.g.*, the large Tavistock and the Wild Astronomical theodolite) the eyepiece micrometer may be turned through 90° about the longitudinal axis of the telescope so that it can be used for observing star passages either in azimuth or in altitude.

The Chronograph. The chronograph (Fig. 25) is an instrument which

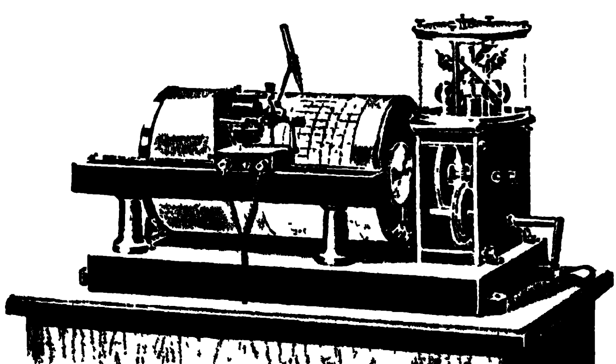


FIG. 25. CHRONOGRAPH.

reproduces the time record of a chronometer (p. 73) in graphical form. A sheet of paper to receive the record is wrapped round a cylinder which is

rotated by the action of a descending weight at the uniform rate of either one or two revolutions a minute. The speed is controlled by a governor. The pen carriage is mounted on a screw which is also rotated by the descending weight, so that the pen moves uniformly along the cylinder. A helix is therefore traced upon the paper, but by means of an electrical connection with a chronometer a sharp break in the line is made every second by the chronometer automatically breaking the circuit. Whole minutes are usually recorded by the absence of the regular mark at the fifty-ninth second. The chronometer may be arranged to break the circuit at the even seconds only, and the whole minutes are then indicated by an additional mark at the fifty-ninth second.

For taking transits with the aid of the chronograph, there is provided in the chronograph circuit a key or button, which the observer holds in his hand and depresses at each passage of the star across a hair. The circuit is thereby broken, and the instants are recorded by the pen in the same manner as the breaks made by the chronometer. The positions

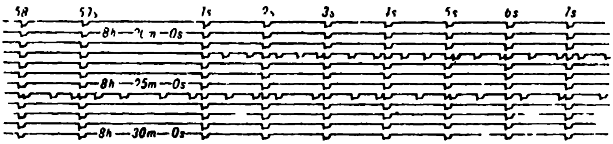


FIG. 26. CHRONOGRAPH RECORD.

of these additional marks relatively to the second or two-second marks can be scaled with considerable precision, since the speed of rotation of the cylinder is sufficiently steady that it may be assumed uniform between adjacent chronometer marks. When an impersonal micrometer is used, the chronograph sheet is automatically marked at the instants at which the movable hair, bisecting the star, reaches the positions corresponding to contacts on the micrometer head (Fig. 26)

In a compact form of the instrument, known as the tape chronograph (Fig. 27), the record is received upon a continuous strip of paper which

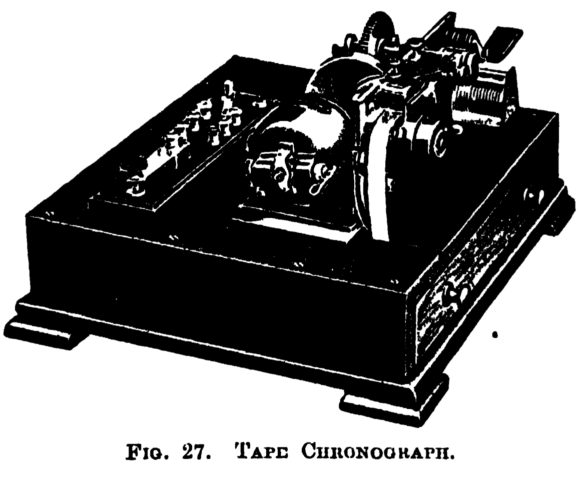


FIG. 27. TAPE CHRONOGRAPH.

is fed through at a uniform rate. The instrument illustrated is operated by a small electric motor, and the speed is regulated by means of a rheostat. The record of the micrometer contacts is made alongside that of the chronometer seconds, instead of being superimposed upon it. In some instruments the armatures carry needle points instead of pens, and a series of fine punctures forms the record.

The Zenith Telescope. This instrument (Fig. 28) is used for precise determinations of latitude by the Talcott method of measuring small differences of meridian zenith distance of pairs of stars (page 113). The telescope is similar in size to that of the portable transit instrument, and is mounted either centrally on, or at one end of a short horizontal axis levelled by a striding level. An eyepiece micrometer, with movable horizontal hair, is an essential feature. The value of one division of the micrometer head is about 0″·5, and the range of the movable hair about 20′. The vertical axis, carrying the supports for the horizontal axis, is made long to ensure a true motion of the telescope in azimuth, its deviation from true verticality being measured by means of one or two chambered latitude levels attached to the telescope and having a sensitiveness of about 0″·8 per mm. The vertical circle or arc is used simply for setting the telescope approximately to any required altitude. The horizontal circle enables the line of sight to be set approximately in the meridian, and azimuth stops are provided for clamping on to the circle so that the telescope may be quickly swung through 180° from a north to a south star. The instrument support must be very stable, and should preferably be of masonry.

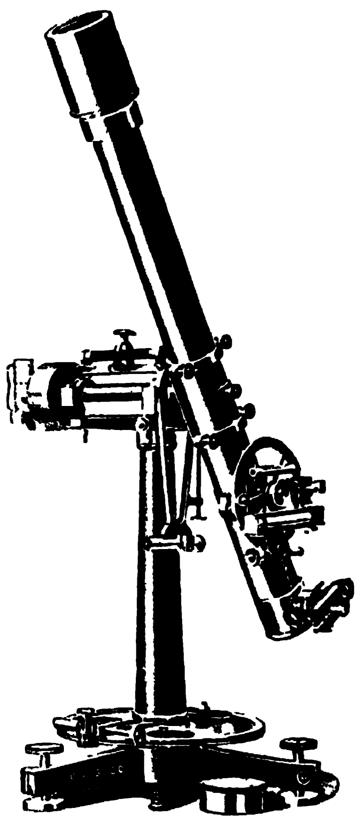


Fig. 28. Zenith Telescope.

In adjusting the instrument, the striding level is used as in the case of the theodolite for the horizontal axis adjustment. If the telescope is mounted centrally on the horizontal axis, the collimation adjustment is performed as for a transit instrument. In the eccentric type of instrument illustrated this adjustment may be made by the double reversal method used for small theodolites, two points being established on either side of the instrument and at a distance apart equal to twice the eccentricity of the telescope. The value of one turn of the micrometer screw is calculated from the sidereal time required for a circumpolar star at or about elongation to move through the angle corresponding to one turn, or it is derived by least squares from the results of the latitude observa-

tions themselves. If the time of passage of a circumpolar star at elongation is used to determine the value of the micrometer screw, then the change in elevation dh of the star corresponding to a change dt in the hour angle, expressed in seconds of arc, is given by $dh = -\cos\delta \,.\, dt$. It is of importance that the movable hair should be truly horizontal when the instrument is levelled. This should be tested by setting the movable hair upon the image of a distant point to one side of the field of view. The instrument being levelled and the telescope clamped in altitude, the point sighted should remain upon the hair while the telescope is moved in azimuth. Otherwise, the eyepiece must be rotated by means of the appropriate adjusting screws.

Although a good deal of modern work in primary latitude determination has been performed by means of the transit instrument, instead of the zenith telescope, the general tendency in recent years has been to replace both instruments by a good geodetic theodolite. A transit instrument can be made equivalent to a zenith telescope by the attachment of a sensitive spirit level to one or both of the setting circles and by using the eyepiece micrometer with the movable hair in the horizontal position. A precise theodolite fitted with a suitable eyepiece micrometer may be similarly adapted for refined latitude observations by fitting a sensitive level.

Hunter Shutter Eyepiece. It has been found by experience that, when estimating times of transit of a star across a hair, nearly every observer has his own personal error so that he observes the passage either too early or too late. The sign and amount of this error is fairly constant with a single observer, but both may differ from those of another observer. The impersonal micrometer (p. 54) has been devised largely to eliminate this source of error but, while such a fitting can be attached to an ordinary portable transit instrument, it is too large and cumbersome for fitting to the average geodetic theodolite. Hence, in order to eliminate as far as possible the occurrence of personal error when an ordinary geodetic theodolite is being used, Dr. J. De Graaf-Hunter has devised a very ingenious piece of apparatus which can be attached to a comparatively small theodolite.

The Hunter shutter eyepiece consists of a movable shutter which operates directly in front of a scale engraved on glass situated in the focal plane of the objective. This scale, which takes the place of the ordinary diaphragm, can be rotated through 90° against stops, so that it can be set for observations of a star passing either a known altitude or a vertical plane. The shutter is electrically controlled to operate every three seconds, when it opens and exposes the star for a period of 70 to 100 milliseconds (thousandths of a second), a period which is long enough for the eye to see the star and note its position relative to the graduations on the scale but at the same time is too short for it to detect any appreciable movement. As the motion of the shutter is controlled by the chronometer, the time of each exposure can be obtained very accurately. The observations therefore consist in noting these times and at the same time recording the reading of the star's position on the scale. Thus, the mean scale reading for a series of rapid observations of the same star corresponding to the mean chronometer time is easily

found, and, after the usual corrections for collimation, bubble reading, refraction, etc., have been applied, the chronometer time of the star passing a known altitude or of transiting a vertical plane can be deduced.

In longitude observations it is necessary to compare the time of shutter operation with radio time signals. For this purpose, an electrical contact connected with the wireless circuit is broken when the shutter operates, the relay through which the timing of the shutter is controlled being set to open the latter every second. Hence, when the time signals are received, those which correspond with the seconds at which the shutter opens are extinguished. Thus, the shutter method does not require any chronograph, which is a very big advantage.

For further particulars of the Hunter shutter eyepiece see *Proceedings of the Royal Society*, Vol. CLXVI (May 19, 1938), No. 925, pp. 197–213. or the *Empire Survey Review*, Vol. IX, No. 63, January, 1947, pp. 20–24. The eyepiece can be obtained as an extra fitting to the Tavistock geodetic theodolite (p. 208) manufactured by Messrs. Cooke, Troughton & Simms, Ltd.

The Astronomical or Nautical Sextant. The framework of this instrument (Fig. 29) is a gun-metal casting, the curved limb of which carries

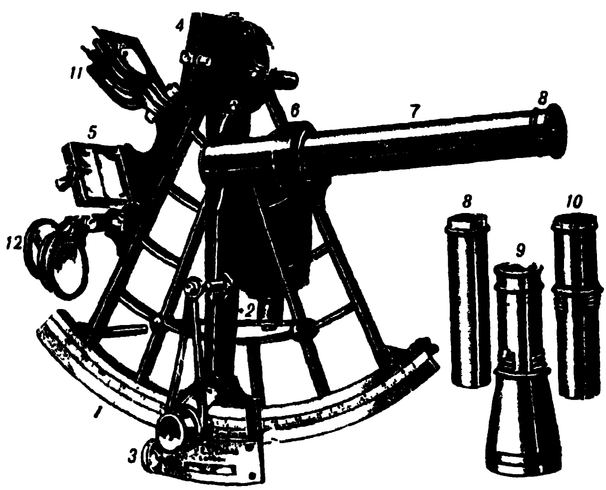


FIG. 29. ASTRONOMICAL SEXTANT.

the graduated arc *1*. At the centre of curvature there is fitted an axis about which the index arm *2* rotates. The latter is provided with a vernier reading against the arc, and is controlled by a clamp and tangent screw *3*. It also carries the index glass *4*, the vertical axis of which is collinear with the axis of rotation of the arm. This glass is wholly silvered, and is fixed in a metal tray in a manner permitting its adjustment perpendicular to the plane of the arc. The horizon glass *5*, mounted on the frame, is silvered on the lower half and of plain glass on the upper half.

The arc has a radius of from 6 to 8 in., and is graduated to 20′ or 10′ according to the size of the instrument. In accordance with the principle of the sextant (Vol. I, page 56), the divisions are figured at twice their actual value so that readings may represent the angles measured. Subdivision is carried to 20″ or 10″ by the vernier, which is usually of the extended type (Vol. I, page 43). The graduation of the arc is continued for a few degrees beyond the zero to form an arc of excess (see index error, page 61).

The telescope is screwed into the collar *6*, formed on a short pillar, called the up-and-down piece. This can be raised or lowered by a screw to vary the position of the line of sight relatively to the upper edge of the silvered part of the horizon glass and so equalise the brightness of the direct and reflected images. Two telescopes accompany the instrument—the long or inverting telescope *7* and the star telescope *9*. The former is generally provided with two eyepieces *8*, of different powers. The reticule usually consists of four lines forming a square in the centre of the field. The star telescope is of low power and wide field, and is convenient in identifying stars and for terrestrial observations. In addition, a plain tube *10*, with a pin-hole sight, is provided, but its utility is chiefly confined to terrestrial observations and to preliminary sights in astronomical work. Two sets of coloured shades, *11* and *12*, are fitted on hinges so that they may be brought into action when observing the sun. When it is desired to reduce the brightness of the direct and reflected images equally, as when making a solar observation by artificial horizon, a dark glass is fixed on the telescope eyepiece.

Testing and Adjustment of Sextant. The geometrical relationships of the sextant which can be established by adjustment are :

(1) The index glass should be perpendicular to the plane of the arc.

(2) The horizon glass should be perpendicular to the plane of the arc.

(3) The line of sight should be parallel to the plane of the arc.

(4) The mirrors should be parallel to each other when the index reads zero.

The following further requirements influencing the accuracy of the instrument are not adjustable, but the errors arising from their non-fulfilment can be ascertained and allowed for.

(5) The axis of rotation of the index arm should pass through the centre of graduation of the arc.

(6) The glasses forming the mirrors and shades should have their faces accurately parallel.

(7) The graduation of the arc should be uniform.

1. Adjustment of Index Glass. *Object.* To set the index glass perpendicular to the plane of the arc.

Test. (1) Clamp the index arm near the middle of the arc.

(2) Place the eye just above the plane of the arc and near the index glass. The part of the arc seen by reflection in the index glass should appear continuous with the arc itself.

Adjustment. If the reflected part appears to rise (fall) from the arc, the glass is leaning forward (backward). Turn the adjusting screw at the back of the glass in the direction thus indicated until the test is passed.

2. Adjustment of Horizon Glass. *Object.* To set the horizon glass perpendicular to the plane of the arc.

Test. (1) Point the telescope on a star.

(2) Move the index arm to either side of the zero. This should cause the reflected image to pass exactly over the direct image.

Adjustment. Turn the adjusting screw at the top of the horizon glass until the test is fulfilled.

Note. Although a star forms the best object for sighting, the test may be performed by observing the sun. If the instrument is used for terrestrial observations, as in the case of the sounding sextant (Vol. I, page 551), it is sufficient to sight any distant straight line, making the direct and reflected images appear continuous. On tilting the instrument, the continuity should be maintained.

3. Adjustment of Telescope. *Object.* To make the line of sight parallel to the plane of the arc.

Test. (1) Fit the inverting telescope, and turn the eyepiece until two of the wires are approximately parallel to the plane of the arc.

(2) Sight one star directly, and bring the reflected image of another, not less than 90° distant, into coincidence on one of the wires.

(3) Move the instrument until the images appear on the second parallel wire. The contact should remain perfect.

Adjustment. Alter the inclination of the telescope by the opposing screws controlling the collar.

Note. A simple alternative test can be performed indoors as follows. Set the sextant on its legs on a table. Place on the arc two small objects of exactly equal height to serve as temporary sights. Sight along them, and mark where the line of sight meets a vertical surface at least 20 ft. distant. Now sight through the telescope, and note how far the telescope line of sight strikes above or below the mark. This difference should be the same as the difference between the height of the temporary sights and the vertical distance between the centre of the telescope and the plane of the arc.

4. Index Error. *Object.* To ascertain the reading of the vernier index when the index glass is parallel to the horizon glass.

The position of the index when the mirrors are parallel, *i.e.* when the direct and reflected images of a very distant object are coincident, is the true zero from which angles are measured on the arc. If this does not coincide with the zero of the graduations, the difference is the index error, positive or negative, which must be applied to all observations alike. As its value is liable to change, it is preferable to ascertain the amount of the correction, rather than attempt to keep the instrument free from error.

Test. First method : Point the telescope at a star and bring the direct and reflected images into coincidence. The reading of the vernier is the index error, which is negative when the vernier index is to the left of the zero, or on the arc, and positive when off the arc (on the arc of excess).

Second method : (1) Set the index at about 30′ on the arc and sight the sun.

(2) The direct and reflected images should appear approximately in contact. Complete the contact of the right and left limbs by the tangent screw, and note the reading.

(3) Set the index at about 30′ off the arc, make the contact as before, and note the reading.

(4) The index error is given by half the difference between the two readings, and is subtractive (additive) when the greater reading is on (off) the arc.

Notes. (1) Since index error is applied to all observations, it must be determined with as great refinement as is employed in the observations, and several determinations may be made and the mean adopted. As the error changes with change of temperature, it is advisable to ascertain its amount on each occasion of observation.

(2) As a check on the observation in the second method, the sum of the readings on and off the arc should equal four times the sun's semi-diameter, as given in the *Nautical Almanac* for the date.

(3) As the arc of excess is read in the opposite direction from the arc itself, care must be exercised in reading the vernier off the arc.

(4) In the case of a sextant used for terrestrial observations, the first method of setting is employed with the sight taken on a distant object.

Adjustment. If it is desired to eliminate the error, clamp the index at zero, and bring the reflected image of a celestial body into coincidence with the direct image by turning the adjusting screw at the base of the horizon glass.

Note. After performing this adjustment, the perpendicularity of the horizon glass should again be tested and, if necessary, re-adjusted, in which case a further adjustment of index error may be found necessary.

5. Centering Error. The error produced by non-coincidence of the axis of rotation of the index arm with the centre of graduation of the arc cannot satisfactorily be detached from the effects of refraction due to non-parallelism of the mirrors and shades, or from errors arising from defective graduation or from flexure. It is therefore usual to group all such residual errors under the name of centering error, and from the results of testing to prepare a table of corrections, for various angles, to be applied in the manner of index error. The examination is most conveniently made by the use of fixed collimators, and in this country it is usual to have the test performed by the National Physical Laboratory, Teddington. Flexure of the instrument may, however, affect the constancy of the corrections, which should therefore be determined from time to time, and the surveyor may have to undertake field tests.

One method consists in observing the angular distance between two stars and comparing the result, after correction for index error and atmospheric refraction, with their calculated distance apart. To facilitate making the refraction correction, the stars selected should lie as nearly as possible on the same vertical circle. Several pairs of stars are thus observed, and the centering error is ascertained for several points on the arc. An alternative method consists in observing latitude by circum-meridian altitudes of north and south stars (page 116), which should be of about equal altitude. The difference between the latitude deduced from the north star and that from the south star represents the centering error corresponding to the double altitude observed in the artificial horizon. In the northern hemisphere the error is positive or negative according as the latitude given by the south star is the greater or smaller, and *vice versa* for the southern hemisphere.

If a good theodolite is available the test can be made by measuring the angles between different distant objects by means of both theodolite and sextant, the angular elevations of the objects also being measured with the theodolite. The angle measured by the latter is the horizontal angle whereas the one measured by the sextant is the angle measured in the inclined plane containing the point of observation and the two objects.

Let α be the horizontal angle between the two objects, h_1 and h_2 their angular elevations and α' the angle measured directly between them with the sextant. Then in the spherical triangle ZAB, Fig. 30,

$$\cos \alpha' = \sin h_1 \sin h_2 + \cos h_1 \cos h_2 \cos \alpha$$
$$= \sin h_2 \sec M \sin (h_1 + M)$$

where $\tan M = \cot h_2 \cos \alpha$. If α' and α are small angles, use.

$$\sin \tfrac{1}{2}\alpha' = \sin \tfrac{1}{2} (h_2 - h_1) \sec \theta$$

where

$$\tan \theta = \frac{\sin \frac{1}{2}\alpha}{\sin \frac{1}{2}(h_2 - h_1)} \cdot \sqrt{\cos h_1 \cdot \cos h_2}$$

Comparison may then be made between the value of α' computed from the above formula and the angle observed with the sextant.

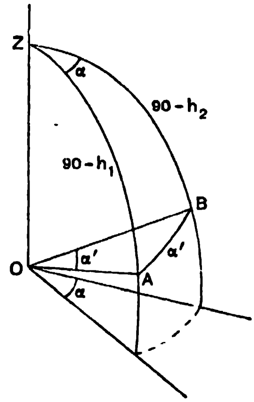


FIG. 30.

The Artificial Horizon. It has not been found practicable to fit the sextant with levelling apparatus so that altitudes may be observed directly, and such measurements are made either by observing the angle of elevation from the sea horizon and applying a negative correction for the dip of the horizon, or by the use of an artificial horizon.

The artificial horizon consists essentially of a horizontal reflecting surface, and the sextant observation of altitude cor..'s in measuring the angle between the celestial body and its image as seen in the reflector. In Fig. 31, let AB be the horizontal reflecting surface, D the position of the sextant, and S a celestial body. DE being a horizontal line, the required altitude h is EDS = BCS. The angle measured is SDS′, and by virtue of the laws of reflection and the parallelism of AB and DE, SDS′ is evidently twice the required altitude.

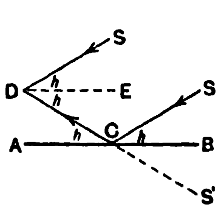


FIG. 31.

There are two general types of artificial horizon according to whether the reflector is a glass plate or the free surface of mercury in a tray. The former instrument consists of a small plate of black glass having a truly plane surface and mounted without strain on a brass frame fitted with three levelling screws. A loose level tube is laid on the surface in setting the plate horizontal. This type of artificial horizon can only be employed for land observation, and is now seldom used. The mercurial instrument has the advantage in the certainty of the reflecting surface being level and in the superior brightness of the reflected image. The commonest form consists of a

shallow iron or wooden tray (Fig. 32) about 6 in. by 3 in. To shield the surface of the mercury from wind, the tray is covered by a collapsible roof with sloping faces of glass plate worked to a uniform thickness and with truly plane surfaces. The mercury is contained in an iron flask when not in use. This type of horizon can be used at sea.

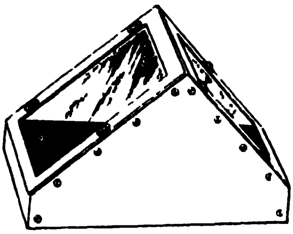


FIG. 32. MERCURIAL ARTIFICIAL HORIZON.

Observing by Artificial Horizon. In the measurement of altitude, the artificial horizon is placed on the ground in front of the observer and at a convenient distance for sighting the reflected image. The latter is viewed directly. By moving the index arm, the celestial body is then brought down until the two images are approximately in contact. This preliminary sighting should preferably be done with the blank tube, or simply through the telescope collar, and when the index arm is clamped at the approximate angle, the inverting telescope is screwed on as quickly as possible, and the two images are made coincident by the tangent screw.

In the case of solar observations the required coincidence consists in making one disc just touch the other. To ascertain whether the measurement is being made to the sun's upper or lower limb, the rule is that, with an inverting telescope, the images are continually overlapping in the forenoon and separating in the afternoon for the upper limb, and *vice versa* for the lower.

When taking repeated sights, except in the case of observations of equal altitudes, the glass roof should be reversed for half the observations as a precaution against the effects of possible non-parallelism of the glass plates. Under all circumstances the index correction is applied to the measured angle *before* halving it.

Notes. (1) Practice is necessary for acquiring speed in observing with the artificial horizon. The use of a sextant stand is helpful, but if this is not available, the observer should sit on the ground and rest the right arm against the knee.

(2) The brilliancy of the reflection depends upon the cleanness and chemical purity of the mercury. Dirty mercury may be cleaned by straining through chamois leather or by pouring it a few times through a paper funnel with a very small opening.

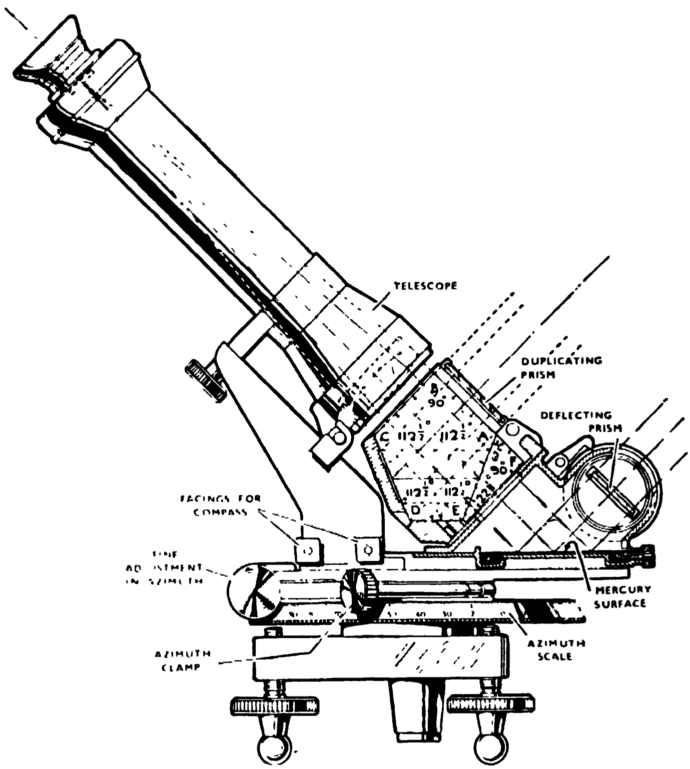
(3) Should the mercury be accidentally spilled, any liquid may be substituted temporarily: viscous liquids such as treacle or heavy oil are preferable to water.

The Prismatic Astrolabe. Any determination of latitude and longitude involves, directly or indirectly, a relation between the position of the star at the moment of observation and the direction of the vertical at the place of observation. In most surveying instruments, such as the transit circle and the theodolite, the apparent direction of the vertical is indicated or controlled by means of a spirit level attached to the instrument. Consequently, the accuracy of the observations depends to a very large extent on the sensitivity and accuracy of the indications of the spirit level. The astrolabe is an instrument which has been designed to make the observations not only independent of the indications of a spirit level but also of the readings on a vertical circle. A spirit level is not necessary since one of the essential rays in the optics of the instrument is reflected

from an artificial horizon. Neither are readings on a vertical circle necessary because all readings are taken at a fixed vertical angle.

The astrolabe in its original form was designed by MM. Claude and Driencourt, who used a 60° prism to control the fixed altitude at which stars are observed. In British practice, the original form has now given way to an improved type designed by Captain T. Y. Baker, R.N., which is manufactured by Messrs. Cooke, Troughton & Simms, and has been

SIDE ELEVATION



45° Prismatic Astrolabe.

FIG. 33.

adopted for standard use by the Admiralty. In this later type of instrument the observations are made at a fixed altitude of about 45°, instead of 60° as in the Claude and Driencourt model. Hence, this instrument is called the 45° Prismatic Astrolabe. One advantage of observing at an altitude of 45° instead of 60° is that there is a greater choice of stars for the lower altitude. Another is that the computations are somewhat simpler.

A form of astrolabe, designed by Mr. E. A. Reeves for use on an ordinary theodolite, can also be obtained.

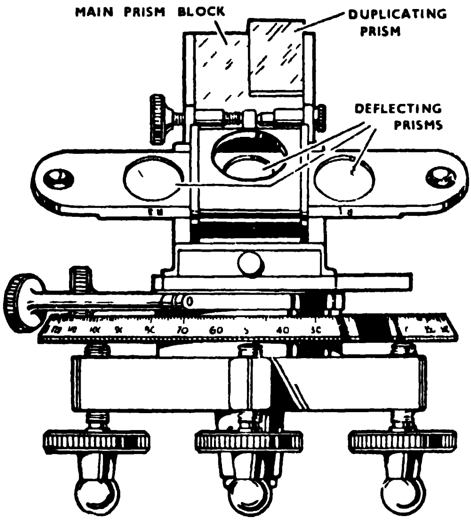
It may be noted that it is not essential that the prism should be so accurately constructed that the observed altitude should be exactly 60° or 45°. So long as all stars in a series are observed at the same altitude — and, apart from ordinary errors of observation, this is secured by the design of the instrument—a small error in the construction of the prism, causing the observed altitude to differ slightly from the nominal altitude, does not matter. Observations to at least three stars are necessary, and the resulting equations take into account, and can be used to determine, the error of the prism.

The astrolabe can only be used to determine latitude, longitude and time ; it cannot be used for observation of azimuth.

Fig. 33 shows a side elevation of the 45° astrolabe and Fig. 34 a front elevation with the telescope removed. In Fig. 33, ABCDEA is a pentagonal, or Prandl, prism in which the faces AE and CD are silvered, but on AE a circular part of the silvering is removed. A small right-angled prism AEF, with an acute angle of $22\frac{1}{2}$°, is cemented on the face AE of the pentagonal prism. The artificial horizon consists of mercury held in a shallow metal tray, and is placed, as shown, in front of the face EF of the acute angled prism. A telescope placed near, and looking down into, the face CB of the pentagonal prism enables magnified images of the star to be seen clearly but serves no purpose other than this.

FRONT ELEVATION

(telescope removed)



45° Prismatic Astrolabe.

FIG. 34.

The principle of the instrument is very simple. A ray of light entering the face AB at right angles to it is reflected horizontally from the face CD and is then reflected again from the face AE, the reflected ray entering the objective at 45° to the horizontal. This gives one image of the star. A second, and direct, image is formed by reflection from the artificial horizon, the reflected ray passing through the unsilvered portion of the face EF of the prism AEF and then through the face CB, no deviation of the ray taking place in its passage through the prism as the faces EF and CB are both parallel and are perpendicular to the ray entering EF. Thus, on looking through the telescope, two images of the star are seen but it will be noticed that they appear to move in opposite directions. The observation is made when they appear to coincide or to lie against each other in the same horizontal line.

As, owing to diffraction, either image of the star is of finite dimensions,

it would be very difficult to judge when the centres were coincident if the two images were superimposed one on top of the other. Consequently, half of the face AB of the pentagonal prism is left uncovered and a weak deviating, or duplicating, prism is placed over the other half with its edge parallel to the edge of the main prism. The effect of this prism is to cause two images, lying close together, to be formed in the pentagonal prism and the instrument is set so that the image formed in the artificial horizon passes midway between these two images. The appearance in the telescope is therefore as shown in Fig. 35, and the observation is taken when the three images appear to lie, close to the vertical wire in the telescope, on a horizontal line. A fine motion adjustment screw is provided to bring the reflected single image between the duplicate images. This screw rocks the whole prism casing about an axis parallel to the telescope axis and situated near the lowest point of the casing.

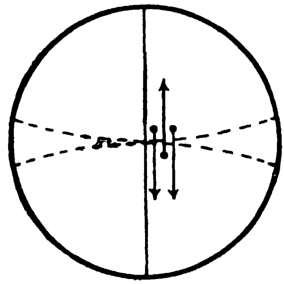


FIG. 35.

Another useful accessory is provided by means of which several observations may be made on the one star. This is done by means of weak prisms interposed in front of the mercury pool. If the angle of deflection caused by one of these prisms is α, the angle between the directly observed star and the reflected image will be $90° + \alpha$ or $90° - \alpha$, according to the way in which the prism is turned. Hence, if it is used in both ways, angular elevations of $45° + \frac{\alpha}{2}$ and $45° - \frac{\alpha}{2}$ will be measured, the mean being 45°. In the latest model of the Cooke, Troughton & Simms instrument three such prisms are fitted, these giving angular deviations of 3′, 9′, and 15′, corresponding to changes of elevation of $1\frac{1}{2}'$, $4\frac{1}{2}'$ and $7\frac{1}{2}'$. These prisms are shown in section in Fig. 33 and in elevation in Fig. 34. They are mounted in a slide which can be rotated through 180° about its horizontal axis and can be pushed into any one of three horizontal positions, so as to expose the different prisms in turn. This slide engages with a spring detent when the prism is correctly placed in the path of the ray. By using these prisms six observations with one star are possible. The edges of the slide are marked R1, R2, R3 and S1, S2, S3 respectively, R1 and S1 referring to the 15′ prism, R2 and S2 to the 9′ prism and R3 and S3 to the 3′ prism. For a rising star, the correct sequence of observations is R1, R2, R3 (reverse), S3, S2, S1. For a setting star, the order is S1, S2, S3 (reverse), R3, R2, R1.

The instrument is provided with a horizontal circle divided into degrees, the sole function of this circle being to assist in finding the star. The circle is friction tight and can be set so that the zero indicates true north when the telescope is in the meridian. This setting is made by means of a tubular magnetic compass attached to the telescope support. A circular level is fixed to the cover plate, but the instrument need not be very carefully levelled as all that is necessary is to ensure that the mercury will not run over the side of the containing metal tray. Levelling itself is done by three levelling screws and fine adjustments in azimuth are

secured by means of a clamp and fine-motion screw fitted between the cover plate and the tribrach.

The stand or stool provided with this instrument is made of mahogany, with stretcher bars for stiffness and an adapter for attachment of the instrument. This stand is only 18 in. high so that observations may be made comfortably from a camp chair. In order to secure rigidity, it is well to set the legs of the stand on a concrete platform and to dig them in slightly while the concrete is still wet. A wind screen should be erected about the instrument and the chronograph and wireless set housed in a tent close to the point of observation.

Preparing a Programme of Work. Before commencing observing it is essential to prepare a suitable programme of work. This means that the approximate azimuths of a number of stars, and the sidereal times when they are at altitudes of 44° 52½′ to 45° 07½′, must be worked out beforehand. This information is needed not only for knowing which stars to observe, and when to observe them, but also for setting the instrument to the proper azimuth and picking up and identifying the stars to be used in the observations. Stars should be observed in sets of four, one in the north, one in the south, one in the east and one in the west quadrant, the azimuths of stars in opposite quadrants differing as closely as possible by 180° and each star being within the middle 30° of the quadrant in which it lies.

The first thing to do is to work out the sidereal time at which it is proposed to commence observing and then to choose suitable stars. For this, the Hydrographical Department of the Admiralty publishes a special diagram (No. 5170. "Diagrams for the Preparation of Star Programmes for the 45° Astrolabe"). From these diagrams the limits of Right Ascension and Declination for the stars in each quadrant which read 45° zenith distance during suitable hours of observation can be obtained. Then, having selected suitable stars, the times and azimuths for reading 45° altitude can either be determined from the diagram and star azimuth tables or else they must be calculated. Naturally, the times to start observing comes a little before the time computed for 45° altitude, since the first observation is taken at an altitude of 44° 52½′ and the last at an altitude of 45° 07½′, or *vice versa* if the star is setting. The first observation will, in fact, be taken at a time 30 $\sec\phi \operatorname{cosec} A$ seconds before the star is at an altitude of 45°, and the last observation the same amount of time later, ϕ being the latitude of the place and A the azimuth of the star.

A very much more convenient method of preparing a programme of observations for the 45° astrolabe is to use a "Mechanical Programme Finder," which is manufactured by Messrs. Cooke, Troughton & Simms. Not only does this extremely useful accessory indicate suitable stars for observation, but the azimuth and local sidereal time at which any one of 956 given stars attains an altitude of 45° can also be obtained from it. It consists of a duralumin disc, 10·6 in. diameter, over which can be fitted, so as to revolve about its centre, one of ten celluloid quadrants giving each degree of latitude from 0° to 60° North and 40° South. The positions of 540 stars with northern declinations are marked on one side of the plate and those of 416 stars with southern

declinations on the other. Each star is numbered and the necessary particulars concerning it are given in a special star list which is supplied with the instrument. With this accessory, sidereal times can be read to 1 minute of time and azimuths to 30 minutes of arc. Hence, it provides an easy and rapid method of selecting suitable stars and of obtaining the necessary data for making out a programme of observation.

If a mechanical programme finder is not available, the selection of stars offered by the *Nautical Almanac* may not be sufficient. In this case a good star catalogue should be used. A suitable and up-to-date one is *Apparent Places of Fundamental Stars,* which is published annually by the Admiralty and gives the apparent places of 1,535 stars. If a 60° astrolabe is used, the preparation of programmes is facilitated by the use of a special 60° star list which is published by the American Geographical Society. This gives, for each degree of latitude between 60° N. and 60° S, the azimuth, local sidereal time, magnitude and right ascension of every star that is available for observation with an astrolabe of this kind.

Observations with the astrolabe should be confined to stars of magnitudes greater than the fourth since the images of smaller stars are very faint when viewed in the mercury pool.

Observing with the 45° *Astrolabe.* Having set up and levelled the instrument some little time before observations are to begin, the surface of the mercury in the artificial horizon should, if necessary, be cleaned by means of the glass rod provided for the purpose. The instrument should then be orientated by means of the tubular compass so that the zero on the circle indicates *true* north. When this has been done the instrument should be set to read the azimuth of the first star to be observed. Three or four minutes before the star is due to reach its proper altitude it will be seen coming into the field of view of the telescope, and, if it is an east star and rising, the slide containing the deflecting prisms should be set to read R1, or, if the star is setting, the slide should read S1. Three images of the star will now be seen, two moving together in one direction and one moving in the reverse direction. These images should be adjusted so that the single one comes midway between the other two. This is done by means of the small adjusting screw on the right which rocks the duplicating prism. The fine azimuth motion should then be used to bring all three images very close to the central wire in the telescope.

At the moment when the three images of the star are seen in the same horizontal line, the key of the chronograph is depressed and the slide containing the deflecting prisms is pushed over to read R2 (or S2). The observation is then repeated, and, in this way, six observations in all are made on the one star with the slide in the successive positions R1, R2, R3 (reverse), S3, S2, S1, or in the order S1, S2, S3, (reverse), R3, R2, R1 if the star is setting instead of rising.

The chronometer times may be checked by noting the differences in time for the R1 and R2 readings, which should be the same as the differences between the S2 and S1 readings. Similarly, the R2—R3 readings should be equal to the S3—S2 readings.

It may be noted that it is better to observe a few sets on each of several nights than it is to observe a large number of sets on one or two nights. If a large number of stars is observed on the same night, the surface of

the mercury may get scummed over while observations are in progress, in which case it can be cleared by means of the glass rod.

Computation of Results. For rough work the mean of the times of the six observations on a single star may be taken as the observed time when the star was at an altitude of 45°. The motion of a star in altitude is not, however, exactly proportional to time. Hence, if the mean of the observed times is taken as the time of observation, it becomes necessary in precise work to apply a second order correction to the zenith distance. This correction is given in seconds of arc by :—

$$\text{Corrn. to Z.D.} = -\left(\frac{1}{2N}\right)\Sigma(\Delta z)^2 \cot A\,(\cot A - \tan\phi \operatorname{cosec} A)\sin 1'',^*$$

where N is the number of observations, A the azimuth of the star measured from north through east, ϕ the latitude of the place, Δz the amount, in seconds of arc, by which the altitude for that particular observation differs from the standard altitude and, as usual, Σ denotes summation, so that $\Sigma(\Delta z)^2$ is the sum of the squares of the individual Δz's. For the later model of the C. T. & S. 45° astrolabe, $N = 6$, and Δz has the values $-7\frac{1}{2}'$ $-4\frac{1}{2}'$, $-1\frac{1}{2}'$, $1\frac{1}{2}'$, $4\frac{1}{2}'$, $7\frac{1}{2}'$, so that in this case the quantity $\left(\frac{1}{2N}\right)\Sigma(\Delta z)^2 \sin 1''$ becomes 0·23 seconds of arc. Hence :—

Correction to zenith distance in seconds of arc =

$$-0{\cdot}23 \cot A\,(\cot A - \tan\phi \operatorname{cosec} A).$$

For logarithmic calculation, the formula may be put in the form :—

$$\text{Correction} = -0{\cdot}23\,\frac{\cot A \cos(\alpha + \phi)}{\sin\alpha \sin A \cos\phi},$$

where $\tan\alpha = \sec A$.

The sign of the trigonometrical co-efficient depends on the signs and magnitudes of $\cot A$, $\operatorname{cosec} A$ and $\tan\phi$, and, if these are given their correct signs, no difficulty need be experienced with regard to the sign of the correction. The trigonometrical term, which may be called A, is given in a diagram, No. 5072, which has been published by the Admiralty. Note, however, that the " Cooke 45° Astrolabe " referred to in example *a* in this diagram is an earlier type of instrument in which eight observations were made instead of six.

After having corrected the zenith distance of 45° by means of the second order term, and applied the corrections for refraction and for the error of the prism when this is known, an assumed value for the latitude is taken and the azimuth and hour angle of each star are worked out from the formulæ :—

$$\tan^2 \tfrac{1}{2}A = \frac{\sin(s - c)\,.\,\sin(s - z)}{\sin s\,.\,\sin(s - p)},$$

$$\tan^2 \tfrac{1}{2}t = \frac{\sin(s - c)\,.\,\sin(s - p)}{\sin s\,.\,\sin(s - z)},$$

* This expression, which only holds for the 45° Astrolabe, can be obtained by using Taylor's Theorem to expand $(z + dz) = \cos^{-1}[\sin\phi \sin\delta + \cos\phi\cos\delta\cos(t + dt)]$ in terms of dt and then making the substitution $dt = dz\,.\sec\phi\,.\operatorname{cosec} A$. If an altitude h other than 45° is used the term in the bracket becomes $(\cot A \tan h - \tan\phi \operatorname{cosec} A)$. See p. 89.

where c = assumed co-latitude of place, z = zenith distance, p = polar distance of star and $s = \frac{1}{2}(c + p + z)$. From the hour angle and the R.A. of the star the computed sidereal time of observation is obtained and this, minus the observed sidereal time, converted into seconds of arc, gives dt_2, the absolute term in the observation equation. Reckoning azimuths clockwise from north from 0° to 360°, the equation may be written :—

$$dt_1 + \cot A \sec \phi d\phi - \operatorname{cosec} A \sec \phi \, dh - dt_2 = 0 \quad \text{for the northern hemisphere.}$$

$$\text{or } dt_1 - \cot A \sec \phi d\phi - \operatorname{cosec} A \sec \phi \, dh - dt_2 = 0 \quad \text{for the southern hemisphere.}$$

where ϕ is always taken as positive, dt_1 is the clock error, or true sidereal time minus observed sidereal time, $d\phi$ the correction to the assumed latitude, and dh a correction to the assumed altitude.*

There are three unknowns in this equation, dt_1, $d\phi$ and dh, and hence at least three equations are needed to obtain a solution. Usually there will be more equations than three, and, in that event, a straightforward solution by least squares is probably the simplest method to use. If the computer is not familiar with the use of least squares, the equations can be solved three by three and the means of all the results taken, but it is better to use a semi-graphic method first described in Ball's *Handbook of the Prismatic Astrolabe* and now included in Close and Winterbotham's *Text-book of Topographical Surveying*. Alternatively, another method, which is really a combination of the semi-graphic method and least squares, is described in Admiralty publication No. H.D. 285, "Notes on the Determination of the Value to be Accepted from Observations for Latitude and Longitude with the Prismatic Astrolabe." The description of the method is sufficiently detailed to enable it to be followed and used by a reader who is not familiar with least squares.

An example of an ordinary solution by least squares from four observations is given in a paper "Instructions for the Use of the Prismatic Astrolabe and Programme Finder," by W. Horsfield and W. A. Erritt, which is published in the *Empire Survey Review*, Vol. III, No. 21, July, 1936. The solution there worked out gives a probable error of $\pm 0''{\cdot}58$ for an unadjusted observation, and $\pm 0''{\cdot}50$ for an observation after adjustment by least squares.

It may be noted that if the error of the prism is not known and has not been allowed for, the quantity dh will contain this error as well as any small errors that there may be in the assumed altitude of the star. Hence, the error of the prism should be fairly well determined after the instrument has been in use for a short time. $d\phi$ gives the correction to the assumed latitude, and, when applied to the latter, gives the corrected latitude ; dt_1 gives the clock error on local sidereal time.

The differential equation used for determining dt, $d\phi$ and dh may be derived as follows :

From the astronomical triangle we have :—

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t.$$

* See *Monthly Notices of the Astronomical Society*, Vol. LXIV, pages 107 to 112, and Captain G. T. McCaw's "Report on the Trigonometrical Survey of Fiji," Vol. II.

Differentiating with respect to z, ϕ and t, we get :—

$$-\sin z\, dz = (\cos\phi \,.\, \sin\delta - \sin\phi\cos\delta\cos t)d\phi - \cos\phi\cos\delta\sin t\, dt.$$

Then, by formula xxi, page 6, the term in brackets is equal to $\sin z \cos A$. Hence, inserting this value and substituting $\sin A \,.\, \sin z$ for $\cos\delta \,.\, \sin t$ in the co-efficient of dt, we have

$$-\sin z\, dz = \sin z \cos A d\phi - \cos\phi \sin A \sin z\, dt,$$

or $$dt \cos\phi \sin A - \cos A d\phi - dz = 0,$$

or, since $dz = -dh$,

$$dt - \cot A \sec\phi\, d\phi + \operatorname{cosec} A \sec\phi\, dh = 0,$$

where A is the azimuth angle, reckoned the shortest way east or west from the elevated pole, and dt is the correction to the hour angle reckoned the shortest way from the zenith. Also, since dt_1 and dt_2 correspond to times measured westward from upper transit, $dt_1 - dt_2 = -dt$ or $+dt$, according as the star is east or west of the meridian, and hence the equation follows.

The method of observing at least three stars at equal altitudes can, of course, be used with a good theodolite provided proper readings are taken on the striding level and the appropriate correction made. When transport is difficult, the question may arise whether it is worth while carrying both a theodolite and an astrolabe or whether to carry one instrument or the other. The theodolite is an " all-purpose " instrument and a good one is capable of doing all that an astrolabe can do, and much else besides. The astrolabe, however, has been specially designed for the special work that it is intended to do, and it is light in weight. Its principal use would therefore appear to be on geographical and other similar surveys where determinations of latitude and longitude only are required and observations for azimuth and of horizontal and vertical angles are not needed. The 45° astrolabe has now been adopted as a standard instrument for use by the Royal Navy in ordinary hydrographical work.

Solar Attachment. The solar attachment is a device whereby the astronomical triangle may be solved mechanically. It was invented in America by Burt, in 1836, in the form of the solar compass, as an improvement on the ordinary compass for setting out boundaries along meridians in land surveys. Several patterns are now made in a form suitable for attaching to an ordinary theodolite. They are used principally for the rough determination of azimuth, but, unless a large number of solar azimuths is likely to be required, it is doubtful whether the saving of labour in reducing

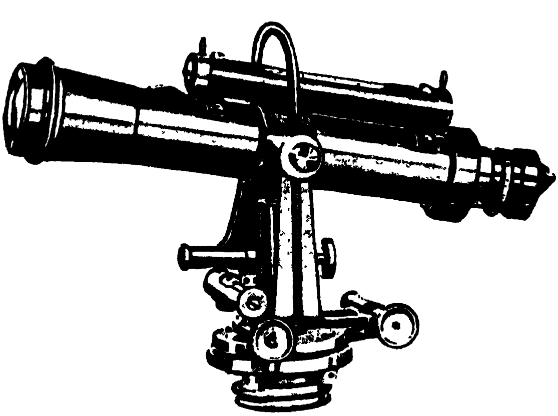


FIG. 36. SOLAR ATTACHMENT.

ex-meridian observations outweighs the disadvantages of extra weight and additional adjustments.

Fig. 36 illustrates a simple telescope form of solar attachment designed by Saegmuller. It is fitted on the top of the theodolite telescope, and differs from other forms in not having an arc on which to set off the sun's declination. The solar telescope, besides having a motion about its horizontal axis, can be rotated about the upright axis of the attachment, called the 'polar axis, which must be accurately perpendicular to the horizontal axis of the theodolite and to the line of sight of the main telescope. A graduated hour-circle, divided in hours with 5^m subdivisions, is fitted round the 'polar' axis in some forms, and serves to give time roughly.

Evidently, if the theodolite is oriented in the meridian, and the co-latitude is set off on the vertical circle, the line of sight of the main telescope will lie in the plane of the celestial equator, and the polar axis in the earth's axis (Fig. 37), parallax being considered negligible. If the solar telescope were parallel to the main telescope, rotation of the former about the polar axis would cause its line of sight to sweep out the plane of the celestial equator. But if the solar telescope is turned about its horizontal axis through an angle equal to the sun's declination for the time of observation, with an allowance for refraction, the solar line of sight may be brought on to the sun by turning the solar telescope about the polar axis. This can be done only when the polar axis is pointing to the pole, so that, when the declination and co-latitude are set off, an observation for meridian consists in turning the solar telescope about the polar axis and the main telescope about the vertical axis until the sun appears in the field of the former. When, on manipulating the respective tangent screws, the sun is placed in the square of the reticule of the solar telescope, the line of sight of the main telescope has been placed in the meridian, and the hour-circle reads the time. In the absence of a declination arc, the declination is set off by first bringing both telescopes into the same vertical plane by sighting any convenient point with each, setting the corrected declination on the vertical circle by depressing the main telescope for a north declination and elevating it for a south declination in the northern hemisphere, and then bringing the solar telescope horizontal by reference to its attached level.

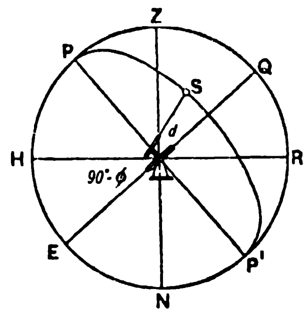


Fig. 37.

Timekeepers. The most precise portable timekeeper is the box chronometer as employed in observatories and on board ship. In field astronomy it is used for primary longitude determinations and in other circumstances where it can be kept either stationary or on board a vessel. The instrument is too delicate to maintain a uniform rate during transport by land. The pocket chronometer is better adapted for land journeys, but is liable to stoppage if subjected to shocks in travelling over rough country. The most serviceable timekeeper for field use is that known as the half-chronometer watch, which with due care gives results of a suitable

accuracy for mapping. A stop-watch is frequently a convenient accessory in observing.

Watch Error and Rate. The result of an observation for time gives, by comparison of the computed time with that indicated by the watch, the watch error at the mean instant of observation. The magnitude of the error is immaterial for purposes of time-keeping. If the watch were regulated to keep mean or sidereal time, the L.M.T., Standard M.T. or L.S.T. for the place of observation could be obtained at any instant by application of the error to the watch reading. It is, however, unnecessary that the error should maintain a constant value. What is expected of a good timekeeper is that, while subject to a particular set of conditions, it should go at uniform speed, so that its error changes uniformly. The daily change of error is known as the rate, which may be either a gaining or a losing one.

The length of time a good timekeeper will maintain a sensibly constant rate depends largely upon the degree of uniformity in the conditions to which it is exposed. It should be wound carefully and at the same hour each day. The rate is liable to variation by change of temperature, no matter how carefully compensated the watch or chronometer may have been, and is also affected by change of atmospheric humidity unless the instrument is enclosed in an air-tight case. It should be kept in either the horizontal or the vertical position, as the rate differs for the two. The standing rate, or rate when the instrument is stationary, differs from the travelling rate developed during transport by land or sea.

Determination of Standing Rate. The value of the standing rate is obtained by taking at the same place at least two sets of time observations separated by an interval of a few days. A test of the constancy of the rate may be made by means of time observations every night or every second night for a period of seven to ten days. Each time determination should, as far as possible, be made in the same manner. The rate is computed from the observed change of error as follows.

Example. A time observation on Oct. 20 at about $22^h\ 21^m$ L.M.T. showed the watch error to be $46^m\ 21^s{\cdot}2$ slow, and at the same place on Oct. 26 at about $20^h\ 13^m$ the error was $45^m\ 49^s{\cdot}4$ slow. Compute the rate, and find the L.M.T. at which the watch reads $6^h\ 42^m\ 54^s{\cdot}2$ on the morning of Oct. 23.

		d	h	m		
	Oct.	26	20	13		
	Oct.	20	22	21		
Interval between observations	=	5	21	52	=	$5^d{\cdot}911$

Change of error $= 46^m\ 21^s{\cdot}2 - 45^m\ 49^s{\cdot}4 = 31^s{\cdot}8$ gained

$$\text{Rate} = \frac{31{\cdot}8}{5{\cdot}911} = 5^s{\cdot}38 \text{ gaining.}$$

To find the error on Oct. 23 at $6^h\ 42^m\ 54^s{\cdot}2$ watch time.

Watch time of determination on Oct. 20 was about $22^h\ 21^m$ less $46^m\ 21^s{\cdot}2$, say $21^h\ 35^m$.

Watch interval $= 23^d\ 06^h\ 43^m - 20^d\ 21^h\ 35^m = 2^d\ 09^h\ 08^m = 2^d{\cdot}38 =$ mean time interval with sufficient accuracy.

Gain $= 2{\cdot}38 \times 5^s{\cdot}38 = 12^s{\cdot}8$.

Required error $= 46^m\ 21^s{\cdot}2 - 12^s{\cdot}8 = 46^m\ 08^s{\cdot}4$ slow

and required L.M.T. $= 6^h\ 42^m\ 54^s{\cdot}2 + 46^m\ 08^s{\cdot}4 = 7^h\ 29^m\ 02^s{\cdot}6$.

Determination of Travelling Rate. Three cases occur, according as the journey—

(1) Lies between two points of which the longitude difference is known.

(2) Returns to the starting point.

(3) Does not return to the starting point.

Case (1) occurs in land journeys when travelling between two points at which longitude difference can be obtained by telegraph or is otherwise known, the travelling rate being required for longitude determinations at intermediate points on the route. The watch error is determined before starting and again on arrival, but the change of error is due in part to the difference between the local times of the two places. The latter, being known, is eliminated, and the residual change falls to be divided by the time interval between the observations, which is given with sufficient accuracy by the watch. In the case of land journeys, the travelling rate conditions include the normal rest at night. If, however, the surveyor is compelled to remain for a few days at a point on the route, and the watch is kept stationary, the circumstance must be allowed for in computing the travelling rate (see *Example* 2 below).

In case (2), since the observations for watch error at start and finish are taken at the same place, the calculation is performed as for standing rate. Case (3) necessitates the determination of the travelling rate before commencement of the journey. The method is the same as for standing rate, except that during the period between the two time observations the watches are carried on a daily march of the same duration as the average daily march is likely to be during the journey, and under as nearly as possible the same conditions. The opportunity should be taken, by means of a series of time observations between those from which the rate is computed, of testing the constancy of the travelling rate, which is more liable to irregularity than the standing rate.

Example 1. A survey party travels eastwards from A to B, the difference of longitude between which is $2° \, 12' \, 20''{\cdot}8$. Before starting, a time observation showed the standard watch to be $6^m \, 14^s{\cdot}2$ fast, and after arrival at B the error was $1^m \, 56^s{\cdot}1$ slow. The interval between the time observations, as shown by the watch, was 9·07 days. Compute the travelling rate.

$$\text{Difference of longitude in time} = \frac{2° \, 12' \, 20''{\cdot}8}{15} = 8^m \, 49^s{\cdot}4.$$

and, since B is east of A, the change of longitude makes the watch less fast by this amount.

But observed change of error $= 6^m \, 14^s{\cdot}2 + 1^m \, 56^s{\cdot}1 = 8^m \, 10^s{\cdot}3$ lost

$\therefore$ change of error due to rate $= 8^m \, 49^s{\cdot}4 - 8^m \, 10^s{\cdot}3 = 39^s{\cdot}1$ gained

$$\text{whence travelling rate} = \frac{39{\cdot}1}{9{\cdot}07} = 4^s{\cdot}31 \text{ gaining.}$$

Example 2. A time observation at A on Nov. 7 at about $20^h \, 52^m$ watch time showed the error of a watch on L.M.T. to be $18^m \, 20^s{\cdot}5$ fast. On the following morning a journey started towards B, and on the evening of arrival on Nov. 15 at about $21^h \, 43^m$ the error on L.M.T. was found to be $6^m \, 49^s{\cdot}6$ fast. The watch was kept stationary while the party remained at B, and on Nov. 17 at about $20^h \, 16^m$ the error was $6^m \, 37^s{\cdot}9$ fast. A return journey to A commenced on the morning of Nov. 18, and A was reached on the evening of Nov. 28, the watch error then being found to be $16^m \, 57^s{\cdot}8$ fast at about $22^h \, 12^m$. Compute the travelling rate.

Interval between observations at A
$= \text{Nov. } 28^d\ 22^h\ 12^m - \text{Nov. } 7^d\ 20^h\ 52^m = 21^d\ 01^h\ 20^m$, of which the rest between the observations at B amounted to
$\text{Nov. } 17^d\ 20^h\ 16^m - \text{Nov. } 15^d\ 21^h\ 43^m = 1^d\ 22^h\ 33^m$

Period during which travelling rate developed
$= 21^d\ 01^h\ 20^m - 1^d\ 22^h\ 33^m = 19^d\ 2^h\ 47^m = 19^d{\cdot}116$

Change of error between observations at A
$= 18^m\ 20^s{\cdot}5 - 16^m\ 57^s{\cdot}8 = 1^m\ 22^s{\cdot}7$ lost
but of this there was lost during the wait at B
$6^m\ 49^s{\cdot}6 - 6^m\ 37^s{\cdot}9 = 11^s{\cdot}7$
$\therefore$ change of error due to travelling rate $= 1^m\ 11^s{\cdot}0$ lost.

$$\text{Travelling rate} = \frac{1^m\ 11^s{\cdot}0}{19{\cdot}116} = 3^s{\cdot}71 \text{ losing.}$$

Note. Instead of having the watches on standing rate while at B, they might have been subjected to travelling conditions by being taken on a march during each of the two days the party remained at B. In calculating the travelling rate, no account would then have to be taken of the wait at B.

Comparison of Watches. When the error of one watch is known, that of another can be obtained by comparing their simultaneous readings. The comparison is best made by two observers, each having one of the watches. At a whole second on his watch, observer A makes a sharp tap, at which B estimates the reading of the other to $0^s{\cdot}1$ or $0^s{\cdot}2$. Each notes his reading. This is repeated two or three times, and then the same number of signals is made by B after an exchange of watches. The precision of such a comparison will probably be not less than that of the known error.

If the comparison is made by a single observer, he must measure the short interval between the reading of one watch and that of the other, either by counting the ticks of the first watch or by means of a stop-watch. The number of ticks per second, usually $2\frac{1}{2}$, $4\frac{1}{2}$ or 5, being known, the watches, A and B, to be compared are laid side by side, and the observer, reckoning from a whole number of seconds on A, starts his count of A with the first tick thereafter. He immediately turns to B, and reads the position of the seconds hand at the instant corresponding to one of the numbers of the count. The time equivalent of the ticks counted is added to the reading of the A watch at the start of the count, and gives the time to be compared with the reading of the B watch. When a stop-watch is used, it is started from $0^m\ 0^s$ when the seconds hand of the first watch reaches a graduation, and is stopped at the instant of reading the second watch. With either method, several repetitions are required.

Comparison of Chronometers. A highly accurate comparison of box chronometers can be made if one keeps mean time and the other sidereal time. Box chronometers beat every half-second, and since sidereal time gains on mean time by nearly 10^s an hour, there is a steadily varying interval between the beats of a mean time chronometer and those of a sidereal time chronometer. The beats synchronise about every 3^m, and each coincidence marks an instant at which the times on the two chronometers differ exactly by their readings taken to half-seconds.

To compare a mean-time with a sidereal time chronometer, each is read by a separate observer at the instant the beats are in unison. Since the readings are required only to half-seconds, the precision of the

comparison is dependent upon the selection by one of the observers of the coincident beat at which both readings are taken. The exact coincidence is difficult to distinguish, since several beats almost synchronise, but with practice it is possible to detect a lack of coincidence as small as $0^s{\cdot}03$, and the probable error is likely to be much less, particularly if two or three coincidences are noted. An observer working alone can make the comparison by watching one chronometer and counting the beats of the other from a whole second near the coincidence. In applying the method to the comparison of two mean time chronometers, each is compared with a sidereal time chronometer, while a mean time chronometer is similarly used as the medium for the comparison of sidereal time chronometers.

GENERAL PROCEDURE IN OBSERVATIONS

The routine in observing depends upon the precision required and the instrument used. The theodolite is generally employed for field determinations, and certain items of routine, common to several observations, are described here.

Observing by Theodolite. In preparing to observe, attention should be given to the stability of the theodolite support. If the instrument must be supported on an ordinary tripod instead of an observing pillar, the tripod should rest on well driven pegs if the ground is at all yielding. Needless to say, the theodolite should be in adjustment (Vol. I, page 92, and Vol. II, page 195), but nevertheless the routine should be designed to eliminate instrumental errors as far as possible. In general, observations must consist of an equal number of face right and face left sights, and each circle reading must be obtained from both micrometers or verniers. In the case, however, of certain observations, such as equal altitudes, the aim is to preserve constant instrumental conditions, and a change of face is not required.

In all astronomical observations where altitudes are observed, the barometer and thermometer should be read at the time of observation in order to provide data for the calculation of the atmospheric refraction correction, and this correction must be applied to the observed altitude (p. 18). The barometric readings may be taken on a good aneroid which has been compared against a standard mercury barometer. In addition, the correction for parallax (p. 19) must, of course, also be applied in the case of observed altitudes of the sun.

Level. Before observing, the vertical axis must be made truly vertical by manipulating the levelling screws and, if necessary, the clip screws until the bubble of the altitude level remains central during a rotation. It is usual to find that this bubble does not preserve a central position throughout a series of observations. The readings of both ends of the bubble should therefore be taken immediately after each observation and before reading the circles. If the bubble is considerably off centre, it should be returned approximately to the central position by the clip screws before taking the next sight. The correction to altitude for dislevelment is as follows.

For a tube graduated in both directions from the centre.

Let ΣE = sum of the successive readings of the eye end of the bubble for the n individual observations, both F.R. and F.L.

ΣO = sum of the corresponding readings of the object glass end

d = value of one division of the level

c = level correction to average altitude.

$$\text{Then } c = \frac{\Sigma O - \Sigma E}{2n} \,.\, d$$

For a tube graduated continuously from the eye end.

Let G = reading of centre graduation.

$$\text{Then } c = \frac{\Sigma(O + E) - 2nG}{2n} \,.\, d$$

In azimuth observations the striding level must be used, and the azimuth correction for horizontal axis dislevelment is computed from the readings of the bubble (page 101).

Note. A field method of determining a mean value of d which is suitable for use with the altitude bubble of a theodolite is described in Vol I, page 37. Professor G. C. Comstock, in his *Text-book of Field Astronomy for Engineers*, has described a very neat method, known as the "Wisconsin method," which can be employed to determine the value of d of a striding level at different parts of the vial. The instrument is set up and levelled with the telescope aligned over one footscrew. This screw is then depressed so that the vertical axis of the instrument is thrown 1° or 2° off the vertical, the angle of inclination i being measured on the vertical circle by bringing the bubble on the micrometer arm back to the centre of its run and then reading the circle micrometers. If the lower part of the theodolite is clamped, and the upper part is swung slowly in azimuth, the bubble on the striding level will move from its central position and it can easily be shown that the angle of depression δ of the bubble will be given by $\tan \delta = \sin \alpha \,.\, \tan i$, or for small values of δ and α, by $\delta = \alpha \,.\, \tan i$, where α is the azimuth angle through which the instrument has been swung from the vertical plane in which the vertical axis has been depressed, that is from the position in which the bubble of the striding level is in its central position. Hence, by taking corresponding readings in the bubble and the horizontal circle, the values of d for different divisions of the level vial can easily be found.

The method may also be used for finding values of d for the altitude bubble. In this case, after the angle of depression of the vertical axis of the instrument has been found, the telescope and vertical circle are swung through 90° and the altitude bubble is brought to the centre of its run, after which readings are taken on the vial for different values of the angle α. The angle of depression of the bubble will then be given as before by the relation $\delta = \alpha \,.\, \tan i$.

Taking Times. Whenever possible, the observer should have the assistance of a recorder, who books the level and circle readings and, when necessary, reads the times of observations. The observer should, as far as possible, allow the celestial body to make its own passage or contact, and a few seconds in advance warn the recorder to be prepared. At the instant of passage or contact he calls out to the recorder, who reads the timekeeper.

Times are most easily taken on a chronometer. Chronometers beat half-seconds, so that the rhythm is not difficult to follow. The best method is to observe the hand and count with the beat up to 10, thus—1 *and* 2 *and* 3 *and* . . . After a little practice, the occurrence of a particular instant can be estimated to $0^s{\cdot}1$ with considerable accuracy. Since the half-chronometer watch or an ordinary watch is more generally

used in field astronomy, times cannot usually be taken more accurately than to about $0^s.2$ In reading a watch it is well to keep count of the ticks while observing the seconds hand Half chronometer watches usually make $4\frac{1}{2}$, and ordinary watches 5 ticks a second

If the observer has not the assistance of an efficient recorder, he must himself take the times In the "eye and ear" method, which is applicable when the chronometer is used, the observer places the chronometer so that it can be heard and seen from the instrument, reads it before the observation, and starts counting from the reading He keeps up a continuous count in unison with the beats, and estimates the instant of the observation from his count, verifying the minutes on the chronometer immediately afterwards With a half-chronometer or ordinary watch, a common method is to hold the watch near the ear while observing and to start counting the ticks at the instant of bisection or contact The watch is then looked at, and the count is stopped when the seconds hand reaches a convenient division The required time is the watch reading at that instant less the time equivalent of the ticks counted

A simpler method is for the observer to use a stop watch, which he starts at the instant of passage or contact He then carries the stop-watch to the chronometer or watch, and stops it when the latter reads an exact second The chronometer reading less that of the stop watch is the required time of the observation

Note-keeping. The observer should maintain a regular order of reading and calling out quantities to the booker, thus level, eye end, object end, vertical circle micro I, micro II, horizontal circle, micro A, micro B Only minutes and seconds are read and booked for the second micrometer or vernier reading of a pair For observations with a small micrometer theodolite, it should be decided at the outset whether the quality of the determination and the state of the instrument as regards adjustment of the micrometers warrants correction of the readings for run (page 197) In refined observations both back and forward readings are taken and columns are provided in the angle book for the entry of run corrections and the means of the corrected readings Quantities entered in addition to those read out by the observer are object observed, face, time, barometer and thermometer readings (for evaluation of the refraction correction) date and place of observations, as well as any further data required in reducing the observations

Relative Merits of Star and Sun Observations. Star observations are likely to yield much more accurate results than those derived from observations of the sun A star appears in the field of the telescope as a point of light, and presents an ideal mark for bisection The great choice of stars of known position makes it possible to multiply, without undue delay, observations on stars well situated for the determination in hand Errors arising from uncertainties in refraction are effectively reduced by pairing and balancing of observations on east and west stars for time and azimuth, and on north and south stars for latitude, more particularly as the observations forming a pair can be taken within such a short time of each other that it is reasonable to assume that the change of refraction between them is negligible Balancing of sun observations for time and azimuth, on the other hand, necessitates a wait from morning till late

afternoon, and it is impossible to make any balance in determining latitude by the sun.

Nevertheless, solar observations are very useful to the surveyor when a refined result is not required, as it is sometimes awkward to have to remain away from camp after dark. In malarial regions, especially, star observations are not generally undertaken if determinations of sufficient precision can be obtained from the sun.

SELECTION AND IDENTIFICATION OF STARS AND THE PREPARATION OF A PROGRAMME OF OBSERVATIONS

Selection of Stars. Before proceeding with astronomical observations in the field, it is always advisable to draw up beforehand a programme of observation giving a list of the stars to be observed, their magnitudes and all the data necessary to enable them to be easily located and observed at chosen times. The preparation of such a list demands a knowledge of the most suitable positions in which to observe a star in order to reduce to a minimum the effects of errors of observation. In general, this means choosing those positions in which the observed quantity is changing most rapidly with respect to the quantity to be determined, a condition which may be expressed mathematically by the requirement that $\Delta Q_o/\Delta Q_r$ should be a maximum, where ΔQ_r is the change in the quantity required corresponding to a change ΔQ_o in the quantity observed. Thus, if altitudes are being observed to determine time, the altitude of a star is changing least rapidly with respect to time when the star is on the meridian and most rapidly when it is on the prime vertical. Hence, we choose stars which will lie as nearly as possible due east or west at the times when the observations are taken. This also follows from the expression

$$\frac{\Delta h}{\Delta t} = -\cos\phi \sin A$$

given on page 92, for this expression is a maximum when $A = 90^\circ$. Again, meridian transits give good determinations of time since a star is changing in azimuth most rapidly with respect to time (and a meridian observation means an observation involving the true azimuth of the line of collimation) when the star is on the meridian.

Errors in quantities besides the one directly observed may also affect the accuracy of computed results. For instance, when altitudes are observed to determine time, we need to know the latitude of the place of observation, since the formula giving the azimuth includes terms in latitude. In this case, it is shown on page 92 that

$$\frac{\Delta t}{\Delta \phi} = \sec\phi \cot A,$$

and consequently the effects of errors in the assumed latitude will affect the derived time least when A is near 90°. Hence, by observing east and west stars we not only minimise the effects of errors in the observed altitude but also those of errors in the assumed latitude.

The effects of errors in observation and in the quantities involved in the computations may also be reduced by careful balancing of stars.

In particular, there is always some doubt about the true value of celestial refraction when altitudes are observed and, unless special precautions are taken, any difference between the real and the computed value of the refraction correction will cause an error in the altitude and in the computations which may affect the quantity that is being determined. For this reason, and because the real value of the refraction correction for very low altitudes is always particularly uncertain on account of unknown variations in atmospheric conditions near the horizon, celestial bodies should not be observed at altitudes less than about 20°. Now in the expressions for $\frac{\Delta t}{\Delta h}$ and $\frac{\Delta t}{\Delta \phi}$ mentioned above, the factor cosec A occurs in the first expression and the factor cot A occurs in the second. Both these factors have different signs for values of A in adjoining quadrants on either side of the meridian. Hence, for the same values of h and ϕ and of Δh and $\Delta \phi$, the sign of Δt will be different for an east star to what it will be for a star oppositely placed to the west of the meridian. From this it follows that, if an observation to an east star is balanced by one to a suitably placed west star at the same altitude, the effects of errors in observed altitude and assumed latitude will be reduced or eliminated when the mean of the computed results of the two observations is taken.

Again, in primary observations for time by meridian transits, there are three corrections to be applied to the observed results, *viz.* azimuth correction $= a \,.\, \sin z \sec \delta$, level correction $= b \,.\, \cos z \sec \delta$ and collimation correction $= c \sec \delta$, where a, b and c are constants, the determination of which is liable to small errors of observation, z is zenith distance and δ is declination (page 86). In applying these formulæ to stars observed above the elevated pole, z is to be reckoned positive when it lies on one side of the zenith and negative when it lies on the other side of the zenith. Consequently, the sign of $\sin z$ will depend on whether the star will cross the meridian north or south of the zenith. It follows therefore, that for stars observed above the elevated pole, the effect of an error in the azimuth correction, due to an error in the value assumed or determined for the constant a, will be reduced if an observation to a north star is balanced by one to a south star for which the quantity $\sin z \sec \delta$ has much the same numerical value as for the north star. Balancing in this way, however, will not reduce or eliminate the effects of errors in the constants b and c since $\cos z$ and $\sec \delta$ do not change sign with change of sign in z or δ. If one star were observed below the elevated pole and the other above it, the rule would need modification since in that case, in order to obtain the correct signs of the corrections, $\sec \delta$ is to be taken as negative for the star observed below the elevated pole (see footnote on page 86).

Hints on the selection and balancing of stars are given in the following pages for each of the more important kinds of observations that are commonly used in field astronomy.

Identification of Stars. Stars of large magnitude may be identified by means of a suitable star chart, such as Norton's *Star Atlas* or the charts provided in the War Office publications *Field Astronomy* and *Text Book of Topographical Surveying*. The use of a star chart is simplified if the

observer familiarises himself beforehand with the appearance of the different constellations and the positions of some of the brightest stars, and useful hints on the use of charts and the identification of the constellations and brightest stars will be found in Chapter XV of *Field Astronomy*. In star charts, the curves corresponding to curves of latitude and longitude on a map are curves of Right Ascension and declination, so that the approximate celestial co-ordinates of each star can easily be read off direct on the chart.

When faint stars are to be used for observation, it will often be impossible to identify them in the telescope from the chart and in this case the easiest and best procedure will be to work out their approximate azimuth and elevation at suitable times, and then pick them up in the field of view of the telescope of the theodolite when the latter is set to the calculated azimuth and elevation. The simplest case is that of meridian transits, since the sidereal time of transit is equal to the R.A. of the star, and the altitude can be found from the simple relations $z = \phi - \delta$ for upper transit and $z = \phi + \delta - 180°$ for lower transit, given, with the relevant rules for signs, on pages 112–113. If a sidereal clock which is keeping sidereal time is used, the clock time of transit will be the star's R.A.: if a mean time clock is used, the mean time of transit can be worked out as in example 10 on page 41. When stars are to be observed away from the meridian, the hour angle corresponding to the time at which it is proposed to observe can be calculated from the star's R.A., and the altitude and azimuth from

$$\sin h = \sin\phi \sin\delta + \cos\phi\cos\delta\cos t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$
$$\sin A = -\sin t \cos\delta \sec h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

where, for logarithmic computation, the first expression may be written

$$\sin h = \frac{\sin\delta}{\cos\theta}\sin(\phi + \theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}$$

in which

$$\tan\theta = \cot\delta\cos t.$$

In some cases, it will be found more convenient to observe a star and identify it later from the data provided by the observations. This can be done when the approximate time of the observation is known and the data available from the observations or from the setting of the instrument enable the altitude and approximate azimuth to be known. From these data, the approximate declination and R.A. can be computed, if necessary, by slide rule or four-figure logarithms and a star found in the catalogue for which the tabulated values agree roughly with these computed values. The objection to this procedure, however, is that, after the work has been completed, it may be found that, after all, the observed star is not one of those for which places are given in the almanac or catalogue. In the case of meridian observations, the observed sidereal time of transit, after due allowance has been made for longitude, will give the star's R.A., and, given an approximate value for the latitude, the observed altitude at meridian passage, or near meridian passage, will give an approximate declination. For stars not on the meridian the relative formulæ are :—

$$\sin\delta = \sin h \sin\phi + \cos h \cos\phi \cos A, \quad . \quad . \quad . \quad . \quad . \quad (D)$$

and

$$\sin t = -\sin A \cos h \sec\delta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E)$$

where, for logarithmic calculation, we can write equation (D) in the form

$$\sin\delta = \frac{\sin h}{\cos\psi}\sin(\phi + \psi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (F)$$

in which

$$\tan\psi = \cot h \cos A.$$

A very simple method of identifying stars not on the meridian is to take three or more timed observations of altitude on one face of the instrument and from these to calculate the rate of change of altitude with respect to time. Then, from

$$\frac{\Delta h}{\Delta t} = -\sin A \cos\phi$$

we have, if Δh is expressed in seconds of arc and Δt in seconds of time :—

$$\sin A = -\frac{1}{15}\frac{\Delta h}{\Delta t}\sec\phi,$$

an expression which has the useful property of being the same for all stars and is independent of the R.A. and declination of the particular star used. Hence, if this value of A and the value of h taken at the middle of the time interval are substituted in equations (D) or (F) and (E), δ and t can easily be calculated and the star found from the catalogue.

In all the above cases it will normally be sufficient to carry out the computations with a slide rule or four-figure logarithms since all we want are values accurate enough to get the star in the field of view or to identify it in the catalogue.

The method of identifying stars by rate of change of altitude with respect to time was devised by Dr. J. De Graaf-Hunter in connection with his methods for the rapid determination of time, latitude and azimuth (pages 130–132). If the horizontal angle between a reference station and the star is observed midway between the two timed observations of altitude, the method also gives a quick and simple means of determining and setting out roughly the direction of the meridian without having to identify any particular star.

Making Out a Programme for Observation. When stars are selected beforehand they should be entered on a list giving in order the time for observation, name or number of star, magnitude, and the approximate azimuth and altitude at the selected times. Stars which are intended to balance one another should be differentiated in some way so that each pair can easily be picked out, a good plan being to enter the particulars of all east or west or north or south stars on one side of the list or sheet and of all the stars to balance them opposite them on the other side. Particulars of more stars than will actually be needed should be entered to allow for breaks in observing time due to passing clouds or other causes.

REDUCTION OF OBSERVATIONS

In practically all cases, the formula for the reduction of the observations is obtained directly from the solution of the astronomical triangle (page 14)

by means of the appropriate formula of spherical trigonometry already given on pages 4 to 6. This simply means the substitution of one or more of the quantities $90° - \phi$, $90° - \delta$ and $90° - h$ for the sides; and A, the azimuth angle, t, the hour angle (reckoned the shortest way, east or west, from upper transit) and q, the parallactic angle, for the angles of the spherical triangle. In formulæ involving s, the half of the sum of the sides, if two of the sides are $90° - \phi$ and $90° - \delta$ it is convenient to use z, the zenith distance, instead of $90° - h$, for the third side. Similarly, if $90° - \phi$ and $90° - h$ are used as two of the sides, it is convenient to use p, the polar distance, instead of $90° - \delta$ as the third side. This avoids the angle 135° being introduced into the formula. An alternative, of course, is to use the co-latitude and polar and zenith distances instead of the latitude, declination and altitude respectively.

Small corrections, or the effect of errors in the observed or computed quantities caused by errors of observation or in the assumed data, may generally be found by simple differentiation of a suitable fundamental formula. In carrying out this differentiation, and in choosing a suitable formula for the process, it is important to decide which quantities are to be held fixed and which can vary. For example, if an error in a computed azimuth arising from an error in observed time is being investigated, and if the formula used for differentiation were $\sin A \sec \delta = \sin t \sec h$ it should not be forgotten that h, as well as A, varies with time, so that the resulting differentiation of A with regard to t would include $\frac{dh}{dt}$. In this case a better formula to use would be :—

$$\tan \delta \cos \phi = \sin \phi \cos t + \sin t \cot A$$

which only involves the two variables, A and t and the two fixed quantities, ϕ and δ. On the other hand, if A were computed from the formula $\sin A \sec \delta = \sin t \sec h$ and a small error dt were made in the value of t used in the computation, and this did not affect the value of h, the error in A could be obtained by differentiating A with regard to t, keeping h and δ fixed in the differentiation.

In certain cases, simple differentiation is not sufficient to determine a correction as it will only give the first order term and the quantities involved may be so great in magnitude that second and even terms of a higher order are appreciable. Here the solution may generally be obtained by using Taylor's or Maclaurin's Theorem (page 7) to expand the function in powers of a known or observed quantity. An example of such a correction is the one to be applied to the mean or standard altitude when six observations of a single star are made, by means of the deflection prisms, with a 45° prismatic astrolabe (page 70), or to the azimuth computed from the mean of the face right and face left ex-meridian altitudes of a star (page 104).

DETERMINATION OF TIME

In determining local time, the chronometer times of the individual observations are noted, and the result gives for the average instant of the observation the error of the chronometer on sidereal or solar time, according as a star or the sun is observed. Determinations are made

from meridian or ex-meridian observations. In the former case, the local sidereal time of the instant of transit of a star is given by the star's right ascension, and the chronometer error is obtained by comparison of that quantity with the chronometer reading at transit. For the sun, the instant involved is that of local apparent noon, which is reduced to local mean time for the evaluation of the watch error on mean time. When a celestial body is observed out of the meridian, the observations are directed to obtaining sufficient data to enable the astronomical triangle to be solved for the hour angle, from which the sidereal or solar time is obtained.

Methods. The principal methods are by :

(1) Transits

(2) Ex-meridian altitudes of stars or the sun

(3) Equal altitudes of stars or the sun.

Time by Star Transits. Observation of the instants of star transits forms the most direct method of obtaining local time, and is that used for primary field determinations. The method can be applied to minor determinations with small theodolites, but is not commonly employed, since the observations necessary to enable the telescope to be placed in the plane of the meridian would themselves yield the approximate watch error.

Primary Determinations. Primary determinations are made by means of a portable transit instrument (page 54) or a first-order theodolite fitted with eyepiece micrometer. With a transit instrument the best results are obtained by automatic registration on a chronograph by means of the transit or impersonal micrometer. Otherwise the instants of the passages of the stars across the successive hairs are transmitted to the chronograph record by key. In the absence of a chronograph, the times are taken by the eye and ear method. For a precise determination, such as would be required in connection with primary telegraphic longitudes, twenty-four to thirty-two star transits are observed. These are arranged in groups of four to six, the telescope being reversed between each group. The Hunter Shutter Eyepiece (page 58) is well adapted for transit observations, particularly for longitude observations in which Greenwich time as well as local time has to be found from the rhythmic time signals (page 126), as the eyepiece has been designed to minimise the effects of constant personal errors and no chronograph is needed.

The approximate direction of the meridian having been set out in advance, the telescope is first placed in that plane preparatory to closer adjustment. An error in azimuth has only a small effect upon the observed time of transit of a star near the zenith, and the chronometer time of transit of such a star is first observed and an approximate value for the chronometer error obtained. The telescope is then directed towards a slow moving or circumpolar star which is about to transit. The chronometer time of transit is computed, allowing for the approximate error, and the middle vertical hair is kept upon the star, the telescope being adjusted in azimuth by the slow motion screws until the chronometer indicates that the star is on the meridian. A second approximation is made by again observing a star transit near the zenith, followed by an adjustment on a circumpolar. The line of sight is now very nearly in the plane of the meridian, and the amount of the residual error is discovered and allowed for in the reduction of the time observations.

The chronometer error on sidereal time cannot be obtained by direct comparison of the right ascensions of the stars with their observed times of transit, because it is impracticable to secure that the instrumental line of sight lies exactly in the plane of the meridian. The observed times are subject to three principal corrections : (1) the azimuth correction, due to the line of sight not being oriented precisely in the plane of the meridian ; (2) the level correction, due to dislevelment of the horizontal axis : (3) the collimation correction, due to the line of sight not being perpendicular to the horizontal axis.

The amounts of these corrections to the observed time of transit of a star of zenith distance z and declination δ are as follows :

$$\textit{Azimuth correction} = a \sin z \sec \delta = aA$$

where a is the error of azimuth in seconds of time. a is considered positive (negative) if the line of sight is too far east (west) when the telescope is pointed south. When the telescope is pointed north, the line ot sight is changed to the other side of the meridian.

$$\textit{Level correction} = b \cos z \sec \delta = bB$$

in which b is the inclination of the horizontal axis in seconds of time. The value of b is given not only by the readings of the striding level in the direct and reversed positions, but also includes the inclination caused by inequality of the trunnion pivots, if appreciable. b is positive (negative) when the west (east) end of the axis is the higher.

$$\textit{Collimation correction} = c \sec \delta = cC$$

where c is the error of collimation in seconds of time, regarded as positive (negative) when the line of sight is to the east (west) of the meridian.

In these corrections the quantities a, b and c represent instrumental errors, while the factors A, B and C depend upon the position of the star, and their values are given in tables of star factors. The signs of A, B and C for the two hemispheres are *

Northern Hemisphere

Position of star	A	B	C
South of zenith	+	+	+
Between zenith and pole	−	+	+
Below pole	+	−	−

Southern Hemisphere

Position of star	A	B	C
North of zenith		+	+
Between zenith and pole	+	+	+
Below pole	−	−	−

The correction bB can be computed for each observation from the readings of the striding level and the pivot inequality constant, and is applied at the outset to the observed time of transit. In the most precise work small corrections are also made for diurnal aberration and chronometer rate. The former arises from the effect of the diurnal rotation of the earth on the relative velocity of light, and the correction to the observed time of transit is given by $0^s{\cdot}02 \cos \phi \sec \delta$, where ϕ is the latitude of the observer, and δ the declination of the star. This correction is negative for all stars observed at upper transit and positive for stars observed at lower transit and is tabulated for different values of ϕ and δ in Table VII

* These signs may be obtained direct from the formulæ by considering z to be positive for stars observed south ot the zenith and negative for stars observed north of it, and by writing $180 - \delta$ instead of δ for stars observed below the pole.

of *Apparent Places of Fundamental Stars.* The rate correction is required when the set of observations before reversal of the telescope extends over a considerably longer or shorter period than that after reversal. The chronometer time of transit so corrected is now in error only by the effects of azimuth and collimation error, and these are determined from the time observations themselves.

Let T = the chronometer reading, corrected for level, aberration and rate

e = the required chronometer error, positive if slow, negative if fast.

The finally corrected chronometer time of transit $= T + aA + cC$, = the chronometer reading which would have been obtained if the line of sight were actually in the meridian = R.A. − e.

$$\therefore\ e = \text{R.A.} - T - aA - cC$$

For each transit we have an observation equation with three unknowns, e, a and c. The number of observations involved in a determination exceeds the number of unknowns, the most probable value of which are therefore most accurately derived by the method of least squares. An approximate solution is obtained with less labour by solving simultaneous equations equal in number to the unknowns, the equations being formed by addition of groups of the observation equations. Examples of this computation will be found in Hosmer's *Geodesy*, page 131, 1930 edition.

The stars forming a set are selected with a view to a good solution of the equations. There may be included in each set one or two slow moving or azimuth stars, from the observation of which a is principally derived. Otherwise all stars are time stars, and contribute equally to the result. The time stars of a set should be situated north and south of the zenith, and their declinations should have such values that the algebraic sum of their A factors is as nearly zero as possible, while their right ascensions should be such that the transits occur at convenient intervals.

The proofs of the expressions for the corrections for azimuth, level and collimation given above are as follows:

(1) *Azimuth Correction.* In Fig. 38 (a) let ZPN be the true meridian. Z the zenith

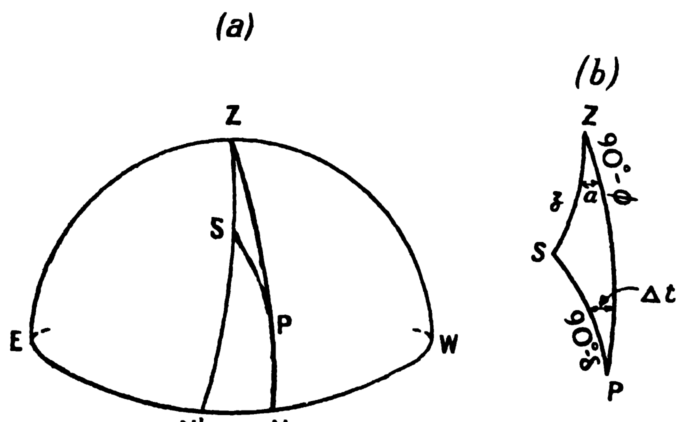


FIG. 38.

and P the celestial pole. Owing to the line of sight not being oriented in the plane of the meridian it will trace out a false meridian, ZSN′, making angle a = SZP with the true meridian ZPN, S being the position of the star when on the cross hair. The required error in time, Δt, will be the angle ZPS.

In the spherical triangle ZSP (Fig. 38 (b)),

$$ZS = z,\ SP = 90^\circ - \delta,\ ZP = 90^\circ - \phi,\ SZP = a,\ ZPS = \Delta t.$$

Hence, by the ordinary sine rule,

$$\frac{\sin \Delta t}{\sin z} = \frac{\sin a}{\cos \delta},$$

or, as Δt and a are small, we can write $\sin \Delta t = \Delta t$ and $\sin a = a$, and so

$$\Delta t = a \sin z \sec \delta$$

Note that, as Fig. 38 is drawn, a is negative because, when the telescope is pointed south, the false meridian and line of sight would lie to the west of the true meridian. In the case shown, the correction is positive since the star would cross the false meridian before it crossed the true meridian. This agrees with the rule of signs given on page 86, the A factor here being negative.

(ii) *Level Correction.* The effect of the level error will be to displace the apparent position Z of the true zenith along the great circle EZZ′W perpendicular to the true meridian ZPN to the position shown at Z′ in Fig. 39. The apparent meridian will therefore be the great circle Z′N, and, when the star is at S on this false meridian, the correction to time will be the angle SPZ = Δt and the observed zenith distance will be Z′S = z. The arc ZZ′ = angle Z′NZ = b, expressed in angular measure, is the level correction b.

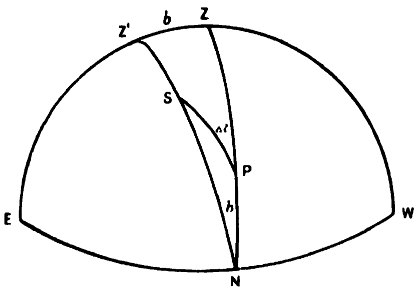


Fig. 39

In the spherical triangle SPN, PS = $90^\circ - \delta$, SN = $90^\circ - z$, SPN = 180° − ZPS = $180^\circ - \Delta t$, and SNP = b. Then

$$\frac{\sin \Delta t}{\cos z} = \frac{\sin b}{\cos \delta}$$

$$\therefore\ \sin \Delta t = \sin b \cos z \sec \delta,$$

or, as Δt and b are small angles,

$$\Delta t = b \cos z \sec \delta.$$

Here the star reaches the false meridian before it reaches the true meridian and hence the correction is positive, b also being positive since the horizontal axis of the instrument is high at its western end.

(iii) *Collimation Correction.* In Fig. 40 the great circle ZPN is the true meridian as before, Z being the zenith and P the celestial pole. As the telescope of the instrument is rotated about the trunnion axis, the line of collimation, instead of tracing out a vertical plane which intersects the celestial sphere in the meridian ZPN, will trace out the surface of a cone with apex O, and this cone will trace out the small circle ASB on the celestial sphere, where the plane of ASB is parallel to the meridian plane. The collimation error will be represented by the angle BON = c on the horizon plane.

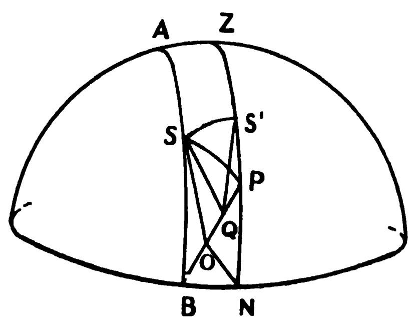


Fig. 40.

When the star is observed, it will be at the point S on the small circle ASB, and, as it moves from S to the point S′ on the true meridian, it will describe a small circle SS′ such that PS′ = PS. As the plane ASB is very close to the plane ZPN and

angle SS′P is a right angle, the length of the arc SS′ will be approximately equal to that of the arc BN, which is equal to $R \times \angle BON = R \times e$, where R is the radius of the sphere.

Let the plane containing the small circle SS′ intersect OP at Q. Then SQP = S′QP is a right angle and the angle SQS′ = SPS′ is the required correction Δt to the hour angle. But SQS′ = SS′/SQ and SQ = $R \sin SOP = R \sin (90° - \delta) = R \cos \delta$. Hence,

$$\Delta t = SQS' = \frac{eR}{R \cos \delta} = e \sec \delta.$$

Minor Determinations by Star Transits. Refinement in the reductions is not warranted when the observations are made with a small theodolite. The setting of the instrument in the plane of the meridian may be performed by zenith and azimuth stars as described above, unless the direction of the meridian is known with sufficient precision. Residual errors of adjustment are approximately eliminated by making the time observations on a pair of stars of about the same altitude, one north and the other south of the zenith.

Time by Ex-meridian Altitudes of Stars. The method of determining time by ex-meridian altitudes of a star or stars or of the sun is that most commonly used by surveyors. In its simplest form, the star observation consists in measuring the altitude of a known star at some distance from the meridian and noting the chronometer time of the measurement. If the observer's latitude is known, the astronomical triangle PZS (Fig. 9) can be solved for the hour angle, from which is derived the local sidereal time corresponding to the chronometer reading. A close approximation to the value of the latitude is required.

Instead of relying upon a single altitude of a star, several altitudes are observed on alternative faces in quick succession, the chronometer time of each being noted. If the observations are completed within a very few minutes it will suffice for most ordinary work if the mean of the chronometer times is taken as the time for the mean altitude. The motion of a star in altitude is not, however, exactly proportional to time. Accordingly, for precise work, if the star is not observed very close to the prime vertical, the mean of the observed altitudes may involve a correction to give the altitude corresponding to the mean of the observed times. The corrected altitude, h, to be used in computing the hour angle at the instant corresponding to the mean of the observed times, is then given by:

$$h = h_m + \frac{\Sigma(\Delta h)^2}{2N} \cot A \,[\tan h_m \cot A - \tan \phi \operatorname{cosec} A] \sin 1''$$

where h_m is the mean of the observed altitudes, A is the azimuth angle of the star measured the shortest way from the elevated pole, ϕ the latitude of the place, N the number of observations and $\Sigma(\Delta h)^2$ is the sum of the squares of the differences in seconds of arc between the observed altitudes and the mean altitude. For computing purposes, the formula may be put into the form

$$h = h_m + \frac{\Sigma(\Delta h)^2}{2N} \cdot \frac{\cot A \cos (\phi + \alpha)}{\sin \alpha \sin A \cos \phi} \sin 1''$$

where $\tan \alpha = \sec A \cot h_m$.

The azimuth angle A, if not already worked out in connection with the preparation of a star programme, may be obtained from :—

$$\tan \frac{A}{2} = \sqrt{\sec s \,.\, \sin (s - h) \,.\, \sin (s - \phi) \,.\, \sec (s - p)}$$

where p is the polar distance $90° - \delta$, measured from the elevated pole, $s = \frac{1}{2}(h + \phi + p)$ and ϕ is always treated as positive. As the term $\Sigma(\Delta h)^2$ is small, only an approximate value for A is required, and the computation of it, and of the trigonometrical co-efficient of $\Sigma(\Delta h)^2$, may be carried out by five, or, if $\Sigma(\Delta h)^2$ itself is small, by four-figure logarithm tables.

When $h = 45°$, the formula for the correction to the mean altitude reduces to the one given on page 70 in connection with the 45° prismatic astrolabe, and, when 60° is substituted for h, it becomes applicable to the 60° prismatic astrolabe when the latter is fitted with deflection prisms or other means of taking observations at altitudes differing by small amounts from the standard altitude. As might be expected from other considerations, it will be seen that the correction vanishes when the star is on the prime vertical.

The following proof of this expression may be taken as an example of the lines followed by proofs of other similar formulæ for which no proof will be given, the results only being stated :—

Let t = hour angle at the instant corresponding to the mean of the observed times, and let the hour angles at each of the observed times be $t + dt_1, t + dt_2, t + dt_3, \ldots t + dt_n$. Let h be the altitude corresponding to t and let the observed altitudes be $h_1, h_2, h_3, \ldots h_n$, so that $h_1 = h + dh_1, h_2 = h + dh_2, \ldots h_r = h + dh_r$. Then h is a function of t defined by :—

$$h = f(t) = \sin^{-1}\left\{\sin \phi \sin \delta + \cos \phi \cos \delta \cos t\right\}$$

By Taylor's theorem (page 7),

$$h + dh_r = f(t + dt_r) = f(t) + dt_r \frac{df}{dt} + \frac{dt^2_r}{2} \frac{d^2f}{dt^2} + \ldots$$

$$\therefore\ h_1 = h + dh_1 = h + dt_1 \frac{df}{dt} + \frac{dt^2_1}{2} \frac{d^2f}{dt^2} + \ldots$$

$$h_2 = h + dh_2 = h + dt_2 \frac{df}{dt} + \frac{dt^2_2}{2} \frac{d^2f}{dt^2} + \ldots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_n = h + dh_n = h + dt_n \frac{df}{dt} + \frac{dt^2_n}{2} \frac{d^2f}{dt^2} + \ldots$$

$$\therefore\ h = \frac{\Sigma h}{N} - \frac{\Sigma dt}{N} \frac{df}{dt} - \frac{\Sigma dt^2}{2N} \frac{d^2f}{dt^2}$$

$$= \frac{\Sigma h}{N} - \frac{\Sigma dt}{N} \frac{dh}{dt} - \frac{\Sigma dt^2}{2N} \frac{d^2h}{dt^2}$$

Since $\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$

$\cos h \, dh = - \cos \phi \cos \delta \sin t \, dt.$

$$\therefore\ \frac{df}{dt} = \frac{dh}{dt} = - \frac{\cos \phi \cos \delta \sin t}{\cos h} = - \cos \phi \sin A.$$

$$\frac{d^2f}{dt^2} = \frac{d^2h}{dt^2} = - \frac{\cos \phi \cos \delta \cos t}{\cos h} - \frac{\sin h \cos \phi \cos \delta \sin t}{\cos^2 h} \frac{dh}{dt}$$

$$= - \frac{\cos \phi \cos \delta}{\cos h} \left[\cos t - \frac{\sin h \cos \phi \cos \delta \sin^2 t}{\cos^2 h}\right]$$

$$= - \frac{\cos \phi \cos \delta}{\cos h} \left[\frac{\sin h - \sin \phi \sin \delta}{\cos \phi \cos \delta} - \frac{\sin h \cos^2 \phi \sin^2 A}{\cos \phi \cos \delta}\right]$$

$$= -\frac{1}{\cos h}[\sin h - \sin\phi(\sin\phi\sin h + \cos\phi\cos h\cos A) - \sin h\cos^2\phi\sin^2 A]$$

$$= -\frac{1}{\cos h}[\sin h\cos^2\phi - \sin\phi\cos\phi\cos h\cos A - \sin h\cos^2\phi\sin^2 A]$$

$$= -\frac{1}{\cos h}[\sin h\cos^2\phi\cos^2 A - \sin\phi\cos\phi\cos h\cos A]$$

$$= -\cos\phi\cos A\,[\tan h\cos\phi\cos A - \sin\phi]$$

Hence,

$$h = \frac{\Sigma h}{N} + \frac{\Sigma dt}{N}\cos\phi\sin A + \frac{\Sigma dt^2}{2N}\cos\phi\cos A\,[\tan h\cos\phi\cos A - \sin\phi]$$

Since the dt's are measured from the mean, $\frac{\Sigma dt}{N} = 0$, so that the second term on the right vanishes.
Hence,

$$h = \frac{\Sigma h}{N} + \frac{\Sigma dt^2}{2N}\cos\phi\cos A\,[\tan h\cos\phi\cos A - \sin\phi]$$

But $\quad dt_r = -\dfrac{dh_r}{\cos\phi\sin A}$ and mean of the h's $= h_m = \dfrac{\Sigma h}{N}$.

$$\therefore \quad h = h_m + \frac{\Sigma dh^2}{2N}\cot A\,[\tan h\cot A - \tan\phi\operatorname{cosec} A]$$

Here the dh's are measured from h, and not from h_m. However, $h - h_m$ is a term of the second order in dh so that $dh - dh_m$ will be a term of the second order and dh^2_m will differ from dh^2 by a term of the fourth order. Hence, we can take Σdh^2 as the sum of the squares of the differences between the observed altitudes and their mean.

An alternative to working out a correction to the mean height would be to apply one to the hour angle, computed from the mean height, to give the corresponding mean hour angle for the different times and altitudes of observation. If t = hour angle computed for mean h, and t_m = arithmetical mean of the different hour angles corresponding to the different times, then :—

$$t_m = t - \frac{\Sigma(\Delta h)^2}{2N}\frac{\cot A}{\cos\phi\sin A}[\tan h\cot A - \tan\phi\operatorname{cosec} A]\sin 1''.$$

Thus, in this case, the correction contains the additional factor $\sec\phi\,.\,\operatorname{cosec} A$ and hence is slightly more laborious to compute than the previous correction.

The star selected for a time observation by ex-meridian altitudes should be one which is changing rapidly in altitude, *i.e.* it should be near the prime vertical. The influence of errors in observed altitude, as well as in the value of the latitude, is a minimum when the star is actually on the prime vertical (page 80). For the avoidance of excessive refraction, the altitude of the star should be at least 20°, and practically it is generally necessary to observe stars at some distance from the prime vertical. For the more effectual reduction of error in the assumed value of the refraction coefficient, as well as of instrumental errors, the observation of an east star should be balanced by an observation of a west star of similar altitude.

If an error of $\Delta h''$ is made in the observation of h, the resulting error in the computed hour angle, reckoned the shortest way, east or west, from

upper transit, and expressed in seconds of time, is given approximately by :—

$$\Delta t^s = -\frac{\Delta h''}{15} \sec \phi \operatorname{cosec} A$$

where A is the azimuth angle measured the shortest way from the elevated pole. Again, simple differentiation of t with respect to ϕ in the fundamental formula, keeping h fixed, will give the error in the computed hour angle resulting from an error $\Delta\phi$ in the assumed latitude of the place. The result, after a little simplification, is :—

$$\Delta t^s = \frac{\Delta \phi''}{15} \sec \phi \cot A.$$

Observation of Ex-meridian Altitudes of Stars. Stars suitable for observation are selected in advance. When a star is on the prime vertical its altitude is given by

$$\sin h = \frac{\sin \delta}{\sin \phi}.$$

This relationship shows that the declination of the star must be of the same sign as the latitude, and must be less than the latitude. In order, however, that h may not be less than 20°, δ should be greater than $\sin^{-1} \sin \phi \sin 20°$. As the star may be observed out of the prime vertical, the foregoing serves to define only roughly the requirements as to declination. The selection must be further based upon the availability of stars for observation at convenient times.

The observation of each star of a pair consists of a series of rapid altitude measurements, each consisting of a face right and a face left observation. The details of each measurement are performed according to the routine described on page 77. The chronometer times are taken to the nearest $0^s{\cdot}2$, or to $0^s{\cdot}1$ if possible.

Computation. The astronomical triangle is conveniently solved for the hour angle t by means of

$$\tan \frac{t}{2} = \sqrt{\cos s \sin (s - h) \operatorname{cosec} (s - \phi) \sec (s - p)}$$

$$\text{where } s = \frac{h + \phi + p}{2}$$

In applying the formula, ϕ is always treated as positive, and p is measured from the elevated pole. The computed value of t is measured the shorter way from the meridian. It is added to or subtracted from the star's right ascension, according as the star is west or east of the meridian, to give the sidereal time corresponding to the mean of the chronometer readings. If the chronometer is a mean time one, the computed sidereal time must be converted to mean time (page 44). The same calculation is performed for the other star of the pair, and the mean of the computed chronometer errors is accepted as the error at the average time of the observations. The observation of two or three pairs of stars with a 5-in. micrometer theodolite should ensure a result well within one second of the truth. The precision of the method is greater in low than in high latitudes.

Example On 1935 March 3, at a place in latitude 15° 12′ 36″ N. and longitude 25° 03′ 20″ E., an observation of γ Orionis (R.A.= $5^h\ 21^m\ 40^s{\cdot}3$, $\delta = +6°\ 17'\ 33''{\cdot}5$) west of the meridian gave a mean corrected altitude of 35° 22′ 10″·5 Find the L.S.T. of the observation

	°	′	″		
h =	35	22	10·5		
ϕ =	15	12	36		
p =	83	42	26·5		
Sum = $2s$ =	134	17	13·0		
s =	67	08	36·5	log cos =	9·5893071
$s - h$ =	31	46	26·0	log sin =	9·7214548
$s - \phi$ =	51	56	00·5	log cosec =	0·1038623
$s - p$ =	−16	33	50·0	log sec =	0·0184068
				Sum =	9·4330310

$$\log \tan \frac{t}{2} = 9{\cdot}7165155$$

$$\frac{t}{2} = 27°\ 30'\ 07''{\cdot}6$$

	h	m	s
t = 55° 00′ 15″·2 =	3	40	01·0
R.A. of star =	5	21	40·3
L.S.T. =	9	01	41·3

Time by Ex-meridian Altitudes of the Sun. In applying the method of ex-meridian altitudes to solar observations, the required balancing is effected by measuring a succession of altitudes both in the morning and afternoon. The requirements as to the position of the sun during observation are the same as for a star, and the most suitable times are between 8 and 9 a.m. and again between 3 and 4 p.m.

The minimum number of observations in each set is four, consisting of a face right and a face left measurement to both the upper and lower limbs, but it is preferable to double this number of observations. The hour angle is computed, as in the case of star altitudes, from

$$\tan \frac{t}{2} = \sqrt{\cos s \sin (s - h) \operatorname{cosec} (s - \phi) \sec (s - p)}$$

but it will be necessary to obtain p by interpolating the value of the sun's declination for the mean instants of the morning and afternoon observations respectively. The interpolation requires a knowledge not only of the longitude of the place of observation, but also of the local time, and, since the latter is being determined, the evaluation of t should really be performed by successive approximations. If, however, local time is known within about two minutes, the value of the sun's declination can be interpolated with sufficient precision for the purpose. On reducing the observations, if a greater discrepancy is discovered between the computed and assumed local times, the former is used for a better interpolation of δ, and the computation of t is repeated with the new value.

The local apparent time corresponding to the mean of the watch times of the forenoon observations is obtained by subtracting the hour angle from 12^h. For the afternoon set, 12^h is added to the hour angle. The conversion to mean time is made by the methods of Chapter I. The mean of the two watch errors thus determined is the error at the average instant of the two sets.

If the utmost possible accuracy is desired, a correction to the mean altitude, similar to that already given on page 89, may be applied if

necessary to each set of F.R. and F.L. observations, but here the Δh's must be corrected for the semi-diameter of the sun.

Example. On 1935 May 10, at a place in latitude 36° 55′ 08″ N. and longitude 57° 53′ 30″ E., the mean observed altitude of the sun, corrected for refraction, parallax and level, was 39° 54′ 13″·8. The mean watch time of the observations was $15^h\ 31^m\ 04^s{\cdot}2$, the watch being known to be about 4^m fast on L.M.T. Find the watch error on L.M.T.

Value of sun's δ at mean instant of observation.

		d	h	m	s
Approx. L.M.T. =	May	10	15	27	04
Longitude =			− 3	51	34
Approx. G.M.T. =	May	10	11	35	30

		°	′	″
From *N.A.*, p. 10, sun's δ at May $10^d\ 0^h$ =	+	17	19	00·2
Change in $11^h\ 35^m\ 30^s$ =	+		7	42·8
Required δ =	+	17	26	43·0
p =		72	33	17·0

	°	′	″		
h =	39	54	13·8		
ϕ =	36	55	08·0		
p =	72	33	17·0		
Sum = $2s$ =	149	22	38·8		
s =	74	41	19·4	log cos =	9·4217074
$s - h$ =	34	47	05·6	log sin =	9·7562534
$s - \phi$ =	37	46	11·4	log cosec =	0·2129003
$s - p$ =	2	08	02·4	log sec =	0·0003013
				Sum =	9·3911624

$$\log \tan \frac{t}{2} = 9{\cdot}6955812$$

$$\frac{t}{2} = 26° 23' 11''{\cdot}9. \quad t = 52° 46' 23''{\cdot}8 = 3^h\ 31^m\ 05^s{\cdot}6.$$

Conversion to L.M.T.

	h	m	s
L.A.T. =	15	31	05·6
Longitude =	−3	51	34
G.A.T. =	11	39	31·6
From *N.A.*, p. 24, G.M.T. of G.A.T. 12^h =	11	56	20·4
Correction = − $0^d{\cdot}014$ × − $2^s{\cdot}8$ =			+0·0
L.A.T. =	15	31	05·6
Sum − 12^h = L.M.T. =	15	27	26·0
Watch time =	15	31	04·2
Watch error on L.M.T. =		3	38·2 fast.

Time by Equal Altitudes of a Star. The mean of the two chronometer times at which a star attains equal altitudes east and west of the meridian represents the chronometer time of its transit. A simple method is thus afforded of obtaining the chronometer reading corresponding to the sidereal time of transit, or right ascension of the star. The determination is independent of the actual value of the altitude, and no correction is required for refraction, but the precision of the result depends upon the refraction having the same value for both observations. The observations should be made when the altitude is changing rapidly, *i.e.* when the star is near the prime vertical, and the method is open to the objection that several hours may elapse between the first and last observations unless the declination of the selected star is nearly equal to the latitude.

The observations are made by means of a theodolite or a sextant and artificial horizon. In place of observing the star only once on either

side of the meridian, a set of four or five altitudes should be taken to reduce errors of observation. Since the essential features of the observation is the equality of the altitudes on either side of the meridian, the theodolite telescope is not reversed, and the bubble of the altitude level on the micrometer arm must be exactly centered either by the level or clip screws at each observation, the angle between the telescope and the level remaining unaltered. For many purposes, where high accuracy is not required, sextant observations will give satisfactory results since the use of the artificial horizon avoids possible errors of level, while the settings are in round figures and no correction for index error is required.

Instead of taking the observations on the same night, they may be made in the morning and evening. If the western altitudes are observed first, the average of the chronometer times represents the chronometer reading at lower transit, the sidereal time of which is 12^h greater than the star's right ascension.

Time by Equal Altitudes of Two Stars. The effect of a possible change in refraction between the two sets of observations at equal altitudes of the same star is greatly reduced, and the inconvenience of waiting is avoided, by making the equal altitude observations on two stars, one east and the other west of the meridian. If it were possible to select two stars having the same declination, the mean of their right ascensions would represent an instant of sidereal time with which the mean of the chronometer readings could be compared for the determination of the chronometer error. Actually the two bodies have different declinations, and, in consequence, a correction must be applied to the mean of their right ascensions.

The observation of a pair of stars may be completed in a few minutes, and several pairs should be used for a good determination. Stars selected to form a pair should have a difference in right ascension of at least 6^h, and, on account of the approximations made in computing the correction for difference of declination, the latter should be kept within from 2° to 5° according to the required refinement of the determination. The stars should also preferably be such as are near the prime vertical at the time they reach the same altitude. Since the sidereal time at which the two stars are simultaneously at the same altitude is approximately the mean of the two right ascensions, the observer can select pairs of stars which will afford convenient intervals between the observation of one pair and that of the next.

If the east star is first observed, the line of sight is set just above it at a few minutes before the time at which the stars are at the same altitude. The chronometer reading when the star crosses the horizontal hair is noted, and the telescope is then turned to the west star. The latter should appear a little above the horizontal hair, and the chronometer time of its passage is likewise observed. If the stars are not selected in advance, a pair may be found by trial of their altitudes.

The correction for difference in declination of the two stars is computed as follows, the latitude being known :—

Let c = correction in time to be applied to the average of the two right ascensions to give the sidereal time corresponding to the average of the chronometer readings T_E and T_W

$t = \frac{1}{2}(\text{R.A.}_E - \text{R.A.}_W) - \frac{1}{2}(T_E - T_W)$

ϕ = observer's latitude, positive or negative according as it is north or south

δ_E and δ_W = declinations of the east and west stars

$\delta = \frac{1}{2}(\delta_E + \delta_W)$

Then $c = \dfrac{\delta_E - \delta_W}{2 \times 15}(\tan\delta\cot t - \tan\phi\,\text{cosec}\,t)$

The proof of this formula is as follows :—

Since h and ϕ are the same for both observations, keep them both fixed when differentiating t with respect to δ in

$$\sin h = \sin\delta\sin\phi + \cos\delta\cos\phi\cos t.$$

Whence :—

$$(\cos\delta\sin\phi - \sin\delta\cos\phi\cos t)\,d\delta = \cos\delta\cos\phi\sin t\,dt.$$

$$\therefore\ dt = (\tan\phi\,\text{cosec}\,t - \tan\delta\cot t)\,d\delta.$$

Then, if t_1 is the hour angle for declination $\delta = \frac{1}{2}(\delta_E + \delta_W)$, $t_1 + dt_1$ will be the hour angle for declination δ_E where

$$dt_1 = (\tan\phi\,\text{cosec}\,t - \tan\delta\cot t)\,(\delta_E - \delta).$$

Similarly, $t_1 + dt_2$ will be the hour angle for declination δ_W where

$$dt_2 = (\tan\phi\,\text{cosec}\,t - \tan\delta\cot t)\,(\delta - \delta_W).$$

Hence, mean observed hour angle =

$$t = t_1 + \tfrac{1}{2}(\tan\phi\,\text{cosec}\,t - \tan\delta\cot t)\,(\delta_E - \delta_W).$$

$$\therefore\ t_1 = t + \frac{1}{2 \times 15}(\tan\delta\cot t - \tan\phi\,\text{cosec}\,t)\,(\delta_E - \delta_W),$$

the 15 in the denominator being introduced to convert the result from arc to time.

It can be shown that the second order term in the expansion for t_1 is negligible for all cases where this particular method would be likely to be used.

Time by Equal Altitudes of the Sun. In applying the method of equal altitudes to solar observations, a series of altitudes is taken about 9 a.m., and the same series is repeated in reverse order about 3 p.m. Each altitude setting on the instrument yields two observations if the instants are noted at which both the upper and lower limbs touch the horizontal hair.

The mean of the times of the forenoon and afternoon equal altitudes does not in this case exactly represent the instant of transit owing to the change in the sun's declination between the observations. The average of the watch times of the observed equal altitudes must therefore be corrected to give the watch time of apparent noon, as follows.

Let c = correction in time to be applied to the average of the watch times of equal altitude to give the watch time of apparent noon

t = half the interval between the times of equal altitude

ϕ = observer's latitude, positive or negative according as it is north or south

δ_E = sun's declination at the average of the morning observations

δ_W = sun's declination at the average of the afternoon observations

$\delta = \frac{1}{2}(\delta_E + \delta_W)$

$$\text{Then } c = \frac{\delta_{I} - \delta_{II}}{2 \times 15} (\tan \delta \cot t - \tan \phi \operatorname{cosec} t)$$

The value of ϕ used in computing the correction need be only approximate. The method is available for afternoon observations followed by a series at the same altitudes on the following morning. The result then gives the chronometer error at apparent midnight. The value of c is then given by

$$c = \frac{\delta_{I} - \delta_{II}}{2 \times 15} (\tan \delta \cot t + \tan \phi \operatorname{cosec} t)$$

Example. On 1935 June 5 in latitude 56° 06′ 11″ N and longitude 4° 39′ 05″ W, equal altitude observations of the sun's lower limb for time gave the results tabulated. Find the error of the watch on standard time (British summer time), the approximate error being 2ᵐ slow.

Altitude Micro. T. F. L.			Watch time			Watch time		
°			h	m	s	h	m	s
49	04	26	11	11	07.0	15	21	31.0
49	23	10	11	14	10.5	15	18	26.5
49	35	32	11	16	13.5	15	16	20.0
49	46	01	11	18	00.0	15	14	35.0
Average watch times			11	14	52.8	15	17	43.1

		°	′	″
From N.A., p. 12, δ at 10ʰ 17ᵐ	$= \delta_{I}$	22	28	07
δ at 14ʰ 20ᵐ	$\delta_{II} =$	+22	29	16
	$\delta_{II} - \delta_{I} =$			1 09
	$\delta =$	+22	29	
	$t =$	2ʰ	01ᵐ	25ˢ

	h	m	s
Mean of watch times	13	16	18.0
c			5.1
Watch time of L.A.N.	13	16	12.9
From N.A., p. 25, G.M.T. of G.A.N.	11	58	10.7
Correction for Longitude 0.0138 × 10.3			0.1
Sum L.M.T. of L.A.N.	11		10.8
Longitude west		18	36.3
Correction for summer time	1		
Sum B.S.T. of L.A.N.	13	18	07.1
Watch time	13	16	12.9
Watch error		1	54.2 slow

Time by Transit of the Sun. If the direction of the meridian is known, a rough determination of time may be made by noting the watch times of transit of the west and east limbs of the sun, the mean of the observed times being the watch time of local apparent noon. If only one limb is observed, the time of transit of the sun's centre is obtained by application of the semi-diameter in sidereal time, which is given for every day of the year in the *Nautical Almanac*.

DETERMINATION OF AZIMUTH

The determination of azimuth, or the direction of the meridian at a survey station, consists in obtaining the azimuth or true bearing of any line from the station, so that the azimuths of all the survey lines meeting there may be derived. An essential feature of the observation is the

measurement of the horizontal angle between a celestial body and a terrestrial signal. In primary determinations the observed azimuth should always be that of a side of the triangulation. Otherwise it is frequently convenient to establish a reference mark, from the observed azimuth of which those of the survey lines are measured.

The quality of the angular work of the survey indicates the necessary precision of the observations. The requirements as to probable error range from about 0″·3, in the case of primary triangulation, to over 1′·0 for determinations made for the frequent control of compass traversing.

Reference Mark. On main triangulation, and on very long lines in primary traverse work, the signal for use as a night signal will usually consist of a special lamp, such as one of those described on pages 158 to 160, centered carefully over one of the main stations. For other classes of work a special signal, called the reference or azimuth mark or referring object (R.O.), is erected. If practicable, it should be placed at a distance of not less than a mile, not only to render negligible small errors of centering of the instrument, but principally to avoid the necessity for changing the focus of the telescope between the sights to the celestial body and to the mark. For the best results the mark should be nearly in the horizontal plane through the instrument, and should be raised at least four feet above the ground. The line of sight to it should be nowhere less than this distance from the ground, to minimise the possibility of lateral refraction.

For solar observations any form of opaque signal is used. In the case of stellar observations, the mark must consist of a lamp placed behind a screen or in a box with an open top and provided with air holes in the bottom. The light shines through a vertical slit or a circular aperture in the face, the effective width of the signal being such as to subtend at the instrument an angle of about 0″·5 to 1″ according to the grade of the determination. For the measurement in daylight of the azimuths of the survey lines from that of the reference mark, an opaque signal must be provided in the vertical of the aperture. This may be arranged by painting the lamp screen with a narrow vertical line centrally with the aperture.

On primary traverses and on minor triangulation schemes, when legs or sides do not exceed two or three miles in length, a target, of a type similar to the one illustrated in Vol. I, page 81, may be used both for the observation of angles and for azimuth observations. This target has a small aperture which can be illuminated by an electric bulb, fitted into an attachment at the back, and worked from a 4-volt battery or accumulator.

Methods. The principal methods of determining azimuth involve observations of:

(1) Close circumpolar stars.
(2) Circumpolar stars at elongation.
(3) Ex-meridian altitudes of stars or the sun.
(4) Hour angles of stars or the sun.
(5) Equal altitudes of stars or the sun.
(6) Circumpolar stars at culmination.

Azimuth by Observations on Close Circumpolar Stars. The most refined determinations are possible by this method, which embraces several

systems of observation, each suitable for primary measurements. The observation consists essentially of the application of a method of precise angle measurement (page 219), by direction or repetition, to the observation of the horizontal angle between the star and the terrestrial signal. Since the position of the star is continually changing, the chronometer times of the individual observations are required, and the chronometer error and rate must be known. From the corrected chronometer times the hour angle of the circumpolar star is obtained, and if the latitude of the place of observation is known, the astronomical triangle PZS can be solved for the azimuth angle A.

When the circumpolar star is at elongation, its motion in azimuth is zero. A refined determination cannot, however, be obtained by observing at the instant of elongation, since repeated measurements are required for the reduction of instrumental and observational errors. But a number of observations may be taken near elongation, preferably before and after. The nearer the star is to elongation, the less is the observation affected by small errors of time, but the influence of an error in the latitude is then a maximum. The latter effect is eliminated by observing the same star near both eastern and western elongation; or, for greater convenience, two stars differing in right ascension by about 12^h are selected, and one is observed near its eastern and the other near its western elongation.

In the measurement of the horizontal angle between the star and the signal, the routine as to changing face, swinging right and left, and number of zeros must be that followed throughout the survey (page 221). One complete measurement involves so many observations that a star may be at some distance from elongation during part of them. The routine of angle measurement should not be curtailed, and, when the time is accurately known, entirely satisfactory results may be obtained by observing a close circumpolar star at any hour angle.

Observation. Suitable stars may be selected from the list of close circumpolars of which the apparent positions are given in *Apparent Places of Fundamental Stars* for every day of the year. Of the northern stars, α *Ursæ Minoris* (*Polaris*) is used whenever possible, and can be observed by daylight with the theodolites used in primary triangulation, if the sun is not too high. The other stars are much fainter; of these 51H *Cephei*, δ, λ and 6B *Ursæ Minoris* are most frequently used. The southern close circumpolars are all of smaller magnitude than the fifth. The stars σ, ν, ζ and ρ *Octantis* are useful, but for determinations by small theodolites it may be necessary to observe β *Hydri* (mag. 2·9) or β *Chamæleontis* (mag. 4·4), which are much farther from the pole.

Apparent Places of Fundamental Stars gives the positions of 26 northern and 26 southern circumpolar stars at upper transits for each day of the year to a hundredth of a second of time in R.A. and a hundredth of a second of arc in declination. The new *Star Almanac for Land Surveyors* gives the positions of 3 northern circumpolar stars—α *Ursæ Minoris* (*Polaris*), 51H *Cephei* and δ *Ursæ Minoris*—and 2 southern circumpolar stars—ζ *Octantis* and δ *Octantis*—at 10-day intervals throughout the year to a second of time in R.A. and a second of arc in declination, and it also contains a special table for *Polaris* which may be used for latitude and

azimuth observations when *Polaris* is observed at any given date and time of day and results are only needed to a tenth of a minute of arc. The *Nautical Almanac* also contains tables for α *Ursæ Minoris* and δ *Octantis* for every day of the year and for other stars at the usual 10-day intervals, but these tables are being discontinued in the 1952 and subsequent volumes.

For observations in the northern hemisphere the times of elongation of *Polaris* are first computed (page 41). If daylight observations are impracticable, it will be ascertained whether it is possible to use *Polaris* during the hours of darkness. If not, it will be necessary to select stars which elongate at a convenient time and such that the right ascensions of paired stars differ by about 12^h. For daylight observations it is necessary to work out rough values of an azimuth and elevation of the star, or obtain these from special tables, and set the instrument accordingly.

When the direction method of angle measurement is used, the azimuth observations may be made in conjunction with the measurement of the horizontal angles of the survey. In each round of angles is included a pointing upon the star. The chronometer time of each bisection of the star is noted to $0^s{\cdot}1$, and the striding level is carefully read. If the instrument is fitted with an eyepiece micrometer, the bisections are made by means of the movable hair and not by the tangent screw. The small angle measured by the eyepiece micrometer lies in the plane containing the line of sight and the horizontal axis of the telescope. It must be reduced to the horizontal by multiplying it by the secant of the altitude, the approximate value of which should therefore be obtained.

If the repetition system is employed, the angle between the star and one of the survey signals is multiplied in the usual manner, notwithstanding that the angle is changing. Division by the number of repetitions gives the mean angle corresponding to the mean of the chronometer times of the observations.

A third method, capable of the highest degree of accuracy, consists in measuring by means of the eyepiece micrometer the small angle between the star and a signal placed nearly in the vertical of the star at elongation. The measurement is independent of readings of the horizontal circle, and a large number of face right and face left pointings may be secured in a short time. The altitude of the star should be observed at intervals for reduction to the horizontal of the angle given by the micrometer readings.

Reduction. The azimuth angle A between the elevated pole and the star at any hour angle t, measured the shortest way from upper transit, is given by

$$\tan A = \frac{\sin t}{\cos\phi\tan\delta - \sin\phi\cos t}$$

$$\text{or } \tan A = \sec\phi\cot\delta\sin t\left(\frac{1}{1-a}\right)$$

$$\text{where } a = \tan\phi\cot\delta\cos t.$$

The latter form is due to Albrecht, and in conjunction with his tables of $\log\left(\frac{1}{1-a}\right)$ is commonly used.

The value to be taken for the hour angle is that corresponding to the mean of the corrected chronometer times of the n observations forming a set. The resulting value of A is then corrected for the curvature of the apparent path of the star by the amount:

$$\text{Curvature correction for set} = \frac{\tan A \sin^2 \delta}{n} \Sigma \frac{2 \sin^2 \frac{1}{2}\Delta t}{\sin 1''},$$

where Δt represents the angular equivalent of the difference in sidereal seconds between the time of an individual observation and the mean of the set. The correction, which is only applicable when the star is at or near elongation, always reduces the numerical value of A whether reckoned eastwards or westwards from the meridian.

The factor $\dfrac{2 \sin^2 \frac{1}{2}\Delta t}{\sin 1''}$ in this formula will be found tabulated as Table XXV in Close and Winterbotham's *Text-book of Topographical Surveying*, or in the companion volume, *Field Astronomy*, both of which are published by H.M. Stationery Office.

If the horizontal axis is inclined during a pointing on the star or the signal, the horizontal circle reading falls to be corrected by

$$\text{Level correction} = \frac{d}{2n}(\Sigma W - \Sigma E) \tan h$$

where d = value of one division of striding level *
ΣW and ΣE = sum of west and east end readings, reckoned from centre, of bubble in direct and reversed positions
h = altitude.

For the northern hemisphere, if the east (west) end of the axis is the higher, the clockwise reading of the circle is too great (small).

In refined determinations, correction is required for the influence of the aberration of light caused by the rotation of the earth. The effect is to place the apparent position of the star east of its true position by the amount $0''{\cdot}32 \dfrac{\cos\phi \cos A}{\cos h}$, the value of which for close circumpolars is always practically $0''{\cdot}32$.

If the observed hour angle is in error by an amount Δt, expressed in seconds of arc, the error in the azimuth angle (or, if necessary, the correction to the azimuth angle due to a correction in the hour angle), also expressed in seconds of arc, will be obtained by differentiating A with respect to t, holding ϕ fixed, in the formula for $\tan A$ given on page 100. The result, after a little simplification, reduces to :—

$$\begin{aligned}\Delta A &= -(\sin\phi - \cos\phi \cos A \tan h)\Delta t \\ &= -\cos\delta \sec h \cos q\, \Delta t \\ &= -\sin A \cos q \operatorname{cosec} t\, \Delta t\end{aligned}$$

where q is the parallactic angle. From the second of these expressions it follows that, for a given error in time, the error in azimuth is least (1) when δ is nearly 90° in value, that is when the star is near the pole,

* For a description of a method of determining the value of d for a striding level see page 78.

and (2) when q is nearly 90° in value, or when the star is near elongation. From the first expression it will be seen that observations of east and west stars will not eliminate the error in azimuth due to the error in time unless the stars are so chosen that the relation

$$2 \tan \phi = \cos A_1 \tan h_1 + \cos A_2 \tan h_2$$

is very nearly satisfied.

If there is an error $\Delta\phi$ in the assumed latitude, the resulting error in azimuth is obtained by differentiating A with respect to ϕ in the formula given on page 100, keeping t fixed. This gives :—

$$\Delta A = \tan h \,.\, \sin A \,.\, \Delta\phi.$$

Hence, pairing of east and west stars will tend to eliminate or to reduce the error in azimuth due to an error in the assumed latitude of the place. Reduction of an error in azimuth due to an error in latitude demands a small value of h and a small value of h decreases the error due to an error in time.

Azimuth by Circumpolar Stars at Elongation. The routine followed in a primary determination by observations of circumpolar stars as just described may be greatly simplified for determinations of secondary accuracy. If *Polaris*, or other circumpolar, is observed within a few minutes on either side of elongation, the motion in azimuth is scarcely perceptible in the telescope of a moderate size theodolite, and the azimuth of the star may be computed as its azimuth at elongation.

The time of elongation is computed in advance (page 41), and the instrument is set up and carefully levelled at least fifteen minutes before the time of elongation. About five minutes before elongation a pointing is made upon the reference mark, and the micrometer readings are taken. The star is then bisected, and after the readings are booked the telescope is reversed, and a second observation is made on the star. Finally, the reference mark is again bisected. A second set may be obtained without having to observe the star at more than five minutes from elongation. When local time is not known, the star must be followed with the vertical hair until it appears stationary in azimuth, and the set is then completed. If the watch error has been carefully determined, several sets may be secured by extending the range of the observations to, say, 20^{m} on either side of elongation, and reducing the position of the star to that at elongation.

When the star is at elongation,

$$\sin A = \frac{\cos \delta}{\cos \phi}$$

where A is the angle measured the shorter way between the elevated pole and the star. An approximate value for the latitude is sufficient.

When the star is observed not actually at elongation but at a small interval not exceeding 30 minutes of time from it, the correction applicable to the value computed from the formula for elongation,

$$\sin A = \cos \delta \,.\, \sec \phi,$$

is given approximately in seconds of arc by

$$c'' = 1{\cdot}96 \tan A \sin^2\delta \,(t_E - t)^2 \,*$$

* This expression is derived from the term of the second order in the expansion of $A_E + \Delta A = f(t_E + \Delta t)$ by Taylor's theorem.

where $t_E - t$ is the sidereal interval in minutes between the time of observation and that of elongation.

With a 5-in. or 6-in. micrometer instrument the error of the mean result from a single star should not exceed 5″. With a large geodetic type theodolite the corresponding error should not exceed 1″.

Example. On 1941 July 6 at a place in latitude 55° 50′ 17″ N., an azimuth observation was made on *Polaris* at eastern elongation, with the results tabulated. The referring object was east of the star. Find its azimuth.

Object	Face	Horizontal Circle Readings					Mean			Angles between R.O. and Star		
		Micro. A			Micro. B							
		°	′	″	′	″	°	′	″	°	′	″
R.O.	L.	246	15	20	15	35	246	15	28	110	41	13
Star	L.	135	34	05	34	25	135	34	15			
Star	R.	315	33	35	33	40	315	33	38	110	41	27
R.O.	R.	66	14	55	15	15	66	15	05			
									Mean	110	41	20

$$\sin A = \frac{\cos \delta}{\cos \phi}$$

From *N.A.*, p. 291, $\delta = +88° 58' 47''\cdot 2$

$\log \cos \delta = 8\cdot 2505492$
$\log \cos \phi = 9\cdot 7493760$
$\log \sin A = 8\cdot 5011732$
$A = 1° 49' 01''$

The star is at eastern elongation,
∴ R.O. is 112° 30′ 21″ east of north,
or azimuth clockwise from north = 112° 30′ 21″

Azimuth by Ex-meridian Altitudes of Stars. This method is very generally used for determinations of other than primary standard. The observation has much in common with that for determining time by ex-meridian altitudes, and the two determinations may be combined if the watch times of the altitudes are noted. A good knowledge of the observer's latitude should be available, and the astronomical triangle is solved for the azimuth angle.

The star should be observed when changing rapidly in altitude and slowly in azimuth. A favourable situation therefore occurs when the star is on the prime vertical, the influence of errors of observed altitude then being small. As in the corresponding observation for time, an east star should be paired with a west star of similar altitude, and suitable stars should be selected in advance.

The routine of observation for either star of a pair consists in first bisecting the reference mark, the readings of the horizontal circle being taken on both micrometers. The telescope is then directed towards the star. Since the star has a slow motion in azimuth and a rapid one in altitude, the telescope should be so pointed that the horizontal hair is in advance of the star. By means of the upper tangent screw the vertical hair is kept upon the star, and the motion in azimuth is stopped when the star reaches the intersection of the hairs. The readings of both micrometers on each circle are noted as well as that of the altitude level. The

telescope is transited, the star again bisected, and the new horizontal angle to the reference mark obtained as before. A second set should be observed in the same manner from a new zero.

The azimuth angle A is conveniently obtained from

$$\tan\frac{A}{2} = \sqrt{\sec s \sin(s - h)\sin(s - \phi)\sec(s - p)}$$

$$\text{where } s = \frac{h + \phi + p}{2}.$$

ϕ is always treated as positive, and p is measured from the elevated pole. The resulting value of A is the horizontal angle between the elevated pole and the average position of the star during the observation, and is measured the shorter way, east or west. This quantity is applied to the mean horizontal angle between the star and the reference mark to give the angle between the latter and the elevated pole. The possibility of confusion as to signs is avoided if the notes include a diagram showing the relative position of the pole, the star, and the reference mark. The calculation of the azimuth of the reference mark is repeated from the other star of the pair, and the mean is accepted. Determinations made in connection with deliberate mapping should be based upon two or three pairs of stars, when the error of the mean is not likely to exceed 5″.

Owing to the curvature of the path of the star, the mean of the two azimuths, when a star is observed face right and face left, is not exactly equal to the azimuth corresponding to the mean altitude. If the mean altitude is taken to compute the azimuth, the correction to be applied to the computed azimuth angle, measured east or west from the elevated pole, is given in seconds of arc by :—

$$\Delta A = \frac{1}{8}\cot q \sec^2 h\,(\sin h - 2\cot A \operatorname{cosec} 2q)\,(\Delta h)^2 \sin 1''$$

where q is the parallactic angle and Δh is the difference in elevation, in seconds of arc, between the two observations. Normally, if the two observations are taken quickly, Δh will be small, and the correction, for all ordinary work, may be, and usually is, neglected.

This formula may be derived by using Taylor's Theorem to expand $A + dA$ in terms of dh in the expression :—

$$A + dA = \cos^{-1}\frac{\sin\delta - \sin\phi\sin(h + dh)}{\cos\phi\cos(h + dh)}.$$

The trigonometrical co-efficient in the formula is then $\dfrac{d^2A}{dh^2}$. The term in dh in the expansion cancels out in the mean.

The parallactic angle, used in the above formula, may be obtained from :—

$$\sin q = \cos\phi \sin A \sec\delta.$$

In computing the correction it is only necessary to use four- or five-figure logarithms, with values for the various angles to the nearest minute or so.

Example. On 1935 June 8 in latitude 56° 06′ 11″ N., ex-meridian altitudes were observed on a pair of stars with the results tabulated. A compass bearing showed the referring object to be about 9° west of north. Find its azimuth.

Star	Mean horizontal angle from R.O.			Corrected mean altitude		
	°	′	″	°	′	″
η *Ursæ Majoris* (west)	80	52	01	66	54	33
δ *Cygni* (east)	102	06	45	60	02	54

η *Ursæ Majoris.* $\delta = +49° \ 38' \ 10''$

		°	′	″			
h	=	66	54	33			
ϕ	=	56	06	11			
p	=	40	21	50			
Sum = $2s$	=	163	22	34			
s	=	81	41	17	log sec	=	0·8399444
$s - h$	=	14	46	44	log sin	=	9·4066926
$s - \phi$	=	25	35	06	log sin	=	9·6353326
$s - p$	=	41	19	27	log sec	=	0·1243683
					Sum	=	0·0063379
					log tan $\frac{A}{2}$	=	0·0031689

		°	′	″	
A	=	90	25	05	west of north
Angle between R.O. and star	=	80	52	01	
R.O. west of north		9	33	04	

δ *Cygni.* $\delta = +44° \ 58' \ 10''$

		°	′	″			
h	=	60	02	54			
ϕ	=	56	06	11			
p	=	45	01	50			
Sum = $2s$	=	161	10	55			
s	=	80	35	28	log sec	=	0·7865380
$s - h$	=	20	32	34	log sin	=	9·5451916
$s - \phi$	=	24	29	17	log sin	=	9·6175282
$s - p$	=	35	33	38	log sec	=	0·0896417
					Sum	=	0·0388995
					log tan $\frac{A}{2}$	=	0·0194497

		°	′	″	
A	=	92	33	54	east of north
Angle between R.O. and star	=	102	06	45	
R.O. west of north		9	32	51	
Mean result R.O. west of north	=	9	32	57·5	

The errors in azimuth due to errors Δh and $\Delta\phi$ in the observed altitude or in the assumed latitude of the place may be found by differentiating A with respect to h, keeping ϕ fixed, or with respect to ϕ, keeping h fixed, in the expression :—

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos A.$$

The results are :—

$$\begin{aligned} \Delta A &= (\tan \phi - \tan h \cos A) \operatorname{cosec} A \Delta h \\ &= \cot q \, . \sec h \, . \Delta h \\ \Delta A &= (\tan h - \tan \phi \cos A) \operatorname{cosec} A \Delta \phi \\ &= \cot t \, . \sec \phi \, . \Delta \phi . \end{aligned}$$

Hence, errors in altitude will have the least effect when q is 90°, or is nearly 90°, *i.e.* when the star is at or near elongation, and errors in the assumed latitude will have the least effect when the hour angle t is equal to or near to 90°. Both conditions cannot be fulfilled at the same

time but we note that, when the star is on or near the meridian it will be moving rapidly in azimuth and slowly in altitude, and when it is on the prime vertical it will be moving slowly in azimuth and rapidly in altitude. Accordingly, the rule commonly given and used is to observe stars on or near the prime vertical. In these circumstances, $A = 90°$ or $270°$. If $A = 90°$, $\Delta A = \tan\phi\Delta h = \tan h\Delta\phi$, so that the errors in the computed azimuth due to errors in both the assumed latitude and the observed altitude are independent of the star and are the same for all stars for the same values of ϕ, h, $\Delta\phi$, and Δh. Moreover, if $A = 270°$, $\Delta A = -\tan\phi\Delta h = -\tan h\Delta\phi$, and consequently taking the mean of the results of observations of east and west stars, observed at the same altitude on or near the prime vertical, will tend to eliminate the effects of errors of assumed latitude and of errors of like sign and magnitude (such as errors in the computed values of the refraction) in the observed altitude. If the stars are not observed on or very near the prime vertical, they should be observed in adjoining quadrants on either side of the meridian, so that $\cot A$ as well as $\operatorname{cosec} A$ has different signs for the two stars. Again, if there should be considerable doubt about the real value of ϕ, and the magnitude of $\Delta\phi$ may be much larger than that of Δh, the star's position should be chosen to minimise the effects of errors in ϕ, *i.e.* the hour angle should be $270°$ or $90°$, in which case $\cos A = \tan h \cot\phi$. If, on the other hand, the error in the observed altitude is likely to be greater than the error in the assumed latitude, it is advisable to observe stars at or near elongation, as in that position the effects of errors in observed altitude are reduced to a minimum.

Azimuth by Ex-meridian Altitudes of the Sun. In finding azimuth by ex-meridian altitudes of the sun near the prime vertical, observations in the morning should be balanced by a similar set in the afternoon. The general features of the measurement are the same as for a star.

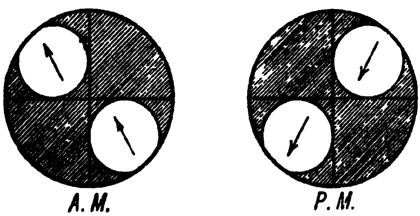


FIG. 41.

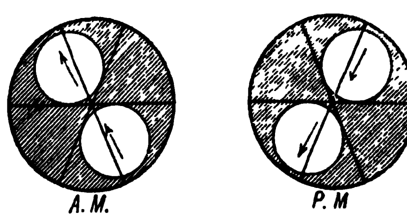


FIG. 42.

Since the required altitudes and horizontal angles are those to the sun's centre, the hairs should be set tangential to two limbs simultaneously. After changing face, the opposite limbs are observed. The appearance at contact presented by the ordinary Ramsden diagonal eyepiece for the two observations of a set is as in Fig. 41. The motion in azimuth is slow, and the vertical hair is kept in contact by the upper slow motion screw, the sun being allowed to make contact with the horizontal hair. If the telescope has no vertical hair, the sun must be placed in the opposite angles as in Fig. 42.

A set of observations consists in sighting the reference mark and then the sun with telescope direct. The telescope is then reversed, and

the opposite limbs of the sun and finally the reference mark are sighted. This set may be followed by a second one in which the limbs should be taken in the reverse order. The time of each altitude measurement is noted, so that the appropriate value of the sun's declination may be interpolated for use in the computation.

The reduction is performed in the same manner as for the corresponding star observation.

As in the case of a star, if the mean of the face right and face left observations is used to compute a single mean azimuth, a correction for the curvature of the sun's path may have to be introduced if the greatest precision possible is to be obtained from the results. This correction is given on page 104 and is applied as in the case of a star. In the case of the sun, however, when observations are made in opposite quadrants of the reticule, an additional correction is necessary because the azimuthal angle of the sun's diameter varies with the altitude. Hence, if there is an appreciable difference in altitude between the face right and face left observations, a correction to the mean azimuth, due to this difference in altitude, may have to be applied. This correction, for the mean of two observations on opposite limbs as in Fig. 41, is given by :—

$$\Delta A = \tfrac{1}{2}\sigma \sin 1' \,.\, \sec h \,.\, \tan h \,.\, \Delta h$$

where σ is the angular semi-diameter of the sun in minutes of arc and Δh is the difference in altitude of the sun's centre between the two observations. If Δh is in seconds of arc, ΔA will also be in seconds of arc. As the sun's altitude increases in the morning and decreases in the afternoon, the correction is to be added to the azimuth angle, measured the shortest way from the elevated pole, for a morning observation and to be subtracted from it for an afternoon observation.

It should be noted that the Δh to be used in computing this correction, as well as that to be used in computing the correction for curvature, is not the actual measured difference in altitude between the two limbs, but the difference in altitude between the two positions of the sun's centre. If the telescope is an ordinary inverting one, as will nearly always be the case, and Figs. 41 and 42 represent the actual views in the telescope, then the first observation in the morning (afternoon) will be taken to the lower (upper) limb and the second to the upper (lower) limb. In this case, Δh is equal to the measured difference in altitude *minus* the sun's angular diameter.

The formula for this correction may be proved as follows :—

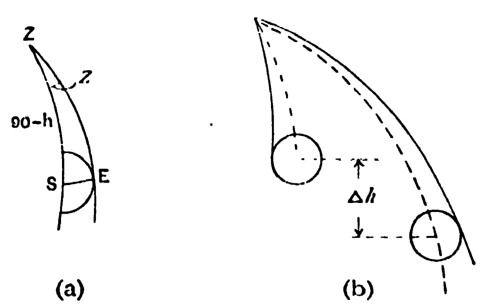


FIG. 43.

In Fig. 43 (*a*) Z is the zenith, S the centre of the sun and E the point where a vertical plan through Z touches the sun's limb. In the spherical triangle ZSE the angle at E is a right angle, the side ZS is equal to $90° - h$, and the angle SZE is the difference in azimuth between S and E. Then, if the angular semi-diameter SE = σ, we have :—

$$\frac{\sin \sigma}{\sin Z} = \frac{\sin (90^\circ - h)}{\sin 90^\circ}$$

or, as σ and Z are both small,

$$Z = \sigma \sec h$$

As the altitude changes by the small amount dh the change in Z will be given by differentiating Z with respect to h. This gives:

$$dZ = \sigma \sec h \,.\, \tan h \,.\, dh.$$

Here σ is in radians and, if it is expressed in minutes of arc, we must multiply it by sin 1′. Hence, for the mean of two observations on opposite limbs, the correction to the azimuth is given by :—

$$\Delta A = \tfrac{1}{2}\sigma \sin 1' \,.\, \sec h \,.\, \tan h \,.\, \Delta h$$

where Δh is the difference in altitude of the sun's centre between the two observations.

In all ordinary work for which sun azimuths are likely to be observed, this correction, as well as the one for curvature of path, is usually neglected. The latter correction, however, may be considerable if Δh is large, and, in such circumstances, it is as well to compute it if accuracy is desired.

Since the method of ex-meridian altitudes is not suitable for bodies observed near the meridian, and if observations to the sun have to be taken in the late morning or early afternoon, it is better then to depend on observations based on time rather than on altitude.

Azimuth by Hour Angles of Stars or the Sun. This method resembles that of ex-meridian altitudes, except that, in place of measuring altitudes, the chronometer times of the observations are recorded. Horizontal angles are measured between the reference mark and east and west stars near the prime vertical. From the corrected mean of the chronometer readings for a series of observations on a star the hour angle corresponding to the mean horizontal angle is obtained.

Various formulæ for A are available, that generally used being

$$\tan A = \tan t \cos M \operatorname{cosec} (M - \phi)$$
$$\text{in which } M = \tan^{-1} \tan \delta \sec t.$$

In applying the formula, t is measured in arc the shorter way from upper transit, and is always positive ; ϕ is positive for north latitudes and negative for south ; A is the azimuth angle from the north part of the meridian measured the shorter way, east or west. In reducing forenoon and afternoon observations on the sun, δ is interpolated for the times of observation.

Largely owing to the fact that a knowledge of the chronometer error is required, this method is less commonly used than those of ex-meridian altitudes or circumpolar stars at elongation. It is, however, a better method than ex-meridian altitudes, as it is not affected by errors of refraction, and near the prime vertical errors of time have very little effect on the result.

In precise work a correction for the curvature of the path of the heavenly body may be applied to the mean of the face right and face left observations. If the mean hour angle is used to compute the azimuth angle the correction to the latter is given by :—

$$\Delta A'' = -\tfrac{1}{8} \sin A \cos \phi \sec^2 h (\cos h \sin \delta - 2 \cos A \cos \phi) \times (\Delta t)^2 \times \sin 1''$$

in which Δt is the difference in time, expressed in seconds of arc, between the face right and face left observations. This expression is zero at culmination, and at elongation it reduces to the form given on page 101 with $n = 2$ and $\frac{1}{4}\Delta t$ written for $\frac{1}{2}\Delta t$ in the expression there.

If this correction is to be applied to solar observations taken in opposite quadrants of the reticule, the Δt to be used is the time interval between the two positions of the sun's centre, not the time interval between the actual observations to the limbs. In this case write P instead of Z in Fig. 43 (a), where P is the celestial pole, and the sides PS and PE of the triangle PSE each become $90° - \delta$. The angle at P is then given by :—

$$P = \sigma \sec \delta.$$

Hence, the correction to be applied to the observed difference in time to obtain Δt is $2\sigma \sec \delta$, taken with its proper sign. If σ is given in minutes of arc the result will also be in minutes of arc.

The errors in the computed azimuth angle arising from errors in the observed hour angle or in the assumed latitude are similar to those given on page 101 for the case of circumpolar star observations by means of hour angles. When the star is on the prime vertical, $A = 90°$, and the formulæ reduce to

$$\Delta A = -\cos q \operatorname{cosec} t \, \Delta t$$
$$\Delta A = \tan h \, \Delta\phi.$$

Azimuth by Equal Altitudes of a Star. By measuring the angle subtended between the reference mark and a star in two positions of equal altitude, the angle between the mark and the meridian is given by half the algebraic sum of the two observed angles. The method is independent of a knowledge of the star's co-ordinates, but, unless the observations are taken before and soon after the culmination of the star, when errors of altitude are serious, it is open to the objection that the determination may involve a wait of several hours.

Several observations are made with the star east of the meridian. The reference mark is first sighted, and the star is bisected by both cross-hairs. An equal number of observations are made with the telescope direct and reversed, and the altitude as well as the horizontal angle is noted for each. For the west series the same altitudes are set on the circle, and for each individual altitude the same face as before is used. At each observation the star is kept bisected by the vertical hair, and the motion in azimuth is stopped when the star reaches the horizontal hair. The algebraic mean of all the horizontal angles represents, without correction, that between the reference mark and the north or south part of the meridian, according to the position of the star.

Azimuth by Equal Altitudes of the Sun. In this case a series of horizontal angles is measured between the reference mark and the sun in the forenoon. In the afternoon a similar series is observed with the sun at the same altitudes. Since the value of the altitude is not required, it is sufficient to observe the upper or the lower limb throughout, but an equal number of sights should be taken on the right and left limbs and with the telescope direct and reversed.

On account of the change in declination in the interval between the morning and afternoon equal altitudes, the mean of the horizontal angles

does not represent that between the reference mark and the meridian. The watch time of each observation should be carefully noted, and the value of the correction is computed as follows.

Let c = angular correction to be applied to the algebraic mean of the observed angles to give the angle between the reference mark and the meridian

t = half the interval between the times of equal altitudes

ϕ = observer's latitude

δ_E = sun's declination at the average of the morning observations

δ_W = sun's declination at the average of the afternoon observations

Then $c = \frac{1}{2}(\delta_W - \delta_E) \sec \phi \operatorname{cosec} t$.

When the sun's declination is changing towards the north (south), the mean of the observed azimuth angles lies west (east) of the meridian for places in north latitudes, and *vice versa* for south latitudes.

This formula may be obtained by differentiating A with respect to δ, keeping h fixed, in :—

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos A.$$

The method of determining azimuth by observations of equal altitudes of star or sun yields inferior results and is seldom used. Its only real virtue appears to be that a result, suitable for rough purposes, may be obtained without computation or the use of tables of any kind.

Azimuth by Circumpolar Stars at Culmination. A rough determination may be obtained by observing the horizontal angle between the reference mark and a circumpolar star on the meridian. The watch error must be known, and the watch times of upper or lower transit of two or more circumpolar stars are computed in advance.

The reference mark is observed shortly before transit of the first star, and the telescope is then pointed to the star, which is kept bisected by the vertical hair up to the computed instant of culmination. The horizontal circle readings are taken, and the reference mark is again bisected as a check on the stability of the instrument. No change of face is possible, and the observation of the second star should be made with the telescope reversed.

The smaller the polar distance of the stars the better. The accuracy is largely dependent upon that with which the time is known.

By observing pairs of stars, each pair consisting of one star above the pole and the other below, with the instrument set very close to the meridian, a primary determination of azimuth may be obtained. For primary work the method is very similar to a primary determination of time as already described on pages 85 to 87. For any star observed very close to the meridian, we have, after the level correction has been applied, the equation :—

$$R.A. = e + T + aA + cC$$

where e is the clock error, T the chronometer reading, a the required error of setting of the instrument in azimuth, $A = \sin z \sec \delta$, c the collimation error and $C = \sec \delta$, the signs of A and C being in accordance with the signs given in the table on page 86. Differentiating the two observations of a pair by the suffixes 1 and 2, and putting $\alpha = R.A.$, the right ascension of the star, we have :—

$$\alpha_1 = e + T_1 + aA_1 + cC_1.$$
$$\alpha_2 = e + T_2 + aA_2 + cC_2.$$

Hence,

$$a = \frac{(\alpha_2 - \alpha_1) - (T_2 - T_1) - c(C_2 - C_1)}{(A_2 - A_1)}$$

α_2 and α_1 are known, T_2 and T_1 are observed and A_2, A_1 and C_2, C_1 can be calculated. Consequently, if we know the value of c, the collimation error of the instrument, or if it is zero, we can at once calculate a. If c is not known, observations of two pairs of stars will enable it to be eliminated from the equations, and its value to be determined.

All the quantities entering into the equations must, of course, be expressed in the same units, arc or time, preferably arc, as a is required in seconds of arc.

It may be noted that, in observing for time, we choose stars culminating north and south of the zenith; in observing for azimuth, we choose stars culminating above and below the elevated pole. In the former case we get a good determination of time and a poor azimuth; in the latter a good determination of azimuth and an indifferent time.

To use this method, it is necessary to be able to set the instrument so that the line of collimation is very approximately in the meridian, and this means either preliminary observations to enable a sighting mark to be laid out with the necessary accuracy, or else the existence of some point which can be used for sighting to and whose azimuth from the point of observation is fairly accurately known. It is not always convenient to take preliminary observations, and a sighting point whose azimuth is accurately known may not already exist on the ground. For secondary work, however, good results can be obtained with a well-collimated instrument set only approximately in the meridian, say by compass within a degree or so.

This approximate method is more fully described, and examples given, in the War Office publication *Field Astronomy*.

For primary work with an instrument provided with eyepiece micrometer with vertical movable hair, it is convenient to have the sighting point set very approximately on the meridian, so that the angle between this point and the line of collimation may be measured by the movable hair of the micrometer. This is often not possible at a trigonometrical station.

Near the equator, of course, it is not possible to use the method at all, either for primary or for secondary work.

Combined Observations for Time and Azimuth by Observations of Two Stars in the Same Vertical Plane. In No. 39, Vol. VI of the *Empire Survey Review* for January, 1941, McCaw has described a method by which azimuth and time can be observed together off the meridian at one set up of the instrument. This method, which is a modification of the one just described, consists in observing the times of passage of two stars—one an equatorial time star and the other a circumpolar star—across the same vertical plane when the theodolite is set at any azimuth suitable for the observation of a circumpolar star. It involves a knowledge of the collimation error of the instrument, and it is suitable for primary determinations of both time and azimuth. A full description of

the method, including a numerical example, will be found in the paper, which shows how the errors of level and collimation may be accurately eliminated from a method otherwise well known.

Corrections to be Applied to Observed Azimuths in First Order Geodetic Work. Although too small to be applied to any work other than that intended for a geodetic survey of the first order, there are two corrections to observed azimuths which, for the sake of completeness, should be mentioned here. The first is the reduction of the computed azimuth for height above sea level of the azimuth mark when this height is large. This correction arises from the fact that the earth is an ellipsoid of revolution, and is not a perfect sphere. The consequence is that the plane which contains the normal drawn through the azimuth station and passes through the azimuth mark does not contain the normal through the latter point. This correction, in seconds of arc, is given by :—

$$+\frac{e^2h}{2a\sin 1''}\cos^2\phi\sin 2A$$

where a is the earth's equatorial radius, $e^2 = \frac{a^2 - b^2}{a^2}$, b being the earth's polar radius, and h is the height above sea level of the azimuth mark, expressed in the same units as a. Taking e and a as defined by the Clarke Spheroid of 1866, and with a and h in metres, the correction becomes :—

$$+\ 0{\cdot}000109h\ .\ \cos^2\phi\sin 2A.$$

The second correction is the correction of the observed azimuth to the mean position of the earth's pole. This correction sometimes becomes necessary on account of the fact that the earth's pole does not remain absolutely steady with reference to the features on the earth's surface, but varies with time. The motion is small and rather irregular and affects observations for latitude and time as well as for azimuth. For the latter it may be as great as half a second of arc in latitude 50°·N. The co-ordinates of the "instantaneous pole" and data for correcting observed azimuths used to be published by the International Geodetic Association but are now published by Professor Cecchini, Directeur de l'Office Central du Service des Latitudes, The Observatory, Turin. The correction, however, is not one which need be applied in anything but the most refined geodetic work.

DETERMINATION OF LATITUDE

Observations for latitude nearly always consist in measuring the altitude of a celestial body when it is either on or near the meridian. The determination of latitude is equivalent to measuring the altitude of the pole, and this may be obtained from the meridian or circum-meridian altitude of a body of known declination.

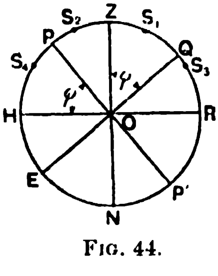


FIG. 44.

The latitude ϕ of the place of observation is related to the meridian zenith distance z of a celestial body of declination δ as follows.

Let ϕ and δ be marked + or − according as they are north or south, and z + or − according as the zenith is north or south of the body for the northern

hemisphere, and the reverse for the southern. Then $\phi = z + \delta$ when the body is at upper transit, as at S_1, S_2 and S_3 in Fig. 44.

If the star is at lower transit, as at S_4, the declination should be taken as the angle from the equator through the pole to the star, *i.e.* QOS_4 or $180° -$ the tabulated declination. Hence $\phi = z + 180° - \delta$.

Methods. The principal observations for determining latitude may be summarised as :

(1) Talcott's method of meridian altitudes of stars.
(2) Circum-meridian altitudes of stars or the sun.
(3) Meridian altitudes of stars or the sun.
(4) Altitudes of *Polaris*.
(5) Ex-meridian altitudes of stars.
(6) Equal altitudes of stars.

Latitude by Talcott's Method. Precise latitude determinations are made by means of the zenith telescope (page 57) by the method devised by Captain Talcott of the U.S. Corps of Engineers about the year 1834.* Less refined determinations may be obtained on the same system by the use of a theodolite fitted with an eyepiece micrometer and having a sensitive altitude level.

The observation consists in measuring the small difference in meridian zenith distance of two stars culminating on opposite sides of the zenith at very nearly the same altitude and within a short time of each other. In Fig. 44 let S_1 and S_2 represent two stars of declination δ_1 and δ_2 respectively, and let their meridian zenith distances in latitude ϕ be respectively z_1 and $-z_2$.

$$\text{Then } \phi = z_1 + \delta_1 = -z_2 + \delta_2$$
$$2\phi = \delta_1 + \delta_2 + z_1 - z_2$$
$$\text{or } \phi = \tfrac{1}{2}(\delta_1 + \delta_2) + \tfrac{1}{2}(z_1 - z_2)$$

The declinations being known, ϕ is therefore derived from the measurement of the difference of the meridian zenith distances.

The method lends itself admirably to precise determinations. The north and south stars selected to form a pair are such as have nearly the same zenith distance, so that $z_1 - z_2$ can be accurately measured by means of the eyepiece micrometer. The refraction correction required is only the difference between those applicable to the individual zenith distances, and the effect of uncertainty in the value of the refraction coefficient is therefore negligible. In addition, small errors of pointing, arising from error of collimation, dislevelment of horizontal axis, or inaccuracies of setting in the meridian, exercise very little influence on the measurement of the small quantity $z_1 - z_2$.

Observations in Talcott's Method. A primary determination having a probable error of measurement not exceeding $\pm 0''\cdot 1$ involves the observation with the zenith telescope of from ten to forty pairs of stars, and requires from one to three or four nights' work. An observing list of pairs of stars is first prepared, and for this purpose the latitude should

* The method was first described by Peter Horrebrow in 1732 but was then overlooked, to be rediscovered and developed independently by Talcott a hundred years later. Hence it is often called the Horrebrow-Talcott method.

be known to the nearest 1′. Suitable stars are selected from a modern star catalogue, and must fulfil the following conditions :—

(1) The zenith distance should not exceed 20° to 25° in order to guard against too great uncertainties in refraction. With a modern catalogue observations may be confined to within about 15° of zenith distance.

(2) The difference in zenith distance of stars forming a pair should be within range of about one-third of the angular field of the micrometer.

(3) The interval between the times of transit of the stars of a pair, or their difference in right ascension, should for accuracy and convenience of observation be not less than $1\frac{1}{2}^m$ or greater than 20^m.

(4) The interval between the transits of the second star of one pair and the first of the next should be not less than 2^m.

(5) The sum of the north zenith distances in a series should balance as nearly as possible that of the south zenith distances to minimise errors arising from inaccurate knowledge of the value of a micrometer division and irregularities in the screw.

The setting of the telescope in zenith distance for the observation of a particular pair of stars is the mean of their zenith distances or half their difference in declination. The setting angle for each pair is tabulated to the nearest 1′ in the observing list, and the right ascensions are also tabulated to the nearest quarter minute to show the sidereal interval between transits and to aid in the identification of faint stars.

The telescope is placed in the plane of the meridian, and the azimuth stops are set by first sighting a previously established meridian mark. The adjustment is finally effected when the times of transit of two or more stars, referred to a sidereal time chronometer of known error, agree with their right ascensions. In observing the first of a pair of stars, the movable hair is placed at the position at which the transit is expected. When the star reaches the middle third of the field, it is bisected, and the micrometer and latitude level readings are taken. The telescope is then swung through 180° in azimuth, the clamp block being brought gently up to the second azimuth stop. The hair is approximately set for the second star, which is similarly observed at transit. In the interval between the two observations of a pair no change must be made in the inclination of the telescope relatively to the latitude level. Occasionally a star has to be observed a little out of the meridian although within the field of the telescope as set. The sidereal time of the observation is then noted so that the observed zenith distance may be reduced to the required meridian zenith distance. In the case of binaries, both stars of the pair are observed.

Reduction.—The fundamental formula

$$\phi = \tfrac{1}{2}(\delta_1 + \delta_2) + \tfrac{1}{2}(z_1 - z_2)$$

in which the subscripts 1 and 2 refer respectively to the south and north stars of a pair, may be rewritten to express $z_1 - z_2$ in terms of the corresponding micrometer readings. It therefore becomes

$$\phi = \tfrac{1}{2}(\delta_1 + \delta_2) + \tfrac{1}{2}(m_1 - m_2)T$$

in which m_1 and m_2 are the micrometer readings for the south and north stars respectively, and T is the angular value of one turn of the micrometer screw. This expression applies if the micrometer readings increase

with increasing zenith distance; otherwise the signs of m_1 and m_2 are reversed. The values of the declinations must be computed for the nights of the observations from the elements given in the star catalogue unless the star happens to be in the *Nautical Almanac* list. In the case of a star below the pole $180° - \delta$ is inserted in the formula in place of δ.

In the reduction of each pair of observations it is necessary to apply corrections for the inclination of the latitude level during the observations and for refraction.

Level Correction. The correction to altitude for a single observation being $\frac{1}{2}d(O - E)$ (page 78), that to be added to zenith distance is $\frac{1}{2}d(E - O)$. The respective corrections for the south and north stars are therefore $\frac{1}{2}d(E_1 - O_1)$ and $\frac{1}{2}d(E_2 - O_2)$, and the correction to $\frac{1}{2}(z_1 - z_2)$ or to ϕ is

$$\frac{d}{4}\Big((E_1 - O_1) - (E_2 - O_2)\Big)$$

The level readings are commonly booked as north and south end readings of the bubble, in which case, writing N_1 for E_1, S_1 for O_1, S_2 for E_2, and N_2 for O_2, the level correction may be expressed as

$$\frac{d}{4}\Big((N_1 + N_2) - (S_1 + S_2)\Big)$$

Refraction Correction. This correction is the difference between the refraction corrections applicable to the individual zenith distances, and, as its value is small, it is sufficient to assume mean atmospheric conditions and to use the formula $r = 58'' \tan z$. If r_1 and r_2 are the respective refraction corrections for the south and north star observations, the additive correction to $\frac{1}{2}(z_1 - z_2)$ or to ϕ is $\frac{1}{2}(r_1 - r_2)$ which, since $z_1 - z_2$ is small, may be evaluated by differentiating r with respect to z and then putting $dz = \frac{1}{2}(z_1 - z_2)$. This gives :—

$$dr = 58''.\tfrac{1}{2}(z_1 - z_2) \sin 1' \sec^2 z = 0''{\cdot}017.\tfrac{1}{2}(z_1 - z_2) \sec^2 z$$

where $z_1 - z_2$ is expressed in minutes.

With the level and refraction corrections applied, the latitude formula therefore becomes, with T expressed in seconds

$$\phi = \tfrac{1}{2}(\delta_1 + \delta_2) + \tfrac{1}{2}T(m_1 - m_2)\,(1 + 0{\cdot}00028 \sec^2 z) + \frac{d}{4}\Big((N_1 + N_2) - (S_1 + S_2)\Big)$$

Reduction to the Meridian. When a star is observed out of the meridian, the line of sight of the telescope being maintained in the meridian, the correction to be applied to the zenith distance of the star, as observed, to give the meridian zenith distance is

$$\sin^2 \tfrac{1}{2}t \sin 2\delta \operatorname{cosec} 1''$$

in which t is the hour angle of the star at the instant of observation, and δ is its declination. Half of this amount represents the correction to the latitude, the sign always being positive in the case of stars of north declination and negative for stars of south declination and stars below the pole. In the event of both stars of a pair being observed out of the meridian, the value of the correction is computed for each. The proof of the formula is as follows :—

In Fig. 45 S is the star observed very close to the meridian. The line of collimation lies along ZP so that zenith distances are measured in the vertical plane through ZP. Draw an arc of a great circle ST perpendicular to the meridian. Then ZT is the measured zenith distance when the star is at S. The declination of the star does not change while it moves from the meridian to the position S. Hence its position when on the meridian will be at Q where PQ = PS = 90° − δ, and ZQ will be its zenith distance when on the meridian. TQ = Δz is thus the correction required.

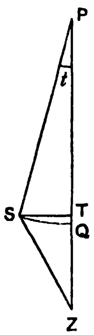


Fig. 45.

In the right-angled spherical triangle PST, angle at P = t, side PS = 90° − δ and side PT = 90° − δ − Δz. Then :—

$$\tan PT = \tan PS \cos P$$

$$\therefore \tan (90° - \delta - \Delta z) = \tan (90° - \delta) \cos t.$$

$$\therefore \tan (\delta + \Delta z) = \tan \delta \sec t$$

Expanding $\tan (\delta + \Delta z)$ by Taylor's Theorem, we have :

$$\tan \delta + \Delta z \sec^2 \delta + \ldots = \frac{\tan \delta}{1 - 2 \sin^2 \frac{1}{2}t}$$

$$= \tan \delta \,(1 + 2 \sin^2 \tfrac{1}{2}t + \ldots)$$

As $\frac{1}{2}t$ and Δz are both very small, we can neglect powers of Δz higher than the first and powers of $\sin \frac{1}{2}t$ higher than the second. This gives :—

$$\therefore \Delta z \sec^2 \delta = 2 \tan \delta \sin^2 \tfrac{1}{2}t$$

$$\therefore \Delta z = \sin 2\delta \,.\, \sin^2 \tfrac{1}{2}t$$

Here Δz will be in radians. To convert it into seconds we have :

$$\Delta z'' = \sin 2\delta \sin^2 \tfrac{1}{2}t \operatorname{cosec} 1''$$

Latitude by Circum-meridian Altitudes of Stars. This is the most accurate of the methods *commonly* used in field determinations for *mapping* purposes. It consists in observing a series of altitudes at noted times of each of several stars for a few minutes before and after transit. From the hour angle of the star at the instant of each observation the measured circum-meridian altitudes are reduced to the meridian, and the mean result furnishes a better measurement of meridian altitude than can be obtained directly.

The effects of an erroneous value for the refraction correction, personal errors of bisection, and instrumental flexure are reduced to a minimum by observing an equal number of north and south stars in pairs of similar altitude. The method requires a good knowledge of the chronometer error.

The stars selected are such as have a meridian altitude not less than 45° to avoid irregular refraction. The reductions are simplified if an upper limit is assigned to the altitudes, and this may be put at 80°. From an approximate knowledge of the latitude of the place the limits of declination of suitable north and south stars are known, and from an approximate value of the longitude the required right ascensions are obtained so that the transits will occur at a convenient time and at suitable intervals. An observing period of about 20^m should be allowed for each star.

The observation of each star is commenced about 10^m before the computed time of transit. The altitudes are observed in rapid succession, the chronometer time of each being recorded to the nearest $0^s{\cdot}1$ or $0^s{\cdot}2$. Observations are continued for about 10^m after transit. The measurements for each star may consist of an equal number of face right and face left

altitudes, but change of face is unnecessary and inadvisable if the observations are adequately paired on north and south stars. There should be the same number of observations on either side of the meridian in order to reduce the effect of an erroneous value of the chronometer error.

The reduction to the meridian is computed as follows :—

Let z_0 = observed circum-meridian zenith distance, corrected for refraction, and positive or negative according as the zenith is north or south of the star

z = required meridian zenith distance

z_1 = approximate meridian zenith distance, deduced from the greatest observed altitudes

ϕ_1 = approximate latitude derived from z_1, positive or negative according as it is north or south

t = hour angle of star when observed

δ = declination of star

$A = \cos\phi \cos\delta \operatorname{cosec} z = \cos\phi_1 \cos\delta \operatorname{cosec} z_1$ very nearly

$$m = \frac{2\sin^2 \frac{1}{2}t}{\sin 1''}$$

Then $z = z_0 - Am$

With greater accuracy, $z = z_0 - Am + Bn$

$$\text{where } B = A^2 \cot z, \text{ and } n = \frac{2\sin^4 \frac{1}{2}t}{\sin 1''}$$

The approximate formula $z = z_0 - Am$ is usually sufficient provided the observations are confined within a period of 10^m on either side of transit and the star is not too near the zenith. The value of A used is that derived from the approximate values of the zenith distance and latitude, and, strictly, a second approximation should be made with the closer values of z and ϕ obtained from the first. Since m is a function of $\sin^2 \frac{1}{2}t$, and t is small, the correction to z_0 varies as the square of the hour angle, and, in consequence, the mean of a number of observed altitudes or zenith distances does not correspond with the average of their chronometer times. The observations cannot therefore be averaged in the usual way.

The values of the variable m should be taken out for the individual observations, either by calculation or from tables for the reduction of circum-meridian observations to the meridian.

If z'_0 = the arithmetical mean of the observed zenith distances, each corrected for refraction

m' = the arithmetical mean of the individual values of m

then $z = z'_0 - Am'$

and $\phi = z + \delta$

The value of z may be worked out for each observation in order that the degree of consistency in the values may be ascertained.

To prove the formula for the correction proceed as follows :—

$$\begin{aligned}\cos z_0 &= \sin\delta \sin\phi + \cos\delta \cos\phi \cos t \\ &= \sin\delta \sin\phi + \cos\delta \cos\phi(1 - 2\sin^2 \tfrac{1}{2}t) \\ &= \cos(\delta - \phi) - 2\cos\delta \cos\phi \sin^2 \tfrac{1}{2}t\end{aligned}$$

In Fig. 45, $z_0 = ZS$; z, the zenith distance on the meridian, is ZQ, and, since $ZQ + QP = ZP$, $z + 90° - \delta = 90° - \phi$, so that $z = \delta - \phi$.*
Hence,

$$\cos z_0 = \cos z - 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2}t$$

$$\therefore\ 2 \sin \frac{z_0 + z}{2} \sin \frac{z_0 - z}{2} = 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2}t.$$

Let $z_0 = z + \Delta z$ Then,

$$\sin\left(z + \frac{\Delta z}{2}\right) \sin \frac{\Delta z}{2} = \cos \delta \cos \phi \sin^2 \tfrac{1}{2}t$$

$\therefore$ By Taylor's expansion

$$\sin \tfrac{1}{2}\Delta z[\sin z + \tfrac{1}{2}\Delta z \cos z - \tfrac{1}{8}(\Delta z)^2 \sin z \ . \ . \ .] = \cos \delta \cos \phi \sin^2 \tfrac{1}{2}t$$

Writing $\sin \tfrac{1}{2}\Delta z = \tfrac{1}{2}\Delta z$, we have:

$$\Delta z \sin z + \tfrac{1}{2}(\Delta z)^2 \cos z - \ . \ . \ . = 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2}t$$

As a first approximation take

$$\Delta z = \frac{2 \cos \delta \cos \phi}{\sin z} \sin^2 \tfrac{1}{2}t.$$

Inserting this value in the second term, and neglecting all succeeding terms in the expression on the left-hand side of the equation, we get: –

$$\Delta z \sin z = 2 \cos \delta \cos \phi \sin^2 \tfrac{1}{2}t - \frac{2 \cos^2 \delta \cos^2 \phi}{\sin^2 z} \cos z \sin^4 \tfrac{1}{2}t,$$

or, for Δz expressed in seconds of arc:

$$\Delta z = \frac{2 \cos \delta \cos \phi}{\sin z \sin 1''} \sin^2 \tfrac{1}{2}t - \frac{2 \cos^2 \delta \cos^2 \phi}{\sin^2 z \sin 1''} \cot z \sin^4 \tfrac{1}{2}t.$$

Tabulated values of the factors m and n will be found in Table XXV in Close and Winterbotham's *Text-book of Topographical Surveying.*

It should be noted that the difference between this correction and the one relating to Talcott's method is that in the one case the zenith distance is measured in the vertical plane containing the zenith and the star, whereas in the other it is measured in the plane of the meridian.

With a 5-in. or 6-in. micrometer theodolite, the error of latitude derived from a single pair of stars should not be more than about 3″. Observation of ten pairs of stars should give the latitude correct to about 1″. If a large geodetic type of theodolite is used the error will be reduced accordingly.

As the star is moving almost at right angles to the meridian, a small error in time will produce but a very small error in z.

Latitude by Circum-meridian Altitudes of the Sun.—In applying the method to solar observations, the precision is much decreased since the observations cannot be paired. The altitudes are measured to the upper and lower limbs alternately. In reducing to the meridian, the value of the declination should, strictly, be computed for the instant of each observation, but practically it is sufficient to use the declination corresponding to the mean instant of the set.

* Note that, as Fig. 45 is drawn, z in the formula $\phi = z + \delta$ is to be taken with a negative sign in order to conform with the convention regarding signs given on page 112.

Example. On 1935 June 15, in longitude 4° 59′ 05″ W., circum-meridian observations of the sun gave the results tabulated. The altitudes given represent the means of the two micrometer readings corrected for level. The error of the watch was known to be $17^m\ 04^s{\cdot}4$ fast on L.M.T. Find the latitude.

Object	Face	Watch Time			Observed Altitude			Pair of Max. Observed Altitudes			Remarks
		h	m	s	°	′	″	°	′	″	
$\underline{\odot}$	L.	12	09	53·0	56	53	06·5				Bar. 30·1 in.
$\overline{\odot}$	R.	12	12	59·6	57	25	43·0				Ther. 72° F.
$\underline{\odot}$	R.	12	15	39·2	56	54	40·0	56	54	40·0	
$\overline{\odot}$	L.	12	18	31·8	57	26	13·0	57	26	13·0	
$\underline{\odot}$	L.	12	21	32·0	56	53	58·5				
$\overline{\odot}$	R.	12	23	55·4	57	24	54·0				
	Means	12	17	05·2	57	09	45·8	57	10	27	

(1) *True mean zenith distance, z'_0.*

		°	′	″
Mean observed altitude	=	57	09	45·8

		″
Mean refraction	=	37·6
Barometer correction	=	+0·1
Temperature correction	=	−1·7

Refraction	=	−		36·0
Parallax 8″·67 × cos 57°	=	+		4·7
True altitude	=	57	09	14·5
z'_0	=	32	50	45·5

(2) *Watch time of L.A.N.*

		h	m	s
From N. A., p. 25, G.M.T. of G.A.N.	=	12	00	06·9
Correction for longitude	=			+0·2
L.M.T. of L.A.N.	=	12	00	07·1
But watch is fast on L.M.T. by			17	04·4
Watch reading at L.A.N.	=	12	17	11·5

(3) *Sun's δ at mean instant of observation.*

		h	m	s
Mean of observed watch times	=	12	17	05·2
Watch time of L.A.N.	=	12	17	11·5
Mean instant of observation is before L.A.N. by				6·3
G.A.T. of observation = $12^h\ 19^m\ 56^s{\cdot}3 - 6^s{\cdot}3$	=	12	19	50·0
From N.A., p. 25, δ at G.A.N.	=	+23	17	11·7
Correction	=	+		2·2
δ	=	+23	17	13·9

(4) *Approximate meridian zenith distance z_1 and latitude ϕ_1.*

		°	′	″
Mean of two greatest observed altitudes	=	57	10	27
Refraction and parallax	= −			31
Approximate meridian altitude	=	57	09	56
z_1	=	32	50	04
δ	=	23	17	14
ϕ_1	=	56	07	18

(5) Am'.

Sun's hour angle at each observation = difference between watch time of observation and watch time of L.A.N.

t		m
m	s	"
7	18·5	104·8
4	11·9	34·6
1	32·3	4·6
1	20·3	3·5
4	20·5	37·0
6	43·9	89·0

Mean = m' = 45·6

$\log m'$	=	1·6590
$\log \cos \phi_1$	=	9·7462
$\log \cos \delta$	=	9·9631
$\log \operatorname{cosec} z_1$	=	0·2658
$\log Am'$	=	1·6341
Am'	=	43″·1

(6) *Latitude.*

		°	′	″
$z = z'_0 - Am'$	=	+32	50	02
δ	=	+23	17	14
ϕ	=	+56	07	16

Latitude by Meridian Altitudes of Stars. The measurement of a meridian altitude does not enable accidental errors of observation to be reduced so effectively as in the case of the multiple observations of circum-meridian altitudes. The reduction of meridian observations, however, is so simple that the number of stars observed can be suitably increased without undue labour.

The direction of the meridian may not be accurately known, and the watch time of transit should be computed in advance. Strictly, one observation of each star is all that can be obtained, but it is usually possible to secure two readings, since the altitude of the star is changing slowly near transit. If the watch error is unknown, the star may be followed with the horizontal hair until it appears stationary in altitude. After the level and circle readings are taken, a second altitude is measured as soon as possible, on the reverse face in the case of an unpaired star.

Errors of observation, refraction and instrument are effectively reduced by observing pairs of north and south stars of similar altitude culminating within a short time of one another. The observation is then similar in principle to Talcott's method, and the pairs of stars are selected on similar lines. The altitudes of the north and south stars of a pair being nearly equal, it is unnecessary to change face, since only the difference of the corrected zenith distances is required in computing the latitude. With a 5-in. or 6-in. micrometer theodolite, the latitude deduced from meridian observations of a pair of stars should be within 4″ or 5″ of the truth.

Latitude by the Meridian Altitude of the Sun. Measurement of the sun's altitude at apparent noon affords only a rough value of the latitude owing to the impossibility of balancing the observation. The method is, however, very useful for rough determinations, and is that employed in navigation.

Both faces must be used in a theodolite measurement. The watch time of apparent noon should be computed in advance, so that the altitude may be taken as nearly as possible on the meridian. Otherwise,

the horizontal hair is kept upon the upper or the lower limb until the greatest altitude is attained, or a series of altitudes is measured in quick succession on alternate faces at about the time of transit, the greatest altitude derived from a F.R. and F.L. pair being accepted as the meridian altitude. The difference between the greatest altitude and the meridian altitude, due to changing declination, is neglected.

Example. Find the approximate latitude given by the following meridian observations of the sun made on 1935 June 13 in longitude 4° 59′ 05″ W. The observations given are the pair showing the greatest altitude.

Object	Face	Altitude								Level		Remarks
		Micro. I			Micro. II		Mean			E	O	
		°	′	″	′	″	°	′	″			
$\underline{\odot}$	L.	56	47	20	47	25	56	47	22·5	13	14	Bar. 30·1 in. Ther. 70° F.
$\overline{\odot}$	R.	57	19	15	19	35	57	19	40	15	12	Level div. = 8″
							57	03	31	28	26	

$$\text{Level correction} = \frac{26-28}{2\cdot 2}\times 8'' = -4''$$

		°	′	″
Mean observed altitude	=	57	03	27
Mean refraction = 37″				
Barometer correction = +1				
Temperature correction = −2				
Refraction	=			−36
Parallax 8″·7 × cos 57°	=			+5
True altitude	=	57	02	56
z	=	32	57	04

Sun's Dec. at L.A.N.

West longitude in time = 19m 56s

		°	′	″
From N. A., p. 25, δ at G.A.N.	=	+23	10	42·9
Correction	=	+		2·8
δ at L.A.N.	=	+23	10	46
z	=	+32	57	04
ϕ	=	+56	07	50

Latitude by Meridian Altitudes of a Circumpolar Star at Upper and Lower Transits. If the altitude of a circumpolar star can be measured at both upper and lower transits, the latitude is given by half the sum of the altitudes separately corrected for refraction. A knowledge of the declination of the star is not required.

The method is of limited value in the field. The two observations are separated by twelve sidereal hours, so that they can be made with small instruments only when the hours of darkness exceed twelve.

Latitude by Altitudes of Polaris. If the error and rate of the chronometer are known, latitude may be determined by observing the altitude of *Polaris* at any instant.

The observation consists in taking a set of alternate face right and face left altitudes in succession and noting the chronometer time of each. The average of, say, two face right and two face left altitudes is

assumed to correspond with the average of the chronometer times, and from these the latitude is computed. Several such sets should be observed, and if the resulting latitudes are consistent, their mean is accepted.

The latitude ϕ is computed from

$$\phi = h - p\cos t + \tfrac{1}{2}p^2 \sin 1'' \sin^2 t \tan h$$

where h = observed altitude, corrected for refraction
p = polar distance, in seconds
t = hour angle = L.S.T. of observation − R.A. of *Polaris*.

The second and third terms of the formula constitute two corrections in seconds of arc, to the measured altitude. The sign of the first is controlled by that of $\cos t$; the second is always positive. Further corrections are omitted since their sum is always less than 1″.

Since the observation of low altitudes is to be avoided on account of uncertainties in refraction, the method cannot be applied in low latitudes. Errors of refraction affect the result by their whole amount unless the observation is paired with an observation of circum-meridian altitudes of a south star of similar altitude, when the accuracy of the determination becomes little inferior to that of circum-meridian altitudes of paired stars.

It is easy to derive the above formula by putting $\phi = h + dh$ and then substituting this value in the equation :—

$$\sin h = \cos p \sin \phi + \sin p \cos \phi \cos t.$$

p and dh are small angles so that we need only take the first few terms in the expansion of the sines and cosines. Neglecting all terms beyond those of the second order we obtain :—

$$-\frac{p^2}{2}\sin h - \frac{dh^2}{2}\sin h + dh\cos h + p\cos h\cos t - p\,dh\sin h\cos t = 0.$$

A first approximation gives $dh = -p\cos t$. Insert this value of dh in the second and last terms and the result, after reducing p and dh from radians to seconds, is :—

$$dh = -p\cos t + \frac{p^2}{2}\tan h\sin^2 t\sin 1''.$$

Example. On 1935 June 8 in longitude 4° 59′ 05″ W., the mean of four observed altitudes of *Polaris* was 55° 21′ 44″, the average of the local mean times being $23^h\ 38^m\ 04^s{\cdot}5$. The barometer and thermometer readings were 30·1 in. and 57° F. respectively. Find the latitude of the station.

$$\phi = h - p\cos t + \tfrac{1}{2}p^2\sin 1''\sin^2 t\tan h$$

				°	′	″	
h							
	Observed altitude		=	55	21	44	
	Mean refraction =	39					
	Barometer correction =	+1					
	Temperature correction =	−1					
	Refraction		=			−39	
	h			55	21	05	
p				°	′	″	
	Dec. of *Polaris*		=	88	57	12·2	
	p		=	1	02	47·8	= 3768
				h	m	s	
	L.M.T.		=	23	38	04·5	
	L.S.T. (as in *Example* 9, p. 40)		=	16	43	49	
	R.A. of *Polaris*		=	1	39	10	
	Hour angle		=	15	04	39	

First correction.

$$\log p = 3{\cdot}5761$$
$$\log\cos t = 9{\cdot}8405n$$
$$\text{Sum} = 3{\cdot}4166$$
$$\text{First correction} = +2610'' = +43'\ 30''$$

Second correction.

$$\log \tfrac{1}{2} = 9{\cdot}699$$
$$2\log p = 7{\cdot}152$$
$$\log\sin 1'' = 4{\cdot}686$$
$$2\log\sin t = 9{\cdot}716$$
$$\log\tan h = 0{\cdot}160$$
$$\text{Sum} = 1{\cdot}413$$
$$\text{Second correction} = +\ 26''$$

$$\phi = 55^\circ\ 21'\ 05'' + 43'\ 30'' + 26''$$
$$= 56^\circ\ 05'\ 01''\ \text{N}.$$

Note. Fig. 46 illustrates the position of the star for the above observation. APM represents the almucantar, or small circle of equal altitudes, through P. The sum of the corrections is represented by the angle SOL.

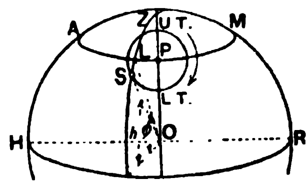


FIG. 46.

Latitude by Ex-meridian Altitudes. Evidently an ex-meridian altitude of a star of known declination would enable the astronomical triangle to be solved for ϕ if the time of the observation were known. An accurate knowledge of t and therefore of the watch error at the instant of observation is required. The observation should consist of altitude measurements of pairs of north and south stars disposed on both sides of the meridian. The method, however, is little used, and is much inferior to that of circum-meridian altitudes, since the effect of errors of time and altitude on the derived latitude increases from a minimum when the body is on the meridian to a maximum when it is on the prime vertical.

An accurate knowledge of the watch error is not required if observation is made of two altitudes, either of the same star or of different stars. In this case, latitude and time can be obtained if the interval between the observations is accurately known from the watch rate. For, if S and S′ be the two stars, triangle PSS′ can be solved for SS′ and angle PSS′. Triangle ZSS′ can be solved for angle ZSS′, which enables triangle PZS to be solved for PZ, the co-latitude, and ZPS, the hour angle of S. The accuracy of the derived latitude is largely dependent upon having one of the stars near the meridian.

Latitude by Equal Altitudes. Several methods of latitude determination are based upon observation of the times of equal altitudes of stars. Until recently these methods had not been applied to any extent in field astronomy, but they have now been successfully employed on the Peru-Bolivia and Brazil-Bolivia boundary surveys, and have been extensively used in Egypt.* Since the actual altitude is not required, the results are independent of circle readings. The observation is that for which the prismatic astrolabe (page 64) has been primarily designed.

* "Modern Methods of Finding the Latitude with a Theodolite," by Dr. Ball. *The Geographical Journal*, June 1917.

Equal Altitudes of Two Stars. The altitude, declination and hour angle of a star, and the latitude of the place have the relationship

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$$

If, therefore, two stars of declinations δ_1 and δ_2 respectively are observed to have the same altitude h with hour angles t_1 and t_2, we have

$$\sin \phi \sin \delta_1 + \cos \phi \cos \delta_1 \cos t_1 = \sin \phi \sin \delta_2 + \cos \phi \cos \delta_2 \cos t_2$$

whence ϕ is derived independently of h.

The stars forming a pair must be situated north and south of the zenith, and should not differ greatly in right ascension. They should be observed near the meridian, and the watch error and rate must be accurately known.

Equal Altitudes of Three Stars. To avoid the necessity for an exact knowledge of the watch error, observation may be made of three or more stars at equal altitudes. Both latitude and time may then be obtained, and this method is better adapted for field determinations than the last. If only three stars are observed they should be approximately 120° apart in azimuth, preferably with one near the meridian. If more stars are used, they may be observed in sets of four, one in each quadrant, as for the case of the prismatic astrolabe (page 68). The watch rate must be known so that the intervals between the observations may be accurately obtained. If these intervals are denoted by i_1 and i_2 for a three-star observation, then

$$\begin{aligned}
\sin h &= \sin \phi \sin \delta_1 + \cos \phi \cos \delta_1 \cos t_1 \\
\sin h &= \sin \phi \sin \delta_2 + \cos \phi \cos \delta_2 \cos (t_1 + i_1) \\
\sin h &= \sin \phi \sin \delta_3 + \cos \phi \cos \delta_3 \cos (t_1 + i_2)
\end{aligned}$$

from which by subtraction are obtained two simultaneous equations for ϕ and t_1. The solution is considerably simplified by assuming values for ϕ and the watch error and computing the value of h for each observation. The results, h_1, h_2 and h_3, will in general differ from the true altitude h because of errors in the assumed values, but if

$$\begin{aligned}
d\phi &= \text{correction to assumed latitude} \\
dt &= \text{correction to watch}
\end{aligned}$$

$d\phi$, dt and h are obtainable from the simultaneous equations

$$\begin{aligned}
h_1 - h &= \cos A_1 \, d\phi + \cos \phi \sin A_1 dt \\
h_2 - h &= \cos A_2 \, d\phi + \cos \phi \sin A_2 dt \\
h_3 - h &= \cos A_3 \, d\phi + \cos \phi \sin A_3 \, dt
\end{aligned}$$

where A is the azimuth of each observation, either observed or computed, reckoned clockwise through 360° from north, and the term containing $d\phi$ becomes negative for places in the southern hemisphere, ϕ being reckoned positive whether north or south. If more than three stars are observed, the reduction may be performed by treating the observations in groups of three as an alternative to the rigorous method of least squares. Alternatively, one of the methods applicable to observations with the prismatic astrolabe may be used instead (page 71).

Instead of observing three different stars, the method may be modified by using two stars and observing one of them twice. The double observation on the same star on opposite sides of the meridian gives the watch error, and the reduction is simplified.

Dr. Ball states that, if carefully carried out, the three-star method applied to a single set of stars with one not more than an hour or so from the meridian and with a level sensitive to 5″ per division, usually gives the latitude to within 3″ of the truth.

This method is the one used in connection with the prismatic astrolabe, the only difference being that, with the theodolite, stars may be observed at any altitude whereas with the prismatic astrolabe the altitude is fixed.

Corrections to be Applied to Observed Latitudes in First-order Geodetic Work. As in the case of observed azimuths, corrections have to be applied to observed latitudes on account of the observing station not being at sea level and for the variation of the position of the earth's pole. Both of these corrections are very small and can safely be neglected in all engineering or topographical work. The first is due to the spheroidal shape of the earth and the manner in which gravity varies with latitude and height above sea level. Its value is given by :—

$$\Delta\phi = -0''\cdot052h \,.\, \sin 2\phi$$

where h is the height of the observing station above sea level in thousands of feet. This correction, which is zero at the equator, is always subtractive numerically no matter whether the latitude is north or south.

The second correction is due to the small variations in the position of the earth's pole with respect to time that have been referred to on page 112, and it is obtained in the same way as the similar correction for azimuths from data provided by the results of observations, covering a long period of time, at certain fixed observatories.

DETERMINATION OF LONGITUDE

Since the longitude of a place is the difference between its local time and that at the reference meridian at the same instant, observation of the difference of time between two places determines their difference of longitude.

Determinations may be classed as relative or absolute. By a relative measurement is meant one in which the longitude of one place is found from the known longitude of another by observing the difference of time between them. In an absolute determination local astronomical observations are made for Greenwich time, and the required longitude is obtained directly from the meridian of Greenwich. Absolute methods are all of inferior precision, and need not be further considered. The only relative method that is now employed is the comparison of local time, as found by astronomical observations, with Greenwich mean time, as broadcast from one or more wireless stations.

Wireless Signals. Greenwich mean time signals are sent out, at stated hours, from a number of powerful transmitting stations, situated in various parts of the world. The *Admiralty List of Radio Signals*, Vol. II, published annually, gives full particulars of the time of emission, wave length, and type of signal sent out by each station. Any changes are notified in the weekly *Notices to Mariners*. Although various types of signal are sent out, the "Standard Rhythmic" signal is the one which is made use of by surveyors for longitude observations because it permits of an easy and accurate comparison between the

wireless signal and a chronometer. The data included in the list of signals which follows have been kindly supplied by the Astronomer Royal. This list only gives a limited selection from the times and stations available, and more complete details will be found in Vol. II of the *Admiralty List of Radio Signals*. In any case, as changes in these signals may take place at any time, it is always well to consult the latest issue of the volume just referred to before any programme for observation is drawn up.

Name of Station	Call Sign	Wavelength Metres	Frequency Kc/s.	Time of Signal (G.M.T.) From h	m	To h	m
Paris-Pointoise	FYP	3,307·6	90·7	8	01	8	06
,,	..	..	..	9	31	9	36
,,	..	..	..	20	01	20	06
..	..	..	..	22	31	22	36
..	TMA3	29·96	10,015	8	01	8	06
..	..	..	..	9	31	9	36
,,	FYA2	40·3	7,430	20	01	20	06
,,	..	..	..	22	31	22	36
,,	TMD	23·33	12,855	20	01	20	06
,,	..	..	..	22	31	22	36
Rugby	GBR	18,750	16	9	55	10	00
,,	..	..	..	17	55	18	00
,,	GIC	34·72	8,640	9	55	10	00
,,	..	..	..	17	55	18	00
,,	GKU3	24·09	12,455	9	55	10	00
..	..	..	..	17	55	18	00
Hamburg	—	41·15	7,290	10	54·30	11	00
Monte Grande	LQC	17·09	17,550	11	45*	11	50*
,,	LSD7	30·61	9,800	11	45*	11	50*
..	LSD2	34·68	8,650	22	45*	22	50*

* Omitting dashes.

The Standard Rhythmic signal consists of a series of Morse dots, which are sent out at the rate of 61 dots per minute of mean time, for five consecutive minutes. The zero of the series, and the end of each minute, is marked by a dash instead of a dot. The duration of a dash is about 0·4 sec., that of a dot about 0·1 sec.

Reception of Wireless Signals. The method of receiving the time signals is as follows. The surveyor must be equipped with either a mean time or a sidereal time chronometer, beating half seconds. We shall assume in the first instance that he has a mean time chronometer. Since the wireless signal consists of 61 beats to the minute, he will hear the wireless beats overtaking the chronometer beats until the two coincide. Coincidence appears to be maintained for a few seconds, and then the wireless beats go ahead of the chronometer. The surveyor must then determine which particular beat of the wireless coincided with which beat of the chronometer. To do this he counts the wireless beats; the first beat

after the dash is "one," the second beat "two," and so on. As the wireless and chronometer beats approach coincidence, he notes the correspondence. For instance, he finds the 28th wireless beat corresponds with the 4th second of the chronometer. The beats appear to coincide exactly from the 30th to the 34th wireless beat. He therefore records coincidence as occurring on the 32nd wireless beat. This beat corresponds with the 8th second on the chronometer.

The following is a record of a complete observation :—

	Rugby Time Signal			Mean Time Chronometer		
	h	m	beat	h	m	s
	9	55	32	9	54	08
	9	56	33	9	55	09
	9	57	33	9	56	09
	9	58	33	9	57	09
	9	59	32	9	58	08
Mean	9	57	32·60	9	56	08·60

The wireless beats must be reduced to seconds ; since there are 61 beats to a minute 32·60 beats equal 32·60 × 60/61 seconds, *i.e.* 32·07 seconds. Hence the Greenwich time of the mean wireless signal was $9^h\,57^m\,32^s{\cdot}07$ and the chronometer time $9^h\,56^m\,08^s{\cdot}60$. The chronometer was therefore $1^m\,23^s{\cdot}47$ slow on Greenwich mean time.

If a sidereal time chronometer is used the coincidences will occur at intervals of approximately 72 sidereal seconds.

Example. The following coincidences were observed on 1933 June 1, with a sidereal time chronometer.

	Rugby Time Signal				Sidereal Time Chronometer		
	h	m	beat		h	m	s
	9	55	16		2	36	42
	9	56	29		2	37	54
	9	57	43		2	39	07
	9	58	56		2	40	19
Mean					2	38	30 50
The mean of the Rugby signals				=	9	56	30 + 36 beats
36 beats = 36 × 60/61 seconds				=			35·41
Mean of Rugby signals				=	9	57	05·41
$9^h\,57^m\,05^s{\cdot}41$ mean time				=	9	58	43·50 S.T.
G.S.T. of 0^h on June 1				=	16	36	06·17
G.S.T. of mean Rugby signal				=	2	34	49·67
Mean of chronometer times				=	2	38	30·50
Chronometer is fast						3	40·83 on G.S.T.

It is possible to observe coincidences on the half-second beat of a chronometer as well as on the full second. But in most instruments the half-second beats do not fall exactly at the half second. It is advisable therefore to use only the full-second beats.

Equipment. Special long-wave wireless receiving sets fitted with frame aerials are made for the reception of time signals. A surveyor in any part of the world equipped with one of these sets should be able to hear the time signals from at least one station.

Precise Reception of Wireless Signals. Accuracy in the reception of time signals is best attained by reducing, or eliminating, the personal errors made by the observer in estimating the moment of coincidence. Personal errors can be eliminated by recording both the chronometer and wireless beats on a chronograph. But the recording apparatus introduces

a time lag which may be a source of serious errors. The best and simplest method is to arrange for the chronometer to short circuit the wireless set on each full-second beat. As coincidence approaches, the wireless dots are progressively shortened until at coincidence one or two are completely suppressed. In this way the exact moment of coincidence can be determined with great accuracy.

Although every precaution is taken to send out time signals at the scheduled time, small errors, amounting to a few hundredths of a second, are unavoidable. The daily corrections to be applied to the signals sent out from the principal stations are published in subsequent issues of the Admiralty *Notices to Mariners* and the *Bulletin Horaire.*

Correction for Time of Travel of Wireless Signals. The velocity of propagation of radio waves is the same as that of light, which is approximately 186,280 miles per second. Hence, if the station of observation is a long distance from the transmitting station, the observed time of reception of the signal should be reduced by 0·000537 secs. per 100 miles of distance between the receiving and transmitting stations. The distance *a* between these two stations can, if necessary, be calculated by slide rule or 4-figure logarithms from the formula

$$\cos a = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta L,$$

or from

$$\tan M = \cos \Delta L \cot \phi_2,$$

$$k = \sin \phi_2 \sec M,$$

$$\cos a = k \sin (\phi_1 + M),$$

where ϕ_1 and ϕ_2 are the latitudes of the two stations and ΔL their difference in longitude. Having found the angle a, express it in degrees and decimals of a degree, when the correction for time of travel of the waves is given direct by

$$\delta t = 0{\cdot}000372 . a \text{ secs.}$$

This correction is usually very small and may be neglected in all but first-order work.

Approximate Determinations of Time and Longitude from Wireless Time Signals. The rhythmic time signals, the use of which is described above, are designed for accurate determinations of time in which the probable error need not exceed a few hundredths of a second of time. These are to be used for accurate work but it very often happens that an approximate longitude, or an approximate Greenwich Mean Time, accurate to a second or so only, is all that is required. When this is so, the choice of stations and of signals is greatly extended, as a number of stations radiate time signals which do not involve the principle of a time vernier, but are emitted at intervals of one second of time. Accordingly, the single seconds can easily be picked out and times can be estimated to tenths of a second.

At present there is no uniform system for the transmission of wireless time signals, but, apart from the rhythmic time signals already described, the following are most commonly used :—

(1) Greenwich Time Signal (as radiated by the B.B.C.).

(2) International (ONOGO) System.

(3) Onogo (Modified) System.

(4) United States (New) System.

These different systems are fully explained in the *Admiralty List of Radio Signals*, Vol. II.

The Greenwich time signals sent out by the B.B.C. consist of six dots or "pips" marking successive seconds, the last dot indicating the exact hour. These signals are emitted from different stations and on different wavelengths throughout the day both on the Home and Overseas Services, certain emissions in the Overseas Service being directed to Africa and certain others to North America. Up-to-date information about these emissions can be obtained on application to the B.B.C. Washington, whose call sign is NSS, also sends signals throughout the day, using different wavelengths on both medium- and short-wave bands. The system employed is the United States (New) System, and the signal, which occupies five minutes, consists of a dash at every second, except that some dashes are omitted at the following times: 29s, 56s, 57s. 58s, and 59s each minute; 51s in the first minute; 52s in the second minute; 53s in the third minute; 54s in the fourth minute; and 51s, 52s, 53s, 54s and 55s in the fifth minute. Hence, the half minute 30s is marked by a single gap preceding it, and the exact minute by a long gap preceding it. The number of the dash in the group ending at 55s indicates the minutes still to be sent. The frequencies used at different times are 122; 4,390; 9,425; 12,630 and 17,000 kc/s, and the times at which these frequencies are used will be found in the Admiralty List.

The Canadian time signals also use the United States (New) System but in this case each signal is a dot instead of a dash.

Much useful information concerning the determination of longitude by receipt of wireless time signals will be found in the War Office publication *Field Astronomy*, which can be obtained from H.M. Stationery Office. This book is a most valuable one to all surveyors called upon to take astronomical observations as it contains a numerical example of the computation of almost every type of observation that the surveyor may have to make.

COMBINED DETERMINATIONS

The principal observations that serve for the determination of more than one quantity are those of ex-meridian altitudes, equal altitudes (for rough determinations only) and observations of two stars on the same vertical for time and azimuth; of equal altitudes for time and latitude, as when using the prismatic astrolabe or a theodolite set for a fixed altitude; and of transits for time, azimuth and latitude. Some of these observations can be taken to the sun, and are frequently used for approximate determinations.

In observing for time and azimuth, the routine is the same as for azimuth alone, except that the watch times of the observations are noted. The method of obtaining time, azimuth and latitude consists in observations similar to those already described in connection with primary observations for time and azimuth by star transits (pages 85 and 110), but with the addition that the elevation of the star as it crosses the vertical hair is accurately observed. In this case, at least four stars

should be observed, one above and one below the pole for azimuth and one above and one below the zenith for time. Rough determinations may be made by observing a series of circum-meridian altitudes with the corresponding watch times and horizontal angles from a referring object. The greatest observed altitude is taken as the meridian altitude, the corresponding watch time as that of transit, and the horizontal angle as the azimuth of the referring object. The accuracy is improved by plotting the circum-meridian altitudes on a watch time base, drawing a smooth curve through them, and interpolating the meridian altitude, the watch time of transit, and the required azimuth.*

Dr. De Graaf-Hunter has recently proposed two simple methods by means of which latitude, longitude and azimuth may be determined together quickly with a minimum of computation and with reasonable accuracy.† The first of these methods is intended primarily for longitude and azimuth but a latitude of moderate accuracy can also be obtained from it. It consists in setting the line of collimation approximately in the meridian (within 5° of it) and observing the altitudes and times of transit of two stars of considerably different declinations (preferably about 90°), with a small time interval between transits. In both methods, the chronometer must be compared with the wireless time signals.

Let L be an assumed longitude for which the required correction is ΔL, and let T_c be the chronometer sidereal local time of transit with all corrections applied, including, of course, that for the assumed longitude. Let ΔA be the error in azimuth due to the vertical hair of the instrument not being exactly on the meridian. Then the star will cross the vertical hair $\frac{1}{15}\,\Delta A \sin z \sec \delta$ seconds later than it will the true meridian (page 86). z here being taken plus if it is south of the zenith and minus if it is north of the zenith. Then, putting

$$t = 15\,(T_c - \text{R.A.}),$$

we have

$$t + \Delta L = \Delta A \sin z \sec \delta.$$

Writing $\omega = \operatorname{cosec} z \cos \delta$ and using suffixes 1 and 2 to differentiate between the observations to the two stars, we have

$$\omega_1 t_1 + \omega_1 \Delta L = \Delta A_1$$

$$\omega_2 t_2 + \omega_2 \Delta L = \Delta A_2.$$

Whence,

$$\Delta L = [(\Delta A_1 - \Delta A_2) - (\omega_1 t_1 - \omega_2 t_2)]/(\omega_1 - \omega_2).$$

In this expression, if the instrument has not been changed between the two observations, $\Delta A_1 = \Delta A_2$ and the term $(\Delta A_1 - \Delta A_2)$ becomes zero. If ΔA_1 is not equal to ΔA_2, the difference $(\Delta A_1 - \Delta A_2)$ will be given by the reading on the horizontal circle. In either case, ΔA_1 or ΔA_2 can be found after ΔL has been computed.

* See "Meridian Diagrams," by C. A. A. Barnes, *Min. Proc. Inst. C. E.*, Vol. CXLI.

† See *Survey of India*, War Research Pamphlets Nos. 7 and 8, or *Proceedings of the 1947 Conference of Commonwealth Survey Officers.*

The latitude is found by reducing the observed zenith distance to the true zenith distance on the meridian by means of the correction given on page 117 for observations for latitude by circum-meridian altitudes, *viz.*

$$m = \frac{2\sin^2 \frac{1}{2} T'}{\sin 1''} \cdot \frac{\cos\delta\cos\phi}{\sin z}$$

where $T' = t + \varDelta L$. Application of this correction to the observed value gives the zenith distance to be used in computing the latitude from meridian zenith distances or altitudes.

For these observations, the instrument can be set roughly in the meridian by the method of observing the rate of change of altitude of any star with respect to time that is described on page 83. In many cases, the whole of the computations can be done with slide rule or with four-figure logarithms.

This method has the advantages that it is simple to use in the field, identification of stars is easy because their observed zenith distances and times of transit give the data for the calculation of an approximate declination and R.A. which enable the star to be identified in the catalogue, and the computations can be done very quickly. Accuracy, of course, is increased by observing an even number of pairs, with change of face between each pair (not between one star of a pair and the other); by keeping the instrument fixed in azimuth between each of the observations of a pair; and by applying the correction for horizontal collimation, $c \sec \delta$, to each observed time of transit. Change of face between pairs will tend to eliminate errors due to dislevelment and lack of perpendicularity between the horizontal and vertical axes. Moreover, provided there is no change of level during the observation of a pair, the observation is not affected by bubble error, a fact which makes the method a particularly suitable one for determining the reduction from astronomical to geodetic azimuth as deduced from the difference between astronomical and geodetic longitude.

In the second method devised by De Graaf-Hunter two stars differing by about 90° in azimuth are used. Each star in turn is intersected on both horizontal and vertical hairs and the times of observation recorded, the observations being repeated on the same face a couple of minutes later. Face is then changed on the instrument and another couple of observations taken to the same pair of stars. If precision is needed, a similar series of observations can be made to other stars or to the same pair. The particular stars used can be identified from the observations by employing the method of calculating a rough azimuth from the time rate of change in altitude described on page 83. Here it is only necessary to calculate the azimuth of one star of a series as, when this azimuth has been obtained, that of the other stars can be found from the readings on the horizontal circle. The rough azimuth is best derived from stars for which $\sin A$ is less than $\frac{1}{2}$, or A is less than 30°. (A large change in A results from a small change in $\sin A$ when A is near to 90°.)

The computation of the results can be done by four-figure logarithms in the following manner: From an assumed value L of the longitude of the point of observation, the observed time of observation and the

R.A. of the star, calculate the hour angle t, Fig. 47, and then compute approximate azimuth and latitude of each star by the formulæ :—

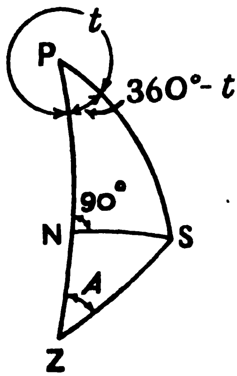


FIG. 47.

$$\sin A = -\cos\delta\sin t\sec h$$
$$\tan PN = \cos t\cot\delta$$
$$\tan ZN = \cos A\cot h$$
$$90^\circ - \phi = \gamma = PN + ZN$$

Denoting the observations to each star by the suffices 1 and 2 and the corrections to the calculated latitude ϕ, assumed longitude L and calculated azimuth A, by $\Delta\phi_1$, ΔL, and ΔA_1 respectively, we have

$$\Delta\phi_1 = (\gamma_1 - \gamma_2)\tan A_1/(\tan A_1 - \tan A_2)$$
$$\Delta L = -(\gamma_1 - \gamma_2)\sec\phi/(\tan A_1 - \tan A_2)$$
$$\Delta A_1 = \Delta L\tan A_1\cot t_1,$$

with similar expressions for $\Delta\phi_2$ and ΔA_2.

It is claimed for this method that the observation of a pair of stars takes about 15 minutes, exclusive of time spent in comparing the chronometer with the radio time signals, and that the results, including identification of stars, can be computed by an experienced observer and assistant in about 25 minutes.

In the formulæ, A is reckoned whole-circle clockwise from north, t whole-circle clockwise from upper transit and ΔL is increment of longitude reckoned positive eastwards.

These formulæ may be proved as follows (Fig. 48.) :—
The correct latitude is given by

$$\phi = \phi_1 + \Delta\phi_1 - \phi_2 + \Delta\phi_2,$$

and the corrected hour angles for the two observations are $(t_1 + \Delta t_1)$ and $(t_2 + \Delta t_2)$.

For the first observation, from the ordinary formulæ for the solution of the astronomical triangle,

$$\sin h_1 = \sin\delta_1\sin\phi_1 + \cos\delta_1\cos\phi_1\cos t_1,$$

and, h_1 and δ_1 being fixed quantities, we have for latitude $\phi = \phi_1 + \Delta\phi_1$.

$$\sin h_1 = \sin\delta_1\sin(\phi_1 + \Delta\phi_1) + \cos\delta_1\cos(\phi_1 + \Delta\phi_1)\cos(t_1 + \Delta t_1)$$
$$= \sin h_1 + \sin\delta_1\cos\phi_1\Delta\phi_1 - \cos\delta_1\sin\phi_1\cos t_1\Delta\phi_1 - \cos\delta_1\cos\phi_1\sin t_1\Delta t_1.$$

Therefore

$$\Delta t_1\cos\delta_1\cos\phi_1\sin t_1 = \Delta\phi_1\{\sin\delta_1\cos\phi_1 - \cos\delta_1\sin\phi_1\cos t_1\}.$$

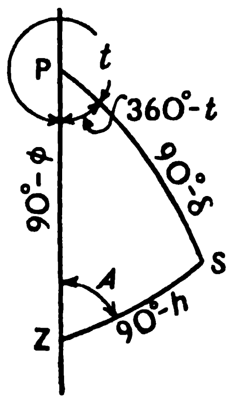


FIG. 48.

But, from equation (xxi), page 6, the expression in brackets on the right hand side of this expression is equal to $\cos h_1\cos A_1$. Also, since the same longitude is assumed in working out the hour angle for each observation, and since, for the same fixed position of the star, an increase in a clockwise or positive direction in the hour angle means an equal rotation anti-clockwise (eastwards and positive) of the meridian from which the hour angle is reckoned, $\Delta t_1 = \Delta t_2 = \Delta L$.

$$\therefore\ \Delta L = \Delta\phi_1\frac{\cos h_1\cos A_1}{\cos\delta_1\cos\phi_1\sin t_1}$$
$$= -\Delta\phi_1\sec\phi_1\cot A_1$$
$$\therefore\ \Delta L\cos\phi_1\tan A_1 = -\Delta\phi_1.$$

Here, since ΔL and $\Delta\phi_1$ are small quantities and ϕ_1 and ϕ_2 only differ by $\Delta\phi_1$ and $\Delta\phi_2$ from ϕ, we can write

$$\Delta\phi_1 = -\Delta L \cos\phi \tan A_1,$$

and

$$\Delta\phi_2 = -\Delta L \cos\phi \tan A_2.$$

$$\therefore \quad \Delta\phi_1 - \Delta\phi_2 = -\Delta L \cos\phi (\tan A_1 - \tan A_2).$$

Again, since $\gamma_1 = 90° - \phi_1 = 90° - (\phi - \Delta\phi_1)$ and $\gamma_2 = 90° - \phi_2 = 90° - (\phi - \Delta\phi_2)$,

$$(\gamma_1 - \gamma_2) = (\Delta\phi_1 - \Delta\phi_2).$$

$$\therefore \quad \Delta L = -(\gamma_1 - \gamma_2) \sec\phi/(\tan A_1 - \tan A_2),$$

and hence,

$$\Delta\phi_1 = (\gamma_1 - \gamma_2) \tan A_1/(\tan A_1 - \tan A_2).$$

Also, from the spherical triangle,

$$\sin A = \frac{\cos\delta \sin(360° - t)}{\cos h} = -\frac{\cos\delta \sin t}{\cos h}$$

$$\therefore \quad \cos A \Delta A = -\frac{\cos\delta \cos t}{\cos h}\Delta t$$

$$= \sin A \cot t \,\Delta t.$$

$$\therefore \quad \Delta A_1 = \Delta L \tan A_1 \cot t_1.$$

POSITION LINES

In addition to the analytical methods already described for deriving astronomical formulæ and computing the results, there is a graphical or semi-graphical method for determining both latitude and longitude from observations of altitudes of two or more stars, timed by means of a chronometer that has been compared with the radio time signals. This method, known as the method of "position lines," was originally applied in 1843 by Capt. T. H. Sumner of the United States Navy to finding the position of a ship at sea and is now much used in navigation. In recent years, however, it has found favour among some surveyors and will be found described in various books on surveying and field astronomy. Accordingly, we propose to give a short description of it here.

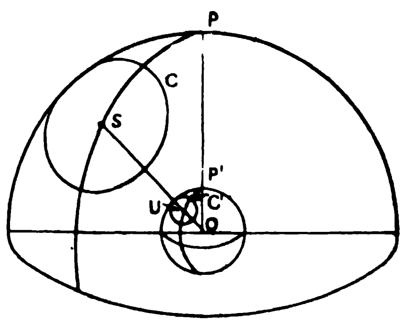


FIG. 49.

In Fig. 49, P is the position of the pole on the celestial sphere and P′ the position of the pole on the earth's surface. S is the position of a star on the celestial sphere and U the point where the line joining O, the centre of both spheres, to S cuts the earth's surface. Then it is obvious that the point S would be the zenith of an observer standing at U. The point U is called the "sub-stellar" or "sub-solar point" according as to whether S is a star or the sun.

Let z be the observed zenith distance and with S as centre describe a small circle C on the celestial sphere of angular radius equal to z. The projection of this circle on the earth's surface will be a small circle C′ having U as centre and angular radius z, and all points on the earth's surface for which the zenith distance of S is z will lie on the small circle

C′, their zeniths lying on the small circle C on the celestial sphere. The small circle C′ is called the " position circle " for the star S. Similarly, if another star is observed, we shall get another position circle and the intersection of this circle and the first one will give the position of the point of observation. Actually, since one circle intersects another in two points, there will be two places which might be the point of observation, but it is always obvious which one should be taken.

Note that, from the figure, it is obvious that the latitude of U on the earth's surface is equal to the star's declination, and the longitude of U, measured westwards from Greenwich, is the Greenwich hour angle of the observed star. In addition, the azimuth of a great circle joining any point on the position circle C′ to the point U will also be the azimuth of the star from that point.

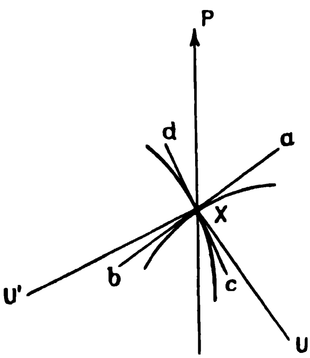


FIG. 50.

In Fig. 50, U and U′ are the sub-stellar points of two stars and X is the point of observation. A circle drawn with radius UX equal to the zenith distance of the star U will give the position circle for the first star and a circle drawn with radius U′X equal to the zenith distance of the second star will be the position circle for that star. The intersection of these two circles will therefore give the position of the point of observation. For all points very close to X the arcs of these circles will practically coincide with their tangents so that, in the immediate vicinity of X we may replace the short flat circular arcs by the tangents aXb and cXd. These two tangents are called the " position lines " of the two stars and we see that the point X is determined by the intersection of the two position lines. Moreover, if XP denotes the direction of the meridian at X, it will be seen that the azimuth of the position line aXb of star S will be the azimuth of U minus 90° and the azimuth of the position line cXd of star S′ will be the azimuth of U′ minus 90°.

Applying these principles to the case of the observation of two ex-meridian stars with azimuths about 90° apart described on page 131, take line AN, Fig. 51, as the meridian of the assumed longitude and assume that Z_1 is the position on this meridian of the point whose latitude, calculated from the observations to the first star by the formula given on page 131, is ϕ_1. Take point Z_2 such that it represents the position of the latitude ϕ_2 calculated from the observations to the second star.

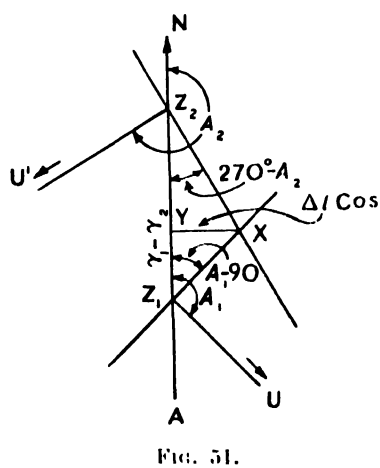


FIG. 51.

Both these points lie on the same meridian since both sets of calculations were based on the same assumed value of L.

Through Z_1 draw line Z_1X perpendicular to the direction of the azimuth of S at Z_1. Then Z_1X is the direction of the position line at Z_1 and the true position of the point of observation will lie somewhere on the line Z_1X. Similarly, if Z_2X is drawn through Z_2 perpendicular to the direction of the azimuth of S′ at that point, the line Z_2X is the position line at Z_2, and the position of the point of observation will lie on Z_2X. Hence X, the point of intersection of the lines Z_1X and Z_2X, will be the point of observation.

Through X draw a line XY perpendicular to the meridian AZ_1Z_2X. Then this line will to all intents and purposes coincide with the arc of the parallel of latitude through X, and, taking the radius of the earth as unity, the length of this parallel will be $\Delta L \cos \phi$, where ϕ is the latitude of X and ΔL is the correction in longitude given by the difference between the true longitude of X and the assumed longitude of the meridian AZ_1Z_2X. Accordingly, we write $XY = \Delta L \cos \phi$, and from the figure we have

$$\gamma_1 - \gamma_2 = Z_1Y + YZ_2 = \Delta L \cos \phi \cot (A_1 - 90^\circ) + \Delta L \cos \phi \cot (270^\circ - A_2)$$

$$= \Delta L \cos \phi (\tan A_1 - \tan A_2)$$

$$\therefore \quad \Delta L = -(\gamma_1 - \gamma_2) \sec \phi / (\tan A_1 - \tan A_2).$$

Also,

$$\Delta\phi_1 = Z_1Y = \Delta L \cos \phi \cot (A_1 - 90^\circ)$$

$$= -\Delta L \cos \phi \tan A_1$$

$$= (\gamma_1 - \gamma_2) \tan A_1 / (\tan A_1 - \tan A_2)$$

Another method of determining latitude and longitude by means of position lines is as follows :—

Assume an approximate latitude and longitude for the place of observation and calculate the approximate hour angle t, from the observed time and assumed longitude, and then calculate the approximate zenith distance z_1 from the formula

$$\cos z_1 = \sin \delta \sin \phi_1 + \cos \delta \cos \phi_1 \cos t_1$$

and an approximate azimuth from

$$\sin A_1 = -\cos \delta \sin t_1 \operatorname{cosec} z_1.$$

This will give the approximate zenith distance SZ in Fig. 52 and the approximate azimuth angle $(360^\circ - A_1)$. Let LM represent the position circle of S, this circle intersecting SZ in M. Then distance $ZM = z_1 - z$, where z is the observed zenith distance, is called the "intercept." Accordingly, we know the azimuth and the angular length of the intercept, and the position line will be a line at right angles to it.

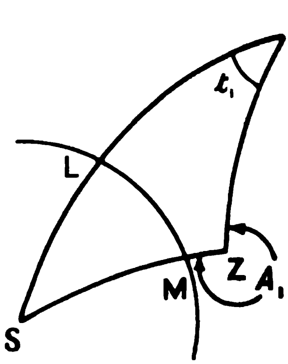


FIG. 52.

On a sheet of paper take Z (Fig. 53), as the assumed position of the point of observation and draw ZP to represent the direction of the meridian. On any suitable scale representing minutes and seconds lay off the intercept ZM, making ZM $= z_1 - z$ and angle PZM $= 360° - A_1$. Through M draw a line mMm′ perpendicular to ZM. Then this line represents the position line of the star. Similarly, for a second star, lay off the intercept ZM′ and position line M′m′, so that the latter intersects the first position line at m′. Then m′ is the true position of the point of observation.

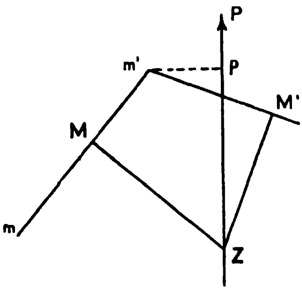


FIG. 53.

From m′ draw m′p perpendicular to ZP and intersecting ZP in p. If $\Delta\phi$ and ΔL are the corrections to the assumed latitude and longitude to give the true values, Zp will represent $\Delta\phi$ on the adopted scale of minutes and seconds and m′p will represent very approximately the parallel of latitude through m′, so that m′p $= \Delta L \cos\phi$. Hence we can write

$$\Delta L = \text{m'p} \sec\phi.$$

$$\Delta\phi = \text{Zp}$$

This method, which is the one commonly employed in navigation, may be used to obtain the most likely result from a series of observations to a number of stars. The position line of each star is plotted for the same assumed values of latitude and longitude. Owing to errors of observation, these lines will not all meet at a point but will form a closed figure (Fig. 54). The most likely position for the point of observation is then the centre of the circle which is most nearly tangential to all the different position lines.

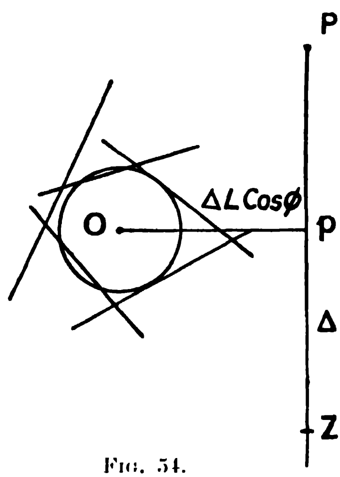


FIG. 54.

EXAMPLES

1. In a rough determination of the watch error on 1935 October 15, at a place in longitude 170° 12′ 15″ E., the sun was observed to transit at $11^h\ 50^m\ 17^s$. Find the error of the watch on L.M.T. and on New Zealand standard time, which is $11^h\ 30^m$ E. of Greenwich. E at October $15^d\ 0^h$ G.M.T. is $+ 13^m\ 54^s{\cdot}0$; change in 24^h is $+ 13^s{\cdot}7$.

2. Describe and explain the method of determining time by an ex-meridian observation of the sun, stating what data are required in the reduction.

On 1935 July 2, at a place in longitude 6° 45′ W., an ex-meridian measurement of the sun gave an hour angle of 68° 15′, the watch time of the observation being $17^h\ 05^m$. The value of the equation of time for 0^h G.M.T. on July 3 is $-3^m\ 47^s{\cdot}9$, the change being $- 11^s{\cdot}5$ per 24^h. Find the error of the watch on G.M.T.

3. On 1935 April 21, at a place in latitude 36° 55′ 08″ N. and longitude 57° 53′ 30″ E, the mean observed altitude of the sun corrected for refraction and parallax was 42° 32′ 14″·3. The mean watch time of the observation was $14^h\ 58^m\ 56^s{\cdot}8$. What was the watch error on L.M.T. ?

Date	G.M.T. of G.A.N.				Dec. at G.A.N.			
	h	m	s	s	°	′	″	″
April 19	11	59	16·95		+10	56	45·6	
				−13·36				+1246·2
20	11	59	03·59		+11	17	31·8	
				−12·93				+1235·2
21	11	58	50·66		+11	38	07·0	
				−12·50				+1223·7
22	11	58	38·16		+11	58	30·7	

4. The observed watch times of equal altitude of *α Aurigæ* (R.A. = $5^h\ 10^m\ 57^s{\cdot}8$) at a place in longitude 28° 04′ E. were $21^h\ 04^m\ 26^s{\cdot}2$ and $24^h\ 58^m\ 03^s{\cdot}4$. Find the error of the watch on L.M.T., if G.S.T. of 24^h G.M.T. for the date was $6^h\ 03^m\ 57^s{\cdot}7$.

5. In a comparison between a S.T. and a M.T. chronometer at a place in longitude 85° W., the readings at a coincidence of beat were respectively $12^h\ 45^m\ 29^s{\cdot}5$ and $22^h\ 06^m\ 07^s{\cdot}5$. The error of the S.T. chronometer was $2^m\ 14^s{\cdot}6$ slow on L.S.T. What was the error of the M.T. chronometer on L.M.T. if G.S.T. of 24^h G.M.T. for the date was $14^h\ 45^m\ 20^s{\cdot}5$?

6. On the afternoon of May 12 (5 p.m. by Greenwich mean time) at a place A, whose latitude was 51° 30′ 20″ N., the altitude of the sun's centre was found to be 23° 05′ 20″, whilst the horizontal angle between the fixed line AB and the direction to the sun's centre was 18° 20′, the sun having crossed the line over an hour before.

Determine the azimuth of the sun from the south at the time of the observation, and the azimuth of the line AB, having given :—

Correction for refraction = 58″ × tangent zenith distance.

Date	Sun's Dec. at 0^h G.M.T.			
	°	′	″	″
May 11	+17	34	53·9	
				+936·0
12	+17	50	29·9	
				+918·1
13	+18	05	48·0	
				+899·8
14	+18	20	47·8	

Sun's horizontal parallax = 8″·7.

7. The following observations of the sun were taken for azimuth of a line in connection with a survey :—

Mean time $16^h\ 30^m$. Mean horizontal angle between sun and referring object 18° 20′ 30″. The sun is west of R.O.

Mean corrected altitude 33° 35′ 10″.

Declination of sun from *N.A.* + 22° 05′ 36″. Latitude of place 52° 30′ 20″.

Determine the azimuth of the line. (Univ. of Lond., 1918.)

8. At a place in latitude 30° 08′ 17″ N. the horizontal angle between *Orionis* (δ = + 7° 23′ 32″, R.A. = $5^h\ 50^m\ 57^s{\cdot}7$) and a referring object was observed to be 84° 35′ 52″, the L.S.T. of the observation being $1^h\ 31^m\ 14^s{\cdot}2$. The star was east of the meridian, and the R.O. was nearly due south of the observer. Find the azimuth to the R.O.

9. A circumpolar star, declination + 80° 17′, right ascension $9^h\ 49^m\ 11^s$, is observed at western elongation in the evening in latitude 60° 04′ N., longitude 127° 30′ W., when its whole circle bearing from a reference line OA is found to be 207° 47′. Find the bearing of OA from the meridian, also the local mean time at which elongation is to be expected, if the G.S.T. of 24^h G.M.T. is $16^h\ 54^m\ 13^s$. The difference between sidereal and mean time intervals may be taken as 10 seconds per hour.

10. On the evening of 1920 December 23 the meridian altitude of *Polaris* was observed to be 57° 03′ 25″, and at lower transit the following morning the observed altitude was 54° 50′ 00″. Find the latitude of the place, taking refraction as 58″ cot h.

11. Describe the procedure for obtaining latitude by observing the meridian altitude of the sun.

An observation gave the meridian altitude of the lower limb of the sun as 28° 57′ 52″ looking towards the south point of the horizon. Apply the following corrections and compute the latitude of the station :—

Sun's semi-diameter . .	16′ 05″
Sun's parallax . . .	8″
Refraction . . .	1′ 47″
Sun's declination . .	−9° 17′ 30″

(Univ. of Lond., 1912)

12. Find the latitude from the following data:—
Observed meridian altitude of *α Gruis*, 40° 16′ 15″.
Declination of *α Gruis*, −47° 22′ 16″·5.
Vertical arc level readings—Eye end, 16½ divisions; object end, 15½ divisions.
Value of 1 division of vertical arc level, 10″.
Refraction may be taken as 1′ 07″·3.
Star is south of the observer. (Inst. C.E., 1915)

13. In an observation on *Polaris* ($\delta = +88°\ 53'\ 06''$, R.A. = $1^h\ 31^m\ 14^s{\cdot}8$) for latitude, the mean corrected altitude was found to be 48° 12′ 16″. The average reading of the sidereal chronometer was $10^h\ 41^m\ 04^s{\cdot}2$, the chronometer being $1^m\ 00^s{\cdot}4$ fast on L.S.T. Find the latitude of the place.

14. On a certain night the first beat of the Eiffel Tower wireless time signals was transmitted at $11^h\ 30^m\ 04^s{\cdot}15$ G.M.T., and was received by a survey party at $12^h\ 59^m\ 40^s{\cdot}28$ on a M.T. chronometer which was $3^m\ 07^s{\cdot}8$ slow on L.M.T. What was the longitude of the receiving station?

15. A time observation at A on Jan. 18 at $22^h\ 42^m$ showed a watch to be 2^m slow on L.M.T. After travelling to B, the longitude of which is 30′ 15″·0 east of A, the watch error on Jan. 25 at $22^h\ 12^m$ was found to be unchanged. Find the travelling rate of the watch. (R.T.C., 1919)

16. The record of the time determinations on a route traverse as given by the reference watch are as follows:—

Place	Date	Approx. Watch Time		Error of Watc Fast	
		h	m	m	s
A	Aug. 14	10	18	24	16·3
B	16	11	26	25	56·7
C	17	10	43	27	03·8
D	19	10	15	28	20·1
E	20	11	08	26	47·3
A	23	10	32	22	44·2

Compute the longitudes of B, C, D and E relative to A as given by this watch.

17. At A, the longitude of which is 20° 10′ 32″·3 E., an observation on July 1 showed a mean time chronometer to be $3^m\ 02^s{\cdot}4$ fast at 22^h chronometer time. On July 5, at the same place of observation, it was found to be $2^m\ 36^s{\cdot}2$ fast at 22^h chronometer time, it having been carried on a daily march in the interval. The chronometer was then transported to B, where on July 8, at 23^h chronometer time, it was found to be $1^m\ 16^s{\cdot}$ slow. Calculate the travelling rate of this chronometer and the value of the longitude of B given by the observations. (R.T.C., 1914)

18. On a certain day in June the altitude of the sun was found to be 45° 27′, the time being noted as $14^h\ 56^m\ 10^s$ G.M.T. The *Nautical Almanac* gave for that time the declination of the sun +22° 35′ 50″, and the equation of time $1^m\ 12^s{\cdot}5$, to be subtracted from apparent solar time. Refraction 1′, and parallax of sun 6″. Latitude at place of observation 51° 30′ N. Find the longitude of the place of observation. (Univ. of Lond., 1919)

REFERENCES

ADMIRALTY. *The Admiralty List of Radio Signals*, Vol II. London. Annually.
— *Apparent Places of Fundamental Stars*. London. Annually.
— "Diagram for Preparation of Star Programmes for 45° Astrolabe." No. 5170. London. 1929.
— "Notes on the Determination of the Value to be Accepted from Observations for Latitude and Longitude with the Prismatic Astrolabe." Publication No. H.D. 285. London.
— 45° Astrolabe. Diagram giving Values of Co-efficient "A." No. 5072. London. 1930.
BALL, J. "Modern Methods of Finding the Latitude with a Theodolite." *The Geographical Journal*, Vol. 49, No. 6, 1917.
— *Astrolabe Diagram*. Cairo, 1919.
BALL, J., and KNOX-SHAW, H. *A Handbook of the Prismatic Astrolabe*. Cairo, 1919.
— "The Prismatic Astrolabe." *The Geographical Journal*, Vol. 54, No. 1, 1919.

BARLOW, C. W. C., and BRYAN, G. H. *Elementary Mathematical Astronomy.* Fifth Edition. Revised by Sir Harold Spencer Jones, London, 1944.

BOSS, K. *Preliminary General Catalogue of* 6188 *Stars for the Epoch* 1900. Washington, 1915.

BOULNOIS, P. K. "Field Longitudes by Wireless." *The Geographical Journal,* Vol. 63, No. 4, 1924.

BOWIE, W. "Determination of Time, Longitude, Latitude and Azimuth." U.S. Coast and Geodetic Survey, Special Publication No. 14. Washington, 1913.

— "Study of Time Errors in Precise Longitude Determinations by the U.S. Coast and Geodetic Survey." U.S. Coast and Geodetic Survey, Special Publication No. 92. . Washington, 1923.

BROWN, R. LL. "Wireless Time Signals and Apparatus." *Report of Proceedings. Conference of Empire Survey Officers.* 1935. London.

Bulletin Horaire du Bureau International de l'Heure. Paris. Monthly.

CHAUVENET, W. *A Manual of Spherical and Practical Astronomy.* Philadelphia, 1891.

CLAUDE, A., and DRIENCOURT, L. *Description et Usage de l'Astrolabe à Prisme.* Paris, 1910.

CLIFFORD, E. H. M. "A Stellar Programme for Time Observations." *Empire Survey Review.* Vol. III, No. 21.

COMSTOCK, G. C. *A Textbook of Field Astronomy for Engineers.* New York, 1902.

COWIE, G. D., and ECKHARDT, E. A. "Wireless Longitude." U.S. Coast and Geodetic Survey, Special Publication No. 109. Washington, 1924.

CRASTER, J. E. E. "Determination of Time and Latitude by the Equal Altitude Method with the Theodolite." *Empire Survey Review.* Vol. I, No. 2.

DOOLITTLE, C. L. *A Treatise on Practical Astronomy as applied to Geodesy and Navigation.* New York, 1900.

DYSON, SIR F. "Time Keeping at Greenwich with Special Reference to Rugby Time Signals." *Report of Proceedings. Conference of Empire Survey Officers,* 1931.

EVANS, S. E. "Observing Polaris in Daylight." *Empire Survey Review.* Vol. IX, No. 69.

FRENCH, O. B. "Longitude Determinations of Great Accuracy." *Eng. News,* Vol. 71, No. 13, 1915.

Hand-book of Professional Instructions for the Trigonometrical Branch. Survey of India Department. Calcutta, 1902.

HAYFORD, J. F. *A Textbook of Geodetic Astronomy.* New York, 1938.

HINKS, A. R. "Wireless Time Signals for the Use of Surveyors." Royal Geographical Society. Technical Series, No. 3, 1925.

HORSFIELD, W., and ERRITT, W. A. "Instructions for the Use of the Prismatic Astrolabe and Programme Finder." *Empire Survey Review.* Vol. III, No. 21.

HOSKINSON, A. J., and DUERKSEN, J. A. "Manual of Field Astronomy." U.S. Coast and Geodetic Survey, Special Publication No. 237. Washington, 1947.

HOTINE, M. "Laplace Azimuths." *Empire Survey Review,* Vol. I, Nos. 1 and 2.

HUNTER, DR. J. DE G. "Surveyors' Astronomical Methods and Some Proposals." *Proceedings of 1947 Conference of Commonwealth Survey Officers.* London, 1949.

— "The Hunter Shutter Eyepiece for Longitude and Azimuth." *Empire Survey Review.* Vol. IX, No. 63.

MAXWELL, P. S. E. "The Forty-five Degrees Prismatic Astrolabe." *Report of Proceedings. Conference of Empire Survey Officers.* 1935.

MCCAW, G. T. "Astronomical Azimuths and their Introduction into Triangulation." *Report of Proceedings, Empire Conference of Survey Officers.* 1928.

— "Corrected Azimuth and Time. Two Stars in the Same 'Vertical' Plane." *Empire Survey Review,* Vol. VI, No. 39.

— "Influence of Geoidal Tilt on Azimuth." *Empire Survey Review.* Vol. III, No. 16.

— "Notes on the Observation of Altitude for Time." *Empire Survey Review.* Vol. II, No. 11.

— "Report on the Trigonometrical Survey of Fiji." Vol. II (Vanua Levu and Taveuni). London, 1917.

— "Wireless Signals." *Report of Proceedings. Empire Conference of Survey Officers.* 1928.

NANSON, M. R. C. "Tabular Azimuth." *Empire Survey Review.* Vol. VI, No. 39.
Nautical Almanac, The. H.M. Stationery Office, London. Annually.
NEWCOMB, S. *A Compendium of Spherical Astronomy.* New York, 1906.
NORTON, A. P. *A Star Atlas and Telescopic Handbook.* London, 1933.
REEVES, E. A. "A Combined Theodolite and Prismatic Astrolabe." *The Geographical Journal,* Vol. 61, No. 1, 1923.
— "A Device for Observing Stars with the Theodolite without Cross-wires or Field Illumination." *The Geographical Journal,* Vol. 64, No. 6, 1924.
— *Hints to Travellers.* Vol. II. London, 1935.
REYNOLDS, J. H. "The Naming of the Stars." *Empire Survey Review.* Vol. III, No. 16, 18 and 19.
RICE-OXLEY, M. K., and SHEARER, W. V. *Astronomy for Surveyors.* London, 1929.
SAMPSON, R. A. "Determination of Longitude by Wireless Telegraphy." *Monthly Notices of the Royal Astronomical Society,* Vol. 80, No. 7, 1920.
SCHOTT, C. A. "On the Determination of an Azimuth from Micrometric Observations of a Close Circumpolar Star near Elongation by means of a Meridian Transit, or by means of a Theodolite with Eye-piece Micrometer." U.S. Coast and Geodetic Survey, Report for 1891, Appendix No. 2. Washington.
SMITH, D. E. "Elimination of Errors of Latitude and Refraction in the Pairing of Azimuth Observations." *Empire Survey Review.* Vol. IV, No. 25.
— "On the Elimination of the Errors of Pairing Azimuth Observations." *Empire Survey Review.* Vol. II, No. 11.
Star Almanac for Land Surveyors. H.M. Stationery Office, London. Annually.
60° Star List. American Geographical Society School of Surveying. Publication No. 4. New York, 1930.
STEWART, L. B. "Certain Useful Forms for the Azimuth Formula." University of Toronto, Faculty of Applied Science and Engineering, School of Engineering Research, Bulletin No. 4. Toronto, 1924.
WADE, E. B. H. "A Method of Determining Small Differences of Latitude." Ministry of Public Works, Egypt, Helwan Observatory Bulletin No. 29, Cairo.
WAR OFFICE. *Field Astronomy.* London, 1932.
WARD, H. R. "Meridian and Latitude Determination." *Engineering News-Record,* Vol. 95, No. 1. 1925.
WRIGHT, T. W., and HAYFORD, J. F. *The Adjustment of Observations.* New York, 1906.

CHAPTER III

GEODETIC SURVEYING

Geodesy. Geodesy in its modern aspect covers a wide field, the chief divisions of the science and the subjects of investigation being: (1) Map making; (2) The dimensions and figure of the earth; (3) The mean density of the earth; (4) Variations in the force of gravity; (5) Deflections of the plumb line caused by the irregular distribution of mass on the earth's surface and in the crust; (6) The structure of the crust; (7) The intensity of magnetic force and magnitude of declination and dip over the earth; (8) Variations in latitude due to changing direction of the rotation axis of the earth; (9) Vertical movements of the crust; (10) Variations in mean sea level; (11) Tidal observations.

The great practical importance of the science of geodesy has led in all civilised countries to the maintenance of a State survey department. The primary object of such an organisation is the production of maps, but the information acquired adds to the data available for the closer determination of the figure of the earth. Every large national survey department also devotes part of its organisation to the furtherance of pure geodesy by conducting investigations on the subjects enumerated above, which, although not so immediately useful to the general user of maps, are of the greatest service in promoting accuracy in practical geodesy and in the sciences of astronomy, physics, geology, and meteorology. The data acquired by national surveys are co-ordinated by the Association of Geodesy, International Union of Geodesy and Geophysics, to which all civilised nations belong, and at whose conferences they are represented.

The civil engineer is not directly concerned with the more purely scientific branches of geodesy, but in conducting extended surveys he may have occasion to employ the methods of geodetic surveying, whether the results approach geodetic accuracy or fall far short of it.

Geodetic Surveying. Geodetic surveying has for its object the precise measurement of the positions on the earth's surface of a system of widely separated points. The positions are determined both relatively, in terms of the length and azimuths of the lines joining them, and absolutely, in terms of the co-ordinates, latitude, longitude, and elevation above mean sea level. These geodetic points form control stations to which cadastral, topographical, hydrographic, engineering and other surveys may be referred, so that their error is limited to that propagated between the geodetic stations.

The area embraced by a geodetic survey is such as to form an appreciable portion of the surface of the earth, and, in consequence, the sphericity of the earth cannot be disregarded. The reductions therefore involve a knowledge of the results of previous investigations of the dimensions of the earth.

GENERAL PRINCIPLES

In order to avoid an embarrassing accumulation of error in parts of a geodetic survey due to the great distances involved, the observations and

reductions must be performed with a much greater degree of accuracy than might appear necessary for ordinary map making. For the latter, or for any kind of extensive survey operations, the principle of working from the whole to the part (Vol. I, page 5) must be scrupulously observed. The first stage is to lay down a system of control points, whose relative positions and heights are very rigidly fixed, and then, from these, to lay down points whose positions need not be so exactly determined, until finally the survey of the detail can be made by methods which, in themselves, are relatively rough and ready.

Until fairly recently, there was only one reliable method of laying down a primary horizontal control, and that was by ordinary triangulation in which the angles of each triangle are measured and two sides calculated from a third side which is either measured or else is calculated from another triangle. In recent years two other important methods have become available. The first is by precise traverse and the other by points fixed by astronomical observations, wireless time signals being recorded for the determination of longitude. A framework control of points fixed by astronomical observations, the advantages and disadvantages of which have already been discussed on pages 47 to 51, can never be so reliable or so accurate as a good system of points established by triangulation or primary traverse. In ordinary triangulation, the only measured distances are the base lines, of which there may be only one or two, and these form a small part only of the total length of the sides or of the chain. By reason of the great difficulty which has hitherto existed in carrying out accurate linear measurements of long lines, especially over rough country, triangulation has long remained the only available method of establishing a reliable primary control for geodetic purposes. The invention of invar, and the use of long wires or thin tapes made either of steel or of invar, have, however, revolutionised the methods of making measurements of long lines in the field with the utmost accuracy. As a result, it is now possible to replace triangulation by primary traverses in many cases where triangulation would either be physically impossible or far too expensive, and these primary traverses, if carefully measured, can be made comparable in accuracy with triangulation of any but that of the highest first-order standard.

In the above, and in what follows, when we speak of triangulation it must be understood, unless expressly stated to the contrary, that we mean ordinary triangulation in which the angles are observed and two sides calculated from a measured base line or a line of known or computed length. In this kind of triangulation the lengths of the sides are controlled by the limits of visibility, it being essential that one end of each line should be visible from the other under suitable atmospheric conditions. This condition sets a limit to the maximum length of side which can be observed, this length seldom exceeding much more than about 100 kilometres, or a little over 60 miles. Recent applications of radar to navigation which were developed during the course of the last World War have, however, been adapted and extended to the measurement of the lengths of very long lines—much longer than those subject to the ordinary limits of visual observations. Consequently, a new type of

triangulation is now possible in which the lengths of the sides of the triangles, instead of the angles, are measured. The main errors in measurements by radar do not in general depend directly on the length of the line but are of much the same order for lines of different lengths. They are too large at present to justify such measurements being used for first-order geodetic control purposes for lines of ordinary length. but, when lines are very long—say 300 to 500 miles—the errors of measurement are little more in magnitude than those propagated over similar distances during the course of ordinary second-order geodetic survey. Consequently, so far as replacing ordinary geodetic methods by radar triangulation is concerned, the latter appears principally to be of use for establishing a relatively few points at great distances apart, or else to connect chains of ordinary triangulation which are separated by wide water or other gaps. Radar triangulation is also now being used to fix the position in space of an aeroplane at the instant when a photograph is taken, the error of fixation normally being too large for ordinary geodetic purposes but not too large for fixing the positions of photographs for air survey purposes. A short explanation of the principles and use of radar triangulation will be found in Chapter IX.

The method of measuring the lengths of lines by optical or electronic means referred to on page 190, and described in Appendix IV, also opens up the possibility of a type of triangulation in which the lengths of the sides, and not the angles, are measured, but here the sides are of normal length. This method, however, is just emerging from the experimental stage and no extensive use has yet been made of it, so that for the present it will still be necessary for the ordinary surveyor or engineer to establish his main control points in the normal way by ordinary triangulation or traverse, or, for less accurate work, by astronomical observations.

Relative Advantages of Triangulation and Precise Traversing. When the country involved is fairly open and is not very flat, triangulation is usually the best method to use. When, however, work has to be carried over country which is covered with forest, or is very enclosed or very flat, primary traversing will often be cheaper and quicker to execute than triangulation. Triangulation covers a definite belt or area of country, whereas a single traverse, unless it closes back on itself, merely establishes a single line of fixed points. On the other hand, triangulation points are usually fixed on tops of hills or mountains, and these may be very difficult of access to the ordinary surveyor who has to use the points later on, whereas traverses usually follow roads or low country, the stations thus being readily accessible and situated moreover in those places where they are most likely to be required. This is one very important advantage which traversing has over triangulation.

As a general rule, a very complete reconnaissance of the country must be made before a scheme for the measurement of a system of triangulation can be drawn up, and it is folly to start observing until this reconnaissance has been completed, or is nearing completion. In forest country, the reconnaissance may be extremely difficult and may involve heavy and expensive clearing before the intervisibility of distant points can be established. In this way, a considerable amount of trial clearing, part of

which will eventually be found to be completely useless for the purpose in view, may have to be done. In traversing it is usually advisable to make a preliminary reconnaissance of the route to be followed, but this reconnaissance involves no extensive clearing or other special difficulties. The line of the traverse will ordinarily follow a main road or a railway, so that travelling is easy and the examination of the country can be carried out rapidly. In addition, the main measurements can be commenced at the same time as, or very shortly after, work on reconnaissance starts, and there is seldom any doubt at the beginning as to whether the scheme will be possible or not.

In traversing, questions of supply and communication are usually much simpler than they are in triangulation, since the latter almost always involves a number of small detached parties, separated and moving at considerable distances apart. Another great advantage of traversing is that, to a very great extent, work is independent of climate, season and weather. This is not the case with triangulation, where good visibility over long distances is essential. In most countries there are certain seasons of the year when visibility is bad, and, during this period, days, weeks and even months may be wasted in waiting for suitable days for observation. This applies not only to actual observing but also to the preliminary reconnaissance, and the season in the year which is suitable for either reconnaissance or observing may be uneconomically limited. The lines measured as traverse legs are generally short compared with the lines used in triangulation and hence traversing can proceed when poor visibility rules out any possibility of sighting over long lines of triangulation.

One great disadvantage of traversing is that mistakes, both in observation and measurement and in computation, are much more easily made than they are in triangulation, and, when they are made, they are not so easy to detect or to rectify. The closing errors of the different triangles are an automatic check on the accuracy of the field work, and other automatic checks exist in the computation of triangulation. No such convenient and useful checks are available in traversing and the chances of undetected gross errors creeping in to the work are therefore far greater. For this, and for other reasons, it is generally advisable, in all ordinary circumstances, to use triangulation in preference to traversing whenever this is possible, even when it means a fairly substantial increase in the cost and an appreciable delay in getting the work completed. In the United States and in Canada, for instance, the rule is that precise traverse should only be adopted in cases where the cost of triangulation would be more than double that of precise traverse. In fact, in more recent years, the tendency has been for the United States Coast and Geodetic Survey to abandon the survey of precise traverses almost altogether. This is because the invention of the Bilby portable observing tower (page 153) has made it much easier to carry out triangulation over flat or wooded country. Another factor is that, owing to the irregular accumulation of error and the less rigid mathematical conditions, traversing has not been found entirely satisfactory as a substitute for first-order triangulation. Unfortunately, the Bilby observing tower would not be of much use in certain types of flat tropical forest country, as the forest "ceiling"

(perhaps 200 ft. high or more) would often be much higher than the tallest Bilby tower, so that, in country of this kind, precise traverse still remains the only means of establishing accurate horizontal control.

When triangulation covers only a very limited area and the sides of the main triangles are very short, such as in the survey of a large town, it is probably better to use traversing than triangulation. This is because of the way in which errors due to faulty centering or sighting tend to propagate themselves when the sides of the triangulation are short. If triangulation is adopted, and sides are uncommonly short, particular attention should be paid to the centering of the instrument and, more particularly, of the signals.

In triangulation, if the organisation is good and a good system of observing is laid down at the beginning, reasonable care and the close observance of instructions will insure good results, but, in traversing, the most meticulous care and close attention to every detail must be exercised at all times if errors are to be avoided and the utmost accuracy obtained. On this account, precise traversing probably demands a more highly skilled and better trained party, together with closer attention to detail on the part of the engineer in charge, than triangulation does.

In the following pages we shall first of all consider the field work of triangulation and in the latter part of the chapter the methods used in precise traversing.

TRIANGULATION

Triangulation is the method of location of a point from two others of known distance apart, given the angles of the triangle formed by the three points. By repeated application of the principle, if a series of points form the apices of a chain or network of connected triangles of which the angles are measured, the lengths of all the unknown sides and the relative positions of the points may be computed when the length of one of the sides is known.

The field work of any triangulation survey therefore possesses the following essential features:

(1) The selection of the stations to form a system of connected triangles;
(2) The measurement of one of the sides, known as the "base line";
(3) The observation of the horizontal angles.

In geodetic and other large surveys, there must be added the astronomical observations necessary to determine the absolute positions of the stations. The geodetic positions and the azimuths of all the lines can be computed if the latitude and longitude of one station and the azimuth of one side of the triangulation are observed. In practice, however, astronomical observations should be made at intervals throughout the survey.

Grades of Triangulation. Triangulation is classified as first-order, second-order, and third-order, or as primary, secondary, and tertiary, according to the degree of accuracy required.

First-Order or *Primary Triangulation* constitutes the highest order of triangulation, and is that employed both for surveys executed primarily for determinations of the earth's figure and to furnish the most precise

control for mapping. As it is independent of external checks, no precaution can be neglected in making the linear and angular measurements or in performing the reductions. The length of base line is from 3 to 20 miles, and that of the sides of the triangles ranges from 10 to over 100 miles. The triangular error should average less than 1 second and should not exceed 3 seconds. The probable error of computed distance will lie between about 1 in 60,000 and 1 in 250,000.

Second-Order or *Secondary Triangulation* is designed to furnish points closer together than those of the primary triangulation. It may cover extensive areas, but, as it is tied to the primary net at intervals, the operations may be conducted with rather less refinement. The lengths of the sides range between about 5 and 25 miles. The triangular error may reach 5 seconds, and the probable error of distance will vary from 1 in 20,000 to 1 in 50,000.

Third-Order or *Tertiary Triangulation* is run between the stations of the secondary system, and forms the immediate control for the detail surveys. The lengths of the sides range from less than a mile to about 6 miles. The triangular error may amount to 15 seconds, and the probable error in the computed sides usually lies between 1 in 5,000 and 1 in 20,000.

Note. The above classification is based only upon the degree of accuracy to be attained. Every survey which can be classed as geodetic involves work of first-order standard, but, on the other hand, the triangulation forming the main control of even an extensive topographical survey may be of secondary, tertiary, or lower grade.

Triangulation Schemes. For a geodetic triangulation of moderate extent the whole area of the survey may be covered by primary triangles, which are extended outwards in all directions from the initial base. In

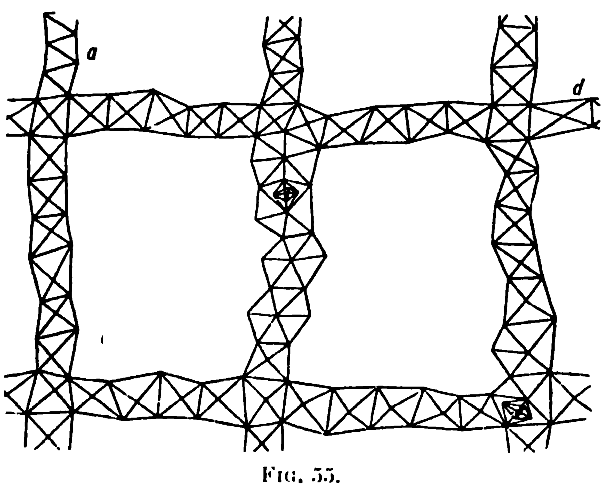


FIG. 55.

very extensive surveys, however, it is necessary to lay out the primary triangulation in two series of chains of triangles, which are usually placed roughly north and south, and east and west respectively (Fig. 55). The areas enclosed remain to be filled by secondary and tertiary triangulation.

The first, or central, system, of which a notable example is the Ordnance Survey of the British Isles, has the merit of affording close control over the secondary work. The second, or gridiron, system has, however, the important advantages that the most favourable country can be selected for the primary triangulation and the work can proceed comparatively rapidly, while the subsequent reduction is less complicated.

Chain Figures. The simplest form of chain is composed of a single system of triangles as at *a* (Fig. 55), but this arrangement, although economical, does not conduce to the requisite precision of primary work, since the number of conditions to be fulfilled in the figure adjustment is relatively small. Much better results are obtained by linking the triangles to form more complex figures. In point of accuracy, the quadrilateral with four corner stations and observed diagonals (*b*, Fig. 55) forms the best figure. When the topography is unsuited for the development of these figures, there may be substituted quadrilaterals, pentagons, or hexagons with a central station (*c*, Fig. 55), which form satisfactory figures.

In any component figure one of the unknown sides is that from which the chain is extended, and the degree of accuracy with which distance is transmitted along the chain depends upon the precision with which that side is computed from the known side. For a given standard of angle measurement, the accuracy of transmission in each figure depends upon the number of geometrical relationships which must be fulfilled by its parts and upon the size of the distance angles, or angles used in the sine ratio calculation of the one side from the other. In order that minimum effect may be produced by uncertainties in the distance angles, they should be such as have small logarithmic differences of sine, *i.e.* they should approach 90°, and should be within the limits, 30° and 150°. In the case of a chain of quadrilaterals, figures very long in the direction of the chain in proportion to their breadth, as at *d* (Fig. 55), should therefore be avoided. For a chain of single triangles the equilateral form is the most advantageous, and a lower limit of 30° should be set upon the angles.

When alternative systems of triangulation are possible, the best scheme may be arrived at by evaluating and comparing the strengths of the proposed chains * and estimating their relative costs.

Base Line Sites. The importance of having bases favourably situated necessitates careful investigation of the relative merits of possible sites. Greater latitude as to the character of the ground surface is permissible when the measurement is to be performed by tape as against rigid bars, but the chief desiderata are :

(1) The site should permit of a line, of length suitable to the requirements of the survey, being laid out over firm and smooth ground with longitudinal slopes not exceeding about 1 in 12 and with the end stations either intervisible at ground level or such that intervisibility can be secured at small expense.

(2) The surrounding country should be suitable for the development of a well-conditioned connection between the base and the main triangulation.

* See Crandall, *Text-book on Geodesy and Least Squares*, or Hosmer's *Geodesy*.

In very flat country a considerable choice of sites may be available, and the location may be made to suit a proposed scheme of triangulation. In rugged country, on the other hand, the choice may be very limited, and the triangulation must be adapted to suit the location of the base line.

Base lines can be measured with a greater degree of precision than can be maintained throughout the triangulation. The uncertainties introduced by errors of angle measurement increase as the work progresses from the initial base line, and check bases should therefore be introduced at intervals. The distance between them should be such that the discrepancy between the measured length and that computed from the preceding case is within the allowable error set for the survey. In extensive primary surveys the base interval ranges from 100 to over 200 miles. In the greater national surveys the number of primary base lines per 100,000 square miles varies from about 1 to 8.

Length of Base Lines. The length of base should depend primarily upon that of the sides of the triangulation, and should bear as large a ratio to the latter as practicable in order to minimise the loss of precision inherent in the connection of a short base to the triangulation system. The length of the great majority of existing bases lies between a tenth and two-thirds of that of the average side of the triangulation. Modern development in the use of tapes and wires has tended to increase the length of bases and some that have been measured in recent years have been anything up to nearly 20 miles in length. In most cases the length is strictly limited by difficulties of the site, but it is an advantage, when this is possible, to secure a base line that approximates in length to at least half that of the average length of side of the main triangles.

Connection of Base Line to Triangulation. In connecting the comparatively short base line to the main triangulation, badly conditioned figures must be avoided by expanding the base in a series of stages. Fig. 56 shows a strong base net connecting the base AB to the main stations G and H. The first expansion yields the line CD, the second EF, and the third the side GH of the main system.

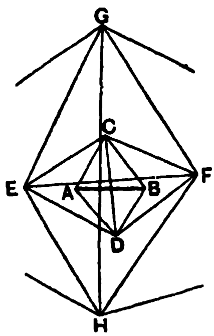


FIG. 56.

Selection of Stations. To lay out a system of triangulation with strong figures so that the survey may be conducted economically demands a careful study of the topography of the country. Stations should be placed upon the most elevated ground, so that long sights through undisturbed atmosphere may be secured at minimum expense for observing towers and signals. Commanding situations are also of advantage for the control of subsidiary triangulation and for possible future extension of the principal system. In very flat country the topography, because of its uniformity, has less influence on the exact location of the stations, and the length of the lines is limited by the cost of erecting high structures to overcome the effect of the curvature of the earth. If such country is heavily wooded, the difficulties are greatly increased, and a considerable amount of clearing may have to be undertaken or precise traversing may have to be substituted for triangulation.

To prevent difficulty of sighting and to avoid the effects of irregular atmospheric refraction, stations should be situated so that lines of sight will not pass over towns, factories, furnaces, or the like, nor graze any obstruction. They must be placed on firm ground where, as far as can be judged, they will remain undisturbed and permanently accessible.

PRELIMINARY OPERATIONS

Reconnaissance. This includes all the operations required in examining the country to be triangulated, fixing sites for base lines, selecting and temporarily marking stations, determining intervisibility, ascertaining the required height of observing towers and signals and the amount and direction of clearing. The economical execution of a triangulation is very largely dependent upon an exhaustive reconnaissance, and the work should be directed by a person whose judgment can be relied upon.

Advantage should be taken of existing maps of the region. If none is available, it will usually be necessary to undertake a rapid preliminary reconnaissance to ascertain the general location of possible schemes of triangulation suited to the topography. These schemes are examined in detail in the reconnaissance proper.

The latter is conducted as a rough triangulation and plotted as the work advances. The equipment carried should be as light as possible, and the instrumental outfit will not exceed the following : small theodolite with magnetic needle, sextant for observing from tree tops, good telescope, two heliotropes for testing intervisibility, prismatic compass, one or two aneroids for levelling, steel tape, drawing instruments and materials and foot irons and rope for climbing trees.

The highest points are occupied, and horizontal and vertical angles are observed to all salient features likely either to serve as triangulation stations or to influence their selection. To facilitate progress, considerable use is made of magnetic bearings as an aid to identifying points previously occupied, and, where convenient, fixes are made by resection. Great care must be exercised to ensure that reliable data are obtained with reference to the necessary height of stations and that stations judged to be intervisible are really so, as a mistake might seriously retard the progress of the observing party.

Plotting may be performed by protracting the angles or by computing the sides, a scale or base being obtained with sufficient approximation from astronomical observations or by subtense stadiometry. The essential features of the topography are sketched in. The possible triangulation schemes are then laid down, their relative strength and cost are examined, and a final decision is reached. Points selected for stations are flagged, and particulars of their exact location are recorded.

During reconnaissance, ample notes should be made of all circumstances likely to influence the economy and rapidity of the operations to follow. These will include information regarding access to stations, means of transport, supplies of food and water, camping ground or nearest suitable accommodation, material for building towers, etc. Descriptive sketches or photographs showing the appearance presented at each station by the surrounding country are useful as an aid to the identification of adjacent

signals, and serve to show the extent of territory which can be controlled from the station. It is useful to figure on these sketches or photographs the observed compass bearings to the more prominent hills or objects that can be seen from the station. The sketches are best made in the form of "panorama sketches."

During the last few years the aeroplane has been extensively used in Canada for making reconnaissances for schemes of triangulation in forest country and the Canadian Geodetic Survey has evolved a specialised technique for this class of work which has met with considerable success.*

Intervisibility and Height of Stations. It is commonly possible to ascertain whether proposed stations are actually intervisible by direct observation either at ground level or from tree tops or guyed ladders. If, however, intervisibility necessitates elevating the station considerably above the ground, the reconnaissance party may not be able to determine by observation the practicability of a proposed site, and the question must be decided by calculation. The height to which both instrument and signal must be raised above the ground depends principally upon the distance between the stations, their relative elevations, and the profile of the intervening country.

Distance between Stations. The deviation of the level line from the horizontal due to the curvature of the earth in a distance D amounts to $\frac{D^2}{2R}$, where R is the mean radius of the earth. Owing to terrestrial refraction, however, a line of sight is not straight, but, except in abnormal cases, is concave to the earth's surface and approximates to a circular arc of radius about seven times that of the earth. The combined effect of curvature and refraction is given by

$$h = (1 - 2k)\frac{D^2}{2R},$$

where k is the coefficient of refraction. The value of k may vary considerably (page 427), but for the present purpose should be taken as not exceeding 0·07 for sights over land and 0·08 over water, unless it is known more exactly for the locality. Taking $k = 0{\cdot}07$, then $h = {\cdot}574D^2$, where h is in feet, and D in miles. The accompanying table gives the value of h in feet for distances from 1 to 80 miles for $k = 0{\cdot}07$.

TABLE OF CURVATURE AND REFRACTION

Distance in Miles.	Curvature and Refraction in Feet.	Distance in Miles.	Curvature and Refraction in Feet.	Distance in Miles.	Curvature and Refraction in Feet.
1	0·6	9	46·5	17	165·8
2	2·3	10	57·4	18	185·9
3	5·2	11	69·4	19	207·2
4	9·2	12	82·6	20	229·5
5	14·3	13	97·0	21	253·1
6	20·7	14	112·5	22	277·7
7	28·1	15	129·1	23	303·6
8	36·7	16	146·9	24	330·5

* See *Civil Engineering*, January, 1932, and *Annual Report of the Director of the Geodetic Survey of Canada for the Fiscal Year ending March 31st, 1931.*

TABLE OF CURVATURE AND REFRACTION—*continued*

Distance in Miles.	Curvature and Refraction in Feet.	Distance in Miles.	Curvature and Refraction in Feet.	Distance in Miles.	Curvature and Refraction in Feet.
25	358·6	44	1110·9	63	2277·5
26	387·9	45	1162·0	64	2350·4
27	418·3	46	1214·2	65	2424·4
28	449·9	47	1267·6	66	2499·6
29	482·6	48	1322·1	67	2575·9
30	516·4	49	1377·8	68	2653·4
31	551·4	50	1434·6	69	2732·0
32	587·6	51	1492·5	70	2811·7
33	624·9	52	1551·6	71	2892·6
34	663·3	53	1611·9	72	2974·7
35	702·9	54	1673·3	73	3057·9
36	743·7	55	1735·8	74	3142·2
37	785·6	56	1799·5	75	3227·8
38	828·6	57	1864·4	76	3314·4
39	872·8	58	1930·4	77	3402·2
40	918·1	59	1997·5	78	3491·2
41	964·6	60	2065·8	79	3581·3
42	1012·2	61	2135·2	80	3672·5
43	1061·0	62	2205·8		

The tabular figures show the elevation above datum of a signal which at various distances can just be seen from datum level, if the intervening ground presents no obstruction, and, conversely, the distance of the visible horizon from a station of known elevation above it.

Relative Elevation of Stations. When the ground below the line of sight is level, the table may be used to give the necessary elevation of a station at known distance, so that it may be visible from another of known elevation. Thus, if h_1 (Fig. 57) is the known elevation of station A above the level ground, the distance D_1, at which the line of sight becomes tangent to the ground, is obtained by interpolation from the table, and the remaining distance, $D_2 = (D - D_1)$, is ascertained. The required elevation, h_2, corresponding to D_2, is then extracted, and, the ground level at B being known, it will be found whether it is necessary to elevate the station above the surface, and, if so, the required height of tower is obtained. If the height of neither station is fixed, then for level intervening ground the minimum total height of scaffolding, when the ground level is the same at both stations, occurs when the towers are of equal height. When the difference between the ground levels does not exceed this height, the stations should be brought to the same level: for greater differences in ground level, the minimum is secured by erecting scaffolding only at the lower station.

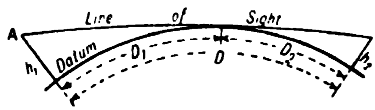


FIG. 57.

The line of sight should not graze the surface at the tangent point but should pass above the strata of disturbed air. In fixing station heights, allowance should be made for keeping it nowhere less than 6 ft. above the ground, and preferably 10 ft. in first-order work.

Example. The elevation of station A is 812 ft., and that of B, 30·5 miles distant, is 857 ft. The intervening ground may be assumed a level plain of elevation 709 ft.

Find the minimum height of signal required at B, so that the line of sight may not pass nearer the ground than 6 ft.

Minimum elevation of line of sight = 709 + 6 = 715 ft.

Taking this elevation as a datum, the elevation of A = 812 − 715 = 97 ft., which, from the table, corresponds to a tangent distance D_1 = 13 miles.

The remainder D_2 = 30·5 − 13 = 17·5 miles, in which distance the ordinate h_2 = 175·7 ft. (by calculation or interpolation from table).

∴ Minimum height of signal above ground at B

$$= 175{\cdot}7 + 715 - 857 = 33{\cdot}7, \text{ say } 34 \text{ ft.}$$

Profile of Intervening Ground. The elevation and position of peaks which might offer obstruction must be ascertained, and a comparison of their elevations with that of the proposed line of sight at the same distances will exhibit whether that line of sight is practicable. A simple solution of this problem, due to Captain G. T. McCaw, is as follows* :—

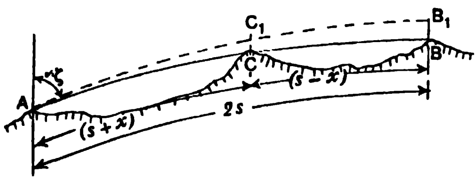


FIG. 58.

Let h_1 be the height of the standpoint A (Fig. 58), and h_2 that of the forepoint B. Let the total distance $AB = 2s$ and let the possible obstruction C occur at distance $(s + x)$ from A and $(s - x)$ from B. Then, if the zenith distance from A to B is ζ, the height h of the line of sight at C is given by :—

$$h = \tfrac{1}{2}(h_2 + h_1) + \tfrac{1}{2}(h_2 - h_1)\frac{x}{s} - (s^2 - x^2)\operatorname{cosec}^2 \zeta \,.\, (\tfrac{1}{2} - k)/R.$$

Usually $\operatorname{cosec}^2 \zeta$ can be neglected as it is approximately equal to unity, but, if necessary, it can be computed with sufficient accuracy from :—

$$\operatorname{cosec}^2 \zeta = 1 + \frac{(h_2 - h_1)^2}{4s^2}.$$

The term $(s^2 - x^2) \,.\, (\tfrac{1}{2} - k)/R$ can be obtained from the table on page 150 by using as argument the distance in miles given by $D = \sqrt{s^2 - x^2}$.

Example. The proposed elevations of two stations A and B, 70 miles apart, are respectively 516 and 1,428 ft. above mean sea level. The only likely obstruction is situated at C, 20 miles from B, and has an elevation of 598 ft. Ascertain by how much, if any, B should be raised so that the line of sight may clear C by 10 ft.

Here $2s$ = 70 miles, $s - x$ = 20 miles. Hence, s = 35 and x = + 15. $h_2 + h_1$ = 1,944, $h_2 - h_1$ = 912.

$$\therefore \tfrac{1}{2}(h_2 + h_1) = 972, \; \tfrac{1}{2}(h_2 - h_1)\frac{x}{s} = +456 \times \frac{15}{35} = +195{\cdot}4. \quad s^2 - x^2 = 1{,}000.$$

$$\therefore D = \sqrt{s^2 - x^2} = 31{\cdot}6 \text{ miles, and, from the table on page 127, } (1-2k)\frac{D^2}{2R} = 573{\cdot}1.$$

Hence, taking $\operatorname{cosec}^2 \zeta = 1$, we have :—

$$h = 972 + 195{\cdot}4 - 573{\cdot}1 = 594{\cdot}3.$$

The line of sight therefore fails to clear C by 598 − 594·3 = 3·7 ft. To clear by 10 ft. it should be raised at C by the amount 13·7 ft. and at B this becomes 13·7 × $\frac{AB}{AC}$ = 13·7 × $\frac{70}{50}$ = 19 ft. Whence, BB_1 = 19 ft., the minimum station height above the ground at B.

* *Empire Survey Review*, Vol. III, No. 21, page 428.

The relative heights of the two towers for minimum total height is best obtained by trial calculation. In the case of a single obstruction the minimum is secured by erecting a tower only at the station near the obstruction.

Observation Towers. When the station must be elevated above the ground, a rigid support is required for the instrument and the signal. It is sometimes possible to utilise church spires or other lofty structures, with or without the addition of scaffolding, but usually the erection of a masonry pier or a timber or steel scaffold is necessary.

Masonry is very suitable for small heights, but otherwise the cost is prohibitive. Timber scaffolds are most commonly adopted, and have been constructed to heights of over 150 ft. Scaffolds (Fig. 59) must be designed in the form of two towers, securely founded and efficiently braced and guyed: the inner tower forms the support for the instrument, and the outer carries a railed platform for the observing party and supports the instrument awning. The two structures must be entirely independent of each other, so that there is no possibility of vibration being transmitted from the outer to the inner. The signal is sometimes erected on the instrument tower, a position necessitating its removal when the station is occupied, and a convenient alternative is to mount it on the observer's scaffold above the instrument. On extensive surveys standard scaffold designs are worked to.*

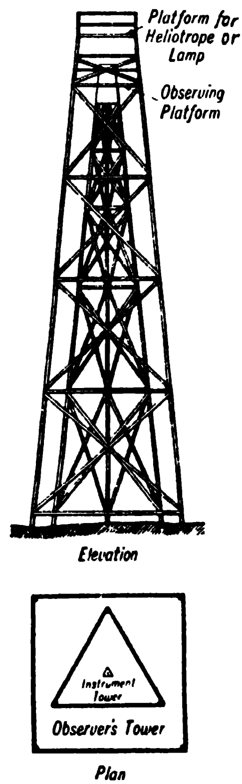


Fig. 59. Observation Scaffold.

In 1927, Mr. J. S. Bilby, of the United States Coast and Geodetic Survey, evolved a "portable" type of observing tower which is built up of steel sections and rods, and, after use at one station, can be dismantled and used at another. Towers of this type have now been in continuous use by the Coast and Geodetic Survey ever since they were invented, and in recent years they have also been used by the Ordnance Survey in the retriangulation of Great Britain. Both the inner and outer towers are tripods, and, as a precautionary measure, the outer tower is provided with three steel wire guys, attached at two-thirds the height above ground and anchored to large screw or angle-iron pickets. The maximum height for which these towers are made is 103 ft.

These Bilby towers, which are constructed in different lengths, are very easily transported, centered, erected and dismantled.†

* For examples see Reports of United States Coast and Geodetic Survey, 1903, Appendix No. 4; 1893, Appendix No. 9, pp. 406–14; 1882, Appendix No. 10; and Special Publication No. 4, 1900.

† For a description of Bilby towers see United States Coast and Geodetic Survey Special Publication No. 158.

Signals. The term signal includes any object or device used to define for the observer the exact position of a station. The various kinds may be classified as opaque or luminous. Opaque signals comprise several forms of mast or target signals, and are used for comparatively short sights. Luminous signals, including the heliotrope and heliograph for day observations and different varieties of lights for night work, are greatly used in first- and second-order triangulation. They are indispensable for long lines, but are also economical for relatively short sights through hazy atmosphere.

A signal of any class should fulfil the following essential requirements.

(1) It should be conspicuous.

(2) It should present a well-defined outline of suitable width for accurate bisection.

(3) It should be capable of being accurately centered over the station mark.

(4) It should exhibit little " phase " (page 227).

Opaque Signals. This is the usual form of signal for sights of less than about 20 miles, although under favourable conditions it may be employed for much greater distances. When the station is not elevated above the ground, the most common arrangement is to have the signal mast supported by a tripod or quadripod trestle (Fig. 60), so that the instrument may be centered over the station mark without disturbing the mast. The legs of the trestle are spiked to stakes driven well into the ground, and may be close-boarded or have canvas stretched between them to protect the instrument against the effects of sun and wind. Tall masts must be secured with wire guys.

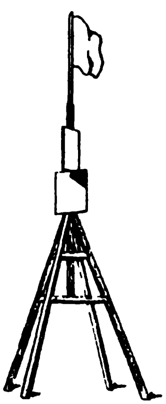


Fig. 60. Quadripod Signal.

The signal should subtend about 1·0 and 1·5 sec. at the observer, corresponding to a width of 0·3 to 0·46 in. per mile, but for short sights considerations of rigidity govern the diameter of the pole and necessitate a greater angular width. On the other hand, a mast to subtend those angles at long distances would be inconveniently heavy for erection, and beyond about 15 or 20 miles the width must be increased by nailing on thin strips of wood or targets of rectangular or diamond shape, formed of boarding or of canvas stretched on a frame. In order that targets may be visible from different directions, they should be fixed in pairs at right angles to each other.

To render a signal conspicuous, its height above the station should be roughly proportional to the length of the longest sight upon it, and usually lies between 10 and 30 ft. It should be of dark colour for visibility against the sky, and should be painted white, or in white and black stripes, against a dark background. To afford greater prominence, the top of the mast should carry a flag, a whitewashed bundle of brushwood, a tin cylinder or cone, or other device. For the longer sights, 15 to 30 miles, the upper part of a quadripod is usually boarded or thatched, as a pole and vanes may then be useless.

Except when viewed against the sky, many forms of signal permit of accurate bisection only when the sun is in the plane of the line of sight.

Under lateral illumination the signal may be partly illuminated and partly in shadow, and, since the observer sees *only the bright portion* and makes his pointing upon it, an error, termed *phase*, is introduced. The effect cannot be avoided in cylindrical signals nor with square masts unless one side of the latter directly faces the observer. In target signals it is caused by the shadow of the upper target falling on the lower. If the direction of the sun is known, the phase correction, in the simple case of a cylindrical signal, may be formulated as on page 227, and the error eliminated.

A phaseless signal is obtained by using a single target normal to the line of sight, an attendant being told off to turn it to face the observer. In the ordinary target, signal phase is reduced by making the depth of the targets great in proportion to their width and by using lozenge forms rather than rectangular. On the other hand, well-defined phase is obtained with a signal consisting of a tin cylinder to reflect the sun's rays, the pointings being made on the bright line.

The Heliotrope and Heliograph. The essential features of the heliotrope are a plane mirror to reflect the sun's rays and a line of sight to enable the attendant to transmit the reflected beam in the direction of the observing station. A very simple heliotrope may be improvised with an ordinary mirror mounted on a horizontal axis, the reflected sunlight being projected over the top of a stake or pole marking the required direction. In the usual forms of the apparatus the mirror is of worked parallel glass with rack motions about both horizontal and vertical axes, and the line of sight may be either telescopic or defined by a sight vane with an aperture carrying cross wires.

In the telescopic form (Fig. 61) the mirror is mounted on the telescope, and two rings are fixed so that the axis through them is parallel to the line of sight. The telescope being directed towards the distant station, the mirror is adjusted until the reflected beam passes centrally through the rings, as evidenced by a concentric annulus of light on the second ring, which is of smaller aperture than the first. In non-telescopic forms, the sight vane may be mounted on the base-board or tripod head supporting the mirror and at about 2 ft. from the latter, or it may be erected on a pole several yards from the station and in the correct line. In aligning the instrument, the cross wires or other mark at

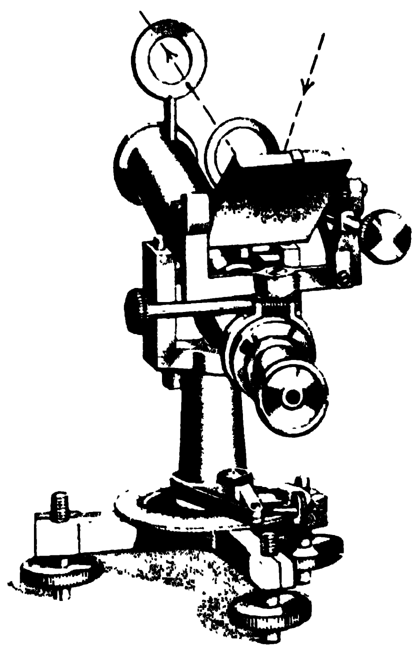


FIG. 61. TELESCOPIC HELIOTROPE.

the centre of the vane are viewed through a small eyehole in the centre of the mirror, formed by removal of the silvering, coincidence being made with the distant station. Alternatively the cross wires and the distant station may be brought into coincidence with the eyehole by viewing their reflexions in the mirror, thus setting the centre of the vane in its correct position. Where the heliograph has to be placed eccentric to the station centre, its centre and that of the vane are most correctly aligned by theodolite, the height of the vane being easily adjusted separately by direct sighting in or through the mirror.

In British Colonial practice, the ordinary military 5-in. heliograph, commonly used in the army for signalling purposes, is very often employed instead of a special heliotrope. This instrument is light and compact, and, in addition to the fittings common to most non-telescopic types of heliotrope, has a tapping arrangement which swings the mirror about its horizontal axis, thus enabling signals in Morse code to be transmitted to a distant station. Its vane is a tiny opaque pillar, with a central mark near its top, unlike the rings or board aperture with cross threads mentioned above.

Another form of heliotrope, used by the Admiralty, is the "Galton Sun Signal." This is of the telescopic variety and it is so arranged that, when the light is thrown on the area occupying the centre of the field of view of the telescope, a small image of the sun is superimposed on the image of the target within the field of the telescope. Hence, it is easy to know immediately when the flash from the "gun" ceases to fall on the target.

Accurate bisections cannot be made when the signal is too bright, and the size of mirror or aperture of vane should not be greater than will show a clear star of light. The size required is proportional to the distance between the stations, but also depends upon the clearness of the atmosphere and the quality of the observing telescope. If D is the length of sight in miles, the effective diameter or width of mirror has varied in different surveys from about 0·05 D to over 0·1 D inches for average atmospheric conditions, corresponding to an angular width of signal of about $\frac{1}{6}$th to $\frac{1}{3}$rd second. In regions of exceptionally clear atmosphere smaller signals have been successfully used. A mirror of 5 or 8 in. diameter is commonly employed, and the effective area is reduced by a diaphragm fitted on the vane or, in the telescopic form, on the outer ring. If a military type of heliograph is used, the "stops" can be rings of cardboard or thin sheet tin, with different diameters of circles cut out from the centre, fastened on the mirror.

Use of Heliotrope. The reflected rays form a divergent beam having an angle equal to that subtended by the sun at the mirror, *viz.* about 32 min. The base of the cone of reflected rays has therefore a diameter of about 50 ft. per mile of distance, and the signal is visible from any point within that base. In consequence, great refinement in pointing the heliotrope is unnecessary, as the signal will be seen provided the error of alignment is less than 16 min.

The heliotrope must be centered over the station mark, and the line of sight directed upon the distant station. Centering is facilitated by means of the "self-centering" beacon introduced by the Survey of

Egypt. The device consists of a brass casting with three radial V-shaped grooves, 120° apart, which receive the feet of the heliotrope or the levelling screws of the theodolite. When the distant station cannot be seen or its direction located within about 16 min. of arc, a reference pole is erected with the top just below the line and at least 100 ft. from the heliotrope. Flashes are sent from the observing stations to enable the direction to be established, or the pole may be lined in by theodolite, as the bearing will usually be known with sufficient accuracy for the purpose. Instead of a simple pole, a much better arrangement is one which is due to McCaw and consists of a board with a circular hole about 5-in. diameter cut in it. The board can be erected on a pole or stand, the centre of the hole, marked by cross threads, being aligned on the distant signal.*

The duty of the heliotroper consists in projecting and maintaining the beam in the proper direction. Because of the motion of the sun, the mirror must be adjusted on its axes about every minute. When the sun's rays cannot be received directly on the mirror, they must be reflected upon it from a second, or "duplex," mirror, which is provided with all forms of heliotrope. This mirror is placed in a convenient position facing the sun, and must be adjusted to follow the sun's motion. To avoid misunderstandings and delays, the observing party carries one or two heliotropes or heliographs, and a simple code of signals is adopted for conveying orders to the heliotropers.

The heliotrope can, of course, be used only in sunshine, whereas the most favourable atmospheric conditions for angle measurement occur

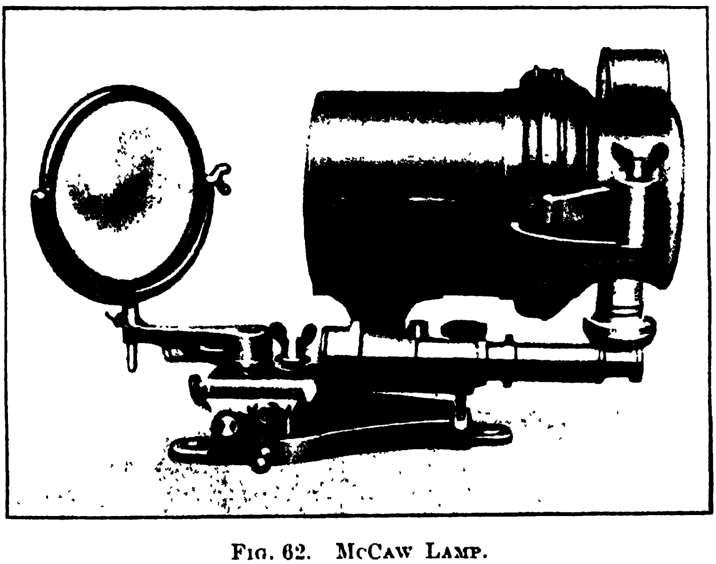


FIG. 62. McCAW LAMP.

on cloudy days. The best results with the heliotrope are obtained from observations taken towards sunset.

* *Geodetic Survey of South Africa*, Vol. III, *North-eastern Rhodesia* (1903–7), H.M.S.O. 1932.

Night Signals. The advantages of night observations (page 219) have been realised since the early days of geodetic triangulation. On the Ordnance Survey of the United Kingdom the longer lines were observed by means of the Drummond or lime light. In the Great Trigonometrical Survey of India blue lights and vase lights * were employed in the earlier work, and latterly Drummond lights and Argand reverberatory lamps with 12-in. parabolic reflectors were adopted.

Various forms of oil lamps, generally with reflectors or optical collimators, have been used for lines of less than about 50 miles. In West Africa, ordinary oil hurricane, or stable, lamps, with a flame about 1 in. wide, have been used for lines up to about 12 miles in length. For lines between 10 and 25 miles Tilley petrol vapour pressure lamps, with mantles about 1 in. long, were found to be simple in use and to give satisfactory results. Neither of these lamps was provided with a reflector. For longer lines either acetylene or electric lamps may be used. A special acetylene lamp, designed by Captain G. T. McCaw, was employed in the measurement of the arc of the meridian in Uganda in 1908–09, the longest sight being 46·6 miles. In West Africa these lamps, which have a candle power of about 600, have been observed up to 50 miles. At home, the same lamps were employed by the Ordnance Survey on the test triangulation in N.E. Scotland in 1910–11, the longest side being 46·9 miles.

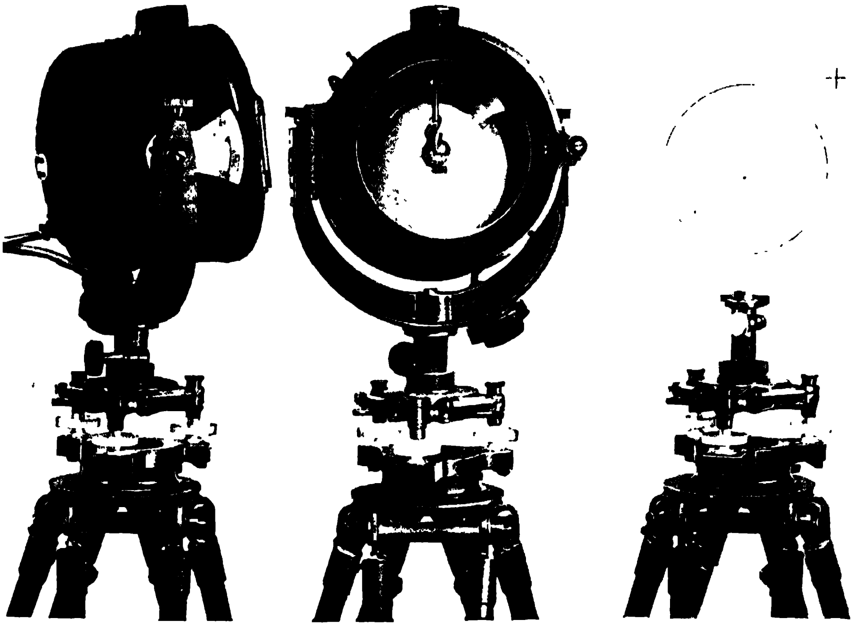
Fig. 62 shows a recent pattern of the McCaw lamp as manufactured by Messrs. E. R. Watts & Sons. It may be noted that the heliograph mirror, used for day work, is swung aside during the night. In this lamp the burner is placed at the centre of curvature of a spherical reflector at the back. In this position it is very approximately at the focus of a converging lens fixed in front of the lamp. The position of the burner is adjustable so that it can be moved along the axis in such a position that the lens throws out a slightly diverging beam of light. The gas is produced in a small metal generator, not shown in the figure, which can be connected to the burner by a length of rubber tubing. Alignment is controlled by a telescope mounted at the side of the lamp, the latter being fitted with means for providing small adjustments in altitude and azimuth. Adjustment in alignment and for width of beam can be carried out by using a special target of the type shown in Fig. 63.

Fig. 63 shows a special type of electrical beacon lamp manufactured by Messrs. Cooke, Troughton & Simms, Ltd., which has been used recently on the re-triangulation of Great Britain. It comprises a bulb, silvered in front and with a compactly wound filament, used in conjunction with a Mangin reflector, the current for the bulb being provided either by a 6-volt dry battery or by an accumulator. The bulbs are interchangeable and are supplied in four strengths—24, 12, 6 and 3 watts—so that the intensity of illumination can easily be varied considerably to suit different conditions. Motions in azimuth and altitude are provided. The setting in azimuth presents little difficulty as a target can be set out with theodolite on the correct alignment and then used to bring the beam to the correct bearing. For setting in altitude, a small vertical arc with a vernier reading to 1 minute of arc may be used. Two spirit levels, for

* See Thuillier and Smyth, *Manual of Surveying for India.*

levelling up the lamp, are fitted to the base, levelling being done by footscrews.

When the filament is in position it should be very approximately



Beacon Lamp Target

FIG. 63.

(By permission of Messrs Cooke, Troughton & Simms.)

at the point of intersection of the horizontal and vertical axes of the lamp, which point also coincides with the focus of the reflecting surface. Small sighting holes are drilled in the body of the lamp to enable this adjustment to be tested and means are provided for centering and focusing the bulbs. Further, since a very slight alteration in the position of the bulb causes a considerable alteration in the direction of the beam, a fine adjustment has to be provided to ensure that, when bulbs are interchanged, the beam maintains its direction with relation to the line of sight, defined by the open sights on the top of the lamp casing, and to the zero of the vertical arc. To test this adjustment, a special target, shown in Fig. 63, can be obtained or made. The target has a cross marked on the top and a ring in the centre indicates the size of a parallel beam, the position of the cross with respect to that of the centre of the ring being the same as the position of the sights with respect to the horizontal axis of the reflector. The target is set up on level ground at a considerable distance from the lamp and very approximately level with it. The test then consists in levelling the instrument and bringing the beam into a position where it is concentric with the ring on the target. The line of sight should then intersect the cross on the target. If it does not do so, the open sight, which is adjustable, should be moved until the line of sight

intersects the cross. If the centre of the target has been set exactly level with the centre of the reflector, and the instrument has been properly levelled, the vertical arc should read zero. If it does not, the exact reading may be taken and used as an index error.

The above tests should be made every time a bulb is changed.

From the point of view of instrument design it is interesting to notice the difference in the optical arrangements of the acetylene and electrical lamps. Both designs try to arrange for as much light as possible to be obtained from front and back of the source of light and then for this light to be thrown out in the form of an almost parallel beam. In the case of the acetylene lamp, the burner is placed at the centre of curvature of the reflector and an image of the back of the flame is thus formed coincident with, or alongside, the front of the flame. The burner is at the focus of the convex lens in front of the lamp. Hence this lens picks up the light from the front of the flame and also that from the image formed by the reflector, and, if the flame were a point of light, the beam thrown out in front would be a parallel beam.

In the case of the electric lamp, the filament is very approximately at the focus of the reflector, not at the centre of curvature. The rays from the front of the filament fall directly on the reflecting silvered surface of the front of the bulb and from this surface are reflected back on to the main reflector at the rear of the lamp. Hence, the rays from both back and front of the filament come from the approximate position of the focus of the reflector and are thrown out from the lamp in the form of an almost parallel beam of light. No lens is therefore needed with this lamp.

If the source of light in each lamp were a point source, and all adjustments were properly made, the beam proceeding from the lamp would be a truly parallel one. Neither source, however, is a point source and its dimensions in comparison with the radius of curvature and size of the reflector are quite appreciably large. Consequently, the beam which is thrown out is not a parallel beam but diverges slightly to the extent of about 5° or so. This is an advantage, provided the divergence is not too great, since it would be very difficult to align a truly parallel beam on a distant target.

For lines of over 50 miles there are available the Drummond light, the electric arc, and the magnesium lamp. The first two are not generally suitable on account of the weight of the apparatus and the necessity for the attendance of a skilled person. The magnesium lamp with parabolic reflector is very portable and easily manipulated, and gives excellent results.* The lamp burns magnesium ribbon, which is delivered by clockwork at any desired rate. The magnesium is somewhat costly for a continuous light, but the expense of this item may be reduced by showing signals intermittently according to a prearranged time-table.

Alignment of night signals is effected by means of the attached telescope or open sights or by the aid of a pole or board with circular hole, as described on page 157, set out on the proper direction and altitude.

Flare Triangulation. During the last war a method of continuing triangulation over wide gaps two to three times the length of normal

* See U.S. Coast and Geodetic Survey Report, 1880, Appendix, No. 8.

visual observation, *i.e.* from about 80 to 250 miles, by means of synchronised theodolite observations to parachute flares dropped near pre-determined positions by aeroplane was devised by Lieut.-Colonel W. E. Browne, M.B.E., R.E. The method was originally proposed for establishing a trigonometrical connection between the south coast of England and the Normandy Coast in connection with the Allied landings in France. For various reasons this was not possible at the time but, immediately the war was over, a test was made over a 90-mile wide gap separating Denmark and Norway. The results of this test proved the practicability of the method, and since then, in 1946, it has been used to establish a trigonometrical connection between Florida and the Bahama Islands.

In Fig. 64, A, B and C are three fixed trigonometrical stations and D, E and F are three others whose positions are to be fixed, the distances from A, B and C to D, E and F being too great to allow of ordinary visual

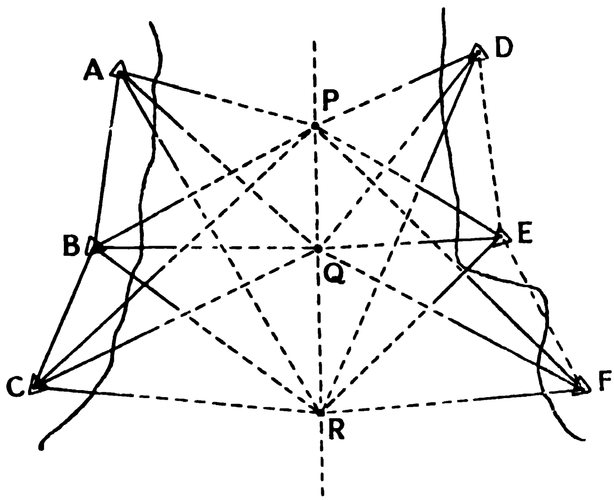


FIG. 64.

observations from the points on the one side of the gap to the points on the other. The method consists in arranging for parachute flares to be dropped in turn by aeroplane at suitable points, P, Q and R, lying on a great circle approximately half way between the fixed stations and the stations to be fixed, and taking simultaneous observations to these flares by theodolites at A, B, C, D, E and F.

For carrying out this method, the most suitable positions for the points P, Q and R are chosen and the officer in charge of the flying operations instructed to drop flares as closely as possible to these positions, radar being used to assist the pilot in determining when he is in the proper position to drop a flare. Shortly before he reaches such a position, the pilot transmits a signal by radio phone which is received by all the observers, who are all equipped with receiving sets and head phones or loud speaker. A short series of warning signals is then sent at intervals until the signal to observe is given, when each observer, who in the mean

time has been following the flare with the cross hairs in his theodolite, clamps his instrument and takes his readings. Ordinarily there is time to take two to four sets of observations to a single flare before the next flare station is reached in about ten minutes time, when the observations are taken to the second flare.

After the three flares have been observed other runs are made and new flares dropped as closely as possible to the previous positions, the complete operation being repeated with change of face and zero on each theodolite between each observation.

In the experiments in the Skagerrak, the aircraft flew at a height of about 8,000 feet and the flares ignited when at a height of about 7,000 feet. In the work connecting Florida and the Bahamas the flares were released at a height of about 20,000 feet.

Unfortunately circumstances prevented the full programme for the Norway-Denmark connection from being completed, but enough was done to prove the practicability of the method and to enable an estimate of the possible accuracy to be formed. In order to test this accuracy, three satellite stations, A′, B′ and C′ in Fig. 65, were used in positions where they could be fixed easily and accurately from the main stations by ordinary direct ground methods. Theodolites were set on these satellite stations as well as on the main stations and flare observations exactly similar to those at the main stations taken at the same times. Hence, the positions of the satellite stations could be fixed by the observations to the flares and the results compared with those obtained by ground measurements. The same procedure was used to test the accuracy of the work connecting Florida and the Bahamas when the following results were obtained :—

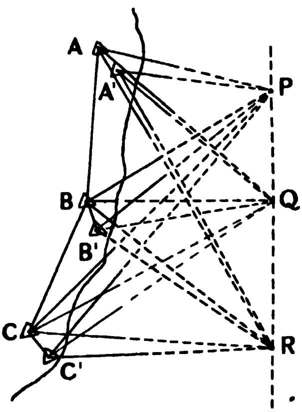


FIG. 65.

Station	Fixed Value Minus Flare Value	Fractional Misclose
A (Pisgah) . .	Lat. (N) = − 0″·082 Long. (W) = + 0″·011	1/72,000
B (Weed) . .	Lat. (N) = − 0″·026 Long. (W) + 0″·033	1/150,000
C (Congress) . .	Lat. (N) = + 0″·031 Long. (W) = − 0″·050.	1/108,000

When the positions of the stations D, E and F as fixed by the flares were compared with the positions observed astronomically, the fractional

closing errors were 1/10,000, 1/30,000 and 1/25,000 respectively. Enough has therefore been done to show that the method is of geodetic value and importance for bridging gaps which could not be bridged by ordinary ground triangulation.

It will be obvious from the description that the successful completion of a flare triangulation scheme demands most careful organisation, involving extensive equipment and a comparatively large number of trained observers. Almost perfect weather conditions are essential, and a long period of waiting for adverse weather to clear away may make the scheme exceedingly expensive in execution.

Radar Triangulation. On pages 142–143 mention has been made of a method of measuring by means of radar the lengths of lines very much longer than any that can be sighted over visually in the ordinary way. By this means, for instance, a considerable extent of country can be covered very quickly for small-scale mapping purposes by a number of very large triangles in which all the sides, and not the angles, are measured, and at the present moment the method is actually being extensively used for this purpose in Canada and Australia. In order to obtain position and orientation, this triangulation must start from widely separated points, fixed by ordinary ground methods, the distance and azimuth between these points being calculated from the known latitudes and longitudes.

The large increase in distance which is made possible by flare and radar triangulation depends on raising one end of the "line of sight" far beyond the heights of ordinary hills on the earth's surface. In both cases this is done by means of a high-flying aeroplane. In the case of flare triangulation, it is the signal—the flare—which is raised, and in the case of radar triangulation it is the recording instrument which is raised, the signals here being special electronic instruments at the ground stations which receive electrical impulses from a transmitter in the aircraft and send them back to a receiver situated alongside the transmitter.

Radar triangulation, which is somewhat above the resources of the ordinary private surveyor as it involves the use of an aeroplane and of complicated and expensive apparatus, as well as the skilled personnel required to operate it, is described at some length in Chapter IX. As will be noted later, radar techniques also have important applications to air survey and to hydrographical work.

Marking of Stations. Stations should be marked in a permanent manner to facilitate their identification for future occupation. The general methods are the same for all grades of triangulation, but various details are adopted in different surveys according to local conditions.

In solid rock it is usual to drill a hole from 3 to 6 in. deep, into which a copper or iron bolt is fixed with cement or lead. A number of arrows should be chiselled in the surrounding rock to point to the station. In ground which can be excavated the mark is made on a stone, preferably foreign to the locality, which, to prevent its being disturbed, is buried to a depth of about three feet and bedded in cement mortar. The centre mark in the stone is a small hole filled with lead, or may consist of a brass screw, an empty cartridge case, or a copper or iron pin cemented in. The sub-surface mark is covered with earth, and a similarly marked

stone is then laid with the upper surface at ground level. The whole is covered with a cairn of stones, or, as in the Survey of India, by an observing pillar of masonry or concrete.* The sub-surface mark is referred to only when there is reason to suspect that the surface mark has been disturbed. To enable the station to be recovered when the surface mark is lost, two or three reference or witness marks should be established at some little distance, and note is made of the bearing and distance of the station from each.

Station Description. Stations, besides being numbered, should be given a concise local name for convenience of reference. The record of the results of the survey should include a description of the site of each station, its marking, and the positions of the witness marks, the description being amplified by sketches as required. In addition, full particulars should be given regarding access routes, camping grounds, subsistence, etc., likely to prove of service to parties occupying the stations.

Standards of Length. The fundamental standard of length for geodetic operations is the International Metre established by the Bureau International des Poids et Mesures. It is marked on three platinum-iridium bars deposited at the Pavilion de Breteuil, Sèvres, and copies have been allotted to various national surveys. This standard differs slightly from the French legal metre, which was intended to represent 1/10,000,000 of the length of a meridian quadrant of the earth.

In Great Britain the legal unit is the Imperial Yard as defined by Act of Parliament in 1855. There are several official copies of the standard, consisting of a bronze bar with gold plugs, the distance between engraved lines on which represents the standard length at a defined temperature. The relationship between the yard and the metre has been established as

1 Imperial Yard = ·91439180 Legal metre †
= ·91439842 International metre.‡

One great disadvantage of fundamental standards of length that are made of metal or other material substances is that it has been found that all such materials are subject to very small secular changes in their dimensions. Accordingly, Maxwell, the famous physicist, suggested that the wave-length of light emitted under certain standard conditions would make a suitable standard. This proposal was first put into execution by the American physicist Michelson who proceeded to determine the number of wave-lengths of the light corresponding to the red line in the spectrum of cadmium vapour which was equivalent to the standard metre.§ He found that, in air at 15° C. and at normal atmospheric pressure, the standard metre was equivalent to 1,553,163·5 wave-lengths of the red cadmium line. In 1906 another determination was made by

* See *Handbook of Professional Instructions for the Trigonometrical Branch*, Survey of India Department, or Ordnance Survey Professional Paper, No. 2.

† Clarke, "Comparison of the Standards of Length." 1866.

‡ See note at end of Table of Constants on page 547.

§ It has recently been found that the red line of cadmium vapour is not entirely satisfactory for this purpose as it is not quite sharply defined. Consequently, it has now been suggested that the metre should be determined in terms of the wave length of a green line emitted under suitable conditions by an isotope of mercury, Hg^{198}, which gives a very sharp spectral line.

MM. Benoît, Fabry and Perot and they obtained the value 1,553,164·13, thus confirming Michelson's figure to within very close limits.

Forms of Base Measuring Apparatus. The two main types of base measuring instruments are (*a*) Rigid bars, (*b*) Flexible apparatus. The former may be divided into :

(1) Contact apparatus, in which the ends of the bars are brought into successive contacts ;

(2) Optical apparatus, in which the effective lengths of the bars are engraved on them and are observed by microscopes which serve to mark their successive positions.

According to the means adopted for reducing the uncertainties of temperature correction, rigid bars may also be classified as :

(1) Compensating, in which by a combination of two or more metals the bar is designed to maintain a constant length under changing temperature ;

(2) Bimetallic, non-compensating, in which the two measuring bars act as a bimetallic thermometer ;

(3) Monometallic, in which the temperature is either kept constant at the melting point of ice or is otherwise ascertained.

Flexible apparatus consists of (1) Steel tapes or wires, (2) Invar tapes or wires, (3) Steel and brass wires.

Comparative Merits of Rigid and Flexible Apparatus. In the earlier days of modern geodesy bars were almost exclusively favoured for base measurement, but of recent years the merits of tapes and wires have become more fully appreciated. The most far-reaching difference between the two classes of apparatus lies in the much greater length of tapes and wires. The smaller number of contacts required has the advantages that :

(1) A wider choice of base sites is available since rougher ground can be utilised, gorges, etc., of smaller width than the tape length offering no obstacle.

(2) The measurement proceeds much more rapidly, and expense is reduced.

(3) By reason of the above considerations, longer bases can be used, and bases of verification can be introduced at closer intervals.

The difficulty of ascertaining the actual temperature of the apparatus is common to all forms, except the iced bar, and is akin to the difficulty of ensuring exact compensation in compensating bars. The temperature of a steel tape cannot be measured with sufficient accuracy by mercurial thermometer except in densely cloudy weather or at night. M. Jäderin, of Stockholm, one of the pioneers in the application of steel tapes to precise measurement, found that better results could be obtained by the employment of two wires, of steel and brass respectively, to form a bimetallic thermometer. The value of flexible apparatus for geodetic measurement has, however, been greatly enhanced by the discovery of the alloy, invar, which, because of its extremely low coefficient of expansion, does away with the necessity for precise temperature determination. With the assistance derived from the increased number and length of base lines rendered possible by tape or wire measurement, the ultimate accuracy is little, if any, short of that attained by the use of rigid apparatus, other than the iced bar, which, however, on account of the time and expense it

involves, is better adapted for the calibration of apparatus than for direct use in the field. For these reasons the use of rigid bars has almost entirely disappeared and the modern practice is to measure base lines with invar tapes or wires "suspended in catenary."

Standardisation of Base Apparatus. Standardisation of base measuring apparatus consists in (1) ascertaining its length, in terms of the allotted prototype bar, under definite conditions of temperature, etc., (2) determining the laws governing its change of length under change of those conditions. Standardisation is performed by the International Bureau and also by survey and standards departments in various countries. In Great Britain it is usually undertaken by the National Physical Laboratory which issues certificates in which the determined length is guaranteed to an accuracy of one part in a million (Class A standardisation).

The utmost refinement is required in standardising apparatus for geodetic measurements. The unknown length is compared with the known standard by means of a comparator, usually in an underground chamber free from vibrations and designed to afford a steady temperature and perfect illumination for microscope reading. The comparator for bar standardisation is a heavy and elaborate apparatus carrying micrometer microscopes. For flexible apparatus it consists of a base, usually not more than 100 ft. long, having the terminals marked on invar plates. It is divided into sections, each of length equal to that of the standard bar, and micrometer microscopes are mounted at the intermediate points and the terminals. The operation of standardising a tape or wire consists in repeatedly measuring the base with it and with the standard bar alternately. Expansion coefficients are determined by immersion of the apparatus in water or glycerine heated by hot water piping and kept in circulation, temperature readings being taken through the liquid.

The National Physical Laboratory at Teddington (often referred to as the N.P.L.) has a 50-metre base for the comparison of tapes and wires used in survey work and a tank of the same length for the determination of coefficients of expansion. An interesting account of this base, and of the methods used, is contained in the section on surveying tapes and wires in Vol. III of *The Dictionary of Applied Physics*. Tapes longer than 50 metres can also be standardised at the National Physical Laboratory.

Standardisation of any form of apparatus may also be performed by measuring with it a base line, the length of which has been determined by standardised apparatus, while the coefficient of expansion may be obtained by making repeated measurements at widely different temperatures.

When sending for standardisation tapes or wires that are to be used for base measurement or precise traversing, it is advisable to send with them all the subsidiary equipment that will be used with them in the field, so that this equipment can be used during the actual laboratory standardisation. This equipment should include the weights or spring balance, and all other gear such as cords, swivels, hooks, etc., that will be used in attaching the tapes to the weights or spring balance in the field. Two or three thermometers should be standardised at the same time, and it is advisable to have the weights or spring balance tested as well. In addition, it is also always helpful to give the laboratory authorities as much information as possible about the methods and apparatus to be used and the

conditions likely to be encountered in the field. This information should include a statement of the tension to be applied, the manner in which the tapes will be supported, and, if this is known, the average approximate temperature likely to be experienced under field conditions.

The tapes or wires that are sent for laboratory standardisation should be kept and used solely as standards for field standardisations and they should never be used for ordinary field measurements. The working tapes for the actual field measurements should then be compared in the field against the standard tapes at frequent intervals and their errors determined. If it is known that the field tapes or wires are of different material from that of the standard tapes or wires, or if they are of invar and of different rollings from the standard tapes, it may be necessary, and it is always advisable, to have their coefficients of expansion determined in the laboratory.

At least two tapes or wires, and if possible three, or even four, should be properly standardised and kept solely for standardising the field tapes. If only two standards are kept, one may alter in length, either as the result of an accident or from some other unforeseen cause, and in the field it may then be impossible to know or to say which is in error. If a third tape is kept as a reserve this contingency is provided for.

Accuracy of Base Measurement. Sources of error in base measurement may be summarised as (1) constant error of standardisation of apparatus, (2) accidental field errors due to uncertainties in reading, levelling, temperature, tension, etc.

The degree of accuracy attained in standardisation limits the final precision of a base measurement, *e.g.* if the probable error of the apparatus is 1/1,000,000, no base measured by it can have a probable error as small as 1/1,000,000. The quality of the field work may be gauged by making repeated measurements of a base with the same apparatus, but the degree of consistency in the results does not indicate the real probable error, which should include all known errors, with allowances for those which cannot be rigidly evaluated.

It may be taken that 1/5,000,000 is the smallest error which can be attained in the standardisation of bar apparatus. Tapes or wires, because of their greater length, cannot be standardised in terms of the short prototype bar with greater precision than about 1/3,000,000. The certificate issued by the National Physical Laboratory does not give the probable error of standardisation, but, for a Class A certificate, merely states that the results are guaranteed to an accuracy of one part in a million. The final uncertainty of field measurement must therefore exceed these limits, and, for geodetic work, the true probable error of a base measurement, taking into account all the different sources of error, usually lies somewhere about 1/500,000. This standard of accuracy cannot be maintained in the computed sides of the survey, but is gradually lowered, by errors of angle measurement, as the triangulation proceeds from the base.

Base Bars. An account of the principal forms of rigid apparatus which have been used in various countries will be found in the references on page 392. These instruments are now mainly of historical interest, but, to illustrate their general features, one example each of compensating,

bimetallic non-compensating, and monometallic apparatus is here described.

The Colby Apparatus. This apparatus (Figs. 66 and 67) was designed by Maj.-Gen. Colby, formerly director of the Ordnance Survey, and was employed by the Ordnance and the Indian Surveys. It is compensating and optical, and consists of an iron and a brass bar, each 10 ft. $1\frac{1}{2}$ in. long, $\frac{1}{2}$ in. broad and $1\frac{1}{2}$ in. deep. These are firmly fixed together at the middle by two steel pins *1*, and the compound bar rests on rollers in a wooden box *7*, mounted on trestles in a manner permitting horizontal and vertical adjustment of the box. The bar is held in the box at the

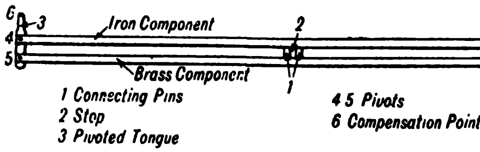


FIG. 66. COLBY APPARATUS.

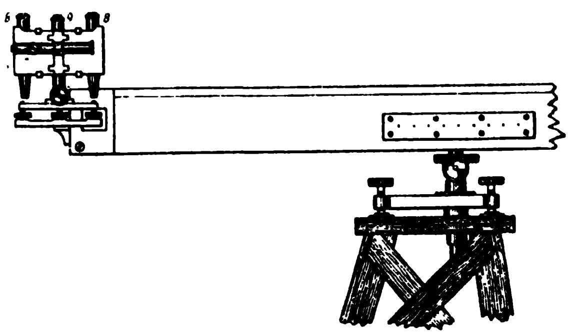


FIG. 67. COLBY APPARATUS.

middle of its length by the stop *2*, and is free to expand or contract under change of temperature. A spirit level is placed on the bar, and is observed through a window in the top of the box.

Near each end of the compound bar is a flat steel tongue *3*, about 6 in. long, which is supported by double conical pivots held in the forked ends of the bars. Each tongue carries at *6* a small platinum plug having a microscopic dot marking the effective end of the bar. To secure compensation, the ratio $6\text{–}4$ is made equal to $\frac{\text{coeff. of lin. expansion of iron}}{\text{coeff. of lin. expansion of brass}}$, so that, since the tongue is free to pivot, the position of the dot remains constant under change of temperature (Fig. 68), the distance between these reference points being 10 ft. Evidently this cannot be strictly true for all temperature changes, but the error is very small.

Six compound bars are simultaneously employed in the field. The gap between the forward mark of one bar and the rear mark of the next is made a constant length of 6 in. by using a compensated bar 7 in. long, the tongues of which carry two vertical microscopes *8*

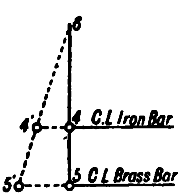


FIG. 68.

6 in. apart. A small telescope *9*, parallel to the microscopes, is fixed at the middle of this bar for use in sighting reference marks on the ground. Each compensated microscope bar is supported on the case containing the main bar, and has provision for horizontal and vertical alignment.

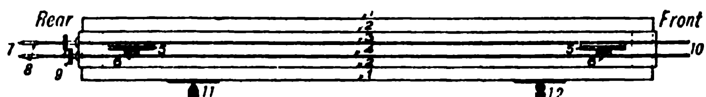
The Eimbeck Duplex Apparatus. The contact apparatus designed by Mr. Eimbeck of the U.S. Coast and Geodetic Survey, and shown diagrammatically in Fig. 69, forms a good example of a bimetallic, non-compensating instrument. The measuring components are two 5 m. long and $\frac{3}{4}$ in. diameter nickel-plated tubes, of steel and brass respectively, the metal of the latter being the thicker so that both may undergo temperature change at the same rate. These are supported within a brass tube of about $2\frac{3}{4}$ in. diameter, which is in turn enclosed within another of about 4 in. diameter mounted on trestles and carrying an aligning telescope and an inclination sector. The inner, or reversing, tube is arranged to rotate about its axis through 180° for reversing the positions of the components to minimise the influence of unequal lateral heating. The

1. Outer Truss Tube. 4 Brass Component. 7. Agate Knife Edges. 10 Agate Planes.
2. Inner Reversible Tube 5. Scale. 8 Contact Slides. 11. Knife Edge Support
3. Steel Component. 6. Vernier 9 Contact Screws. 12 Roller Support

FIG. 69. EIMBECK APPARATUS.

components project beyond the tubes at both ends, and each is fitted with contact surfaces of agate in the form of a vertical plane at one end and a horizontal knife-edge at the other. The contacts, steel to steel and brass to brass, are made by turning the screws *9*, which move the components against the action of springs.

Two bars are employed in the field, and, when the contacts are made, the forward bar remains stationary while the other is carried forward and brought into line and contact. The inner tube with the components is reversed at regular intervals. Unless the temperature happens to be that at which the components are of equal length, one will continually gain on the other, and, when the difference becomes inconvenient in making the contacts, the brass component is set forward or backward relatively to the steel. The shift is measured on scales attached near each end of the steel component, and is read through windows in the tubes against verniers on the brass component. The relative positions of the components at the beginning and end of a measurement are read on these scales.

Two measures of a base are thus obtained by means of the steel and the brass components respectively, and from the instrumental constants and the difference between the results the true measure, corrected for temperature, is deduced. The constants obtained by calibration are the expansion coefficients and the length and temperature of the components when they are of equal length. To furnish a check, each bar is provided with three mercurial thermometers, from the indications of which the base length may be obtained from the result given by either component independently of the other.

The Woodward Iced Bar Apparatus. This apparatus, designed by Prof. Woodward, formerly of the U.S. Coast and Geodetic Survey, to obviate uncertainties of temperature correction and compensation, is monometallic and optical. The bar is of steel, 5·02 m. long, 32 mm. deep, and 8 mm. broad, with the upper half cut away for a length of about 2 cm. at each end to enable the terminal lines, engraved on platinum-iridium plugs, to be placed on its neutral surface or surface of no strain. For the alignment of the bar, similar plugs, carrying a centre line, are inserted in its top surface.

The bar is supported at intervals of 0·5 m. in a Y-shaped trough, and is controlled at each support by one vertical and two lateral screws (Fig. 70). The trough is mounted on two bogies, which run on a portable narrow gauge track made in 5-m. sections. It is completely filled with crushed ice, which is kept in close contact with the bar by reason of the form of the trough and the vibration caused in rolling the bogies forward. The mounting of the trough on the bogies permits of vertical, longitudinal, and lateral movements, with fine adjustment, the gradient of the bar being given on a sector. The successive positions of the ends of the bar are marked by micrometer microscopes mounted on trestles or stout posts. The microscopes are set vertical by levelling screws, and can be given a small motion along and transverse to the line of measurement. Micrometer readings are estimated to ·0001 mm.

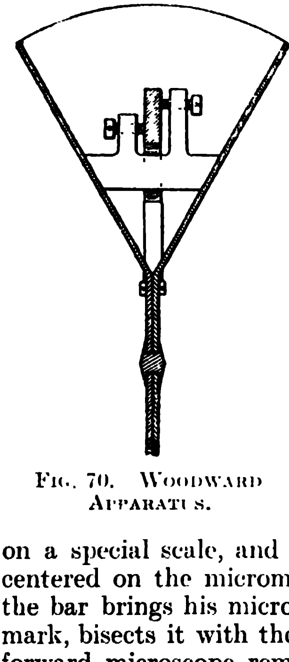


Fig. 70. Woodward Apparatus.

In measuring a base, the microscopes having been erected approximately 5 m. apart, the small distance between the base terminal and the first microscope is read through the latter on a special scale, and the bar is then adjusted until its rear mark is centered on the micrometer wires. The observer at the other end of the bar brings his microscope into position for sighting the forward end mark, bisects it with the micrometer wires, and notes the reading. The forward microscope remains stationary while the bar is rolled forward until its rear mark is focused under that microscope.

Measurement by Flexible Apparatus. Wires and tapes applied in precise measurement are usually from 24 m. to 100 m. long. They are stretched under a constant tension, and, to avoid uncertainties arising from friction and from unevenness of the ground surface, they must be suspended in the air. The most accurate method, and that adopted with the shorter lengths, is to suspend the wire or tape in one catenary between end supports. Under constant conditions of temperature, tension, etc., the straight line distance between its terminal marks is invariable, and forms the standardised measuring unit, alterations in the conditions being the subject of corrections. Long tapes are sometimes stretched over one or several intermediate supports.

The question as to whether wires or tapes are to be preferred is a

matter of opinion. Wires expose less surface to the action of wind, while tapes have the advantages that twist is easily detected and that the terminal graduations can be engraved directly on the tape, whereas a wire must carry attached scales. Wires also have a great tendency to coil up on themselves, and consequently are less easily handled and are more easily damaged than tapes, which, moreover, are less likely to take a permanent set when they are coiled on drums. Largely for these reasons most surveyors prefer to use tapes.

The increasing importance of the catenary method of precise measurement, and the now all but universal application of invar thereto, necessitates a description of the apparatus and methods usually employed.

Invar. The researches of Dr. Guillaume, of the Bureau International des Poids et Mesures, on various properties of the nickel-steel alloys, led to the discovery in 1896 that, as the proportion of nickel increases to about 36%, the value of the coefficient of expansion decreases to a minimum. This least expansible alloy has been called *Invar*. It possesses the smallest coefficient of expansion of any metal or alloy known, and its application to precise linear measurement has greatly reduced errors arising from inexact knowledge of the temperature of base apparatus.

The coefficient of expansion of invar varies in different specimens. It depends not only upon the percentage of nickel but also upon the temperature, and is influenced by the proportions of carbon, manganese, silicon, etc., present, and by thermal and mechanical treatment. By suitable treatment, invar having a negative coefficient of expansion may be produced: the value of the coefficient rarely exceeds ·0000005 per 1° F.

Owing to the way in which the coefficient of expansion of invar differs with different rollings, every invar tape should have its own coefficient determined, although, if it is known that several tapes are made from the same rolling, the coefficient of one may be determined and those of the remainder of the batch assumed to have the same value as it. This assumption, however, has been found unjustifiable in Uganda and elsewhere, and even in the same tape there may be marked absence of homogeneity. Sometimes the makers provide a small specimen, made from the same rolling as the tape, which can be used for the approximate determination of the coefficient of expansion.

Invar exhibits phenomena of change of length, both of a permanent and a temporary character, the existence of which must be recognised in base apparatus. The permanent change manifests itself in a slow increase with age which continues for years at a decreasing rate, the length of a tape or wire appearing to tend towards a definite limit. This effect is considerably reduced, although not eliminated, by subjecting the tapes or wires to a process of artificial ageing (*étuvage*), which consists in annealing them by exposure for several days to the temperature of boiling water. The temperature is thereafter slowly lowered to reach 25° C. at the end of about three months. Fig. 71 shows results obtained at the Bureau International for the increase of length of invar bars of which the final temperature of annealing was 40° and 25° C. respectively, the bars being thereafter kept at the temperature of the laboratory. Because of the comparatively rapid growth at first, it is advisable to

allow as long a period as possible to elapse between *étuvage* and standardisation of apparatus.

The temporary phenomena occur with change of temperature. When invar which has remained for some time at a certain temperature is subjected to a rise of temperature, it expands according to the coefficient of expansion, but, on continued exposure to the higher temperature, a

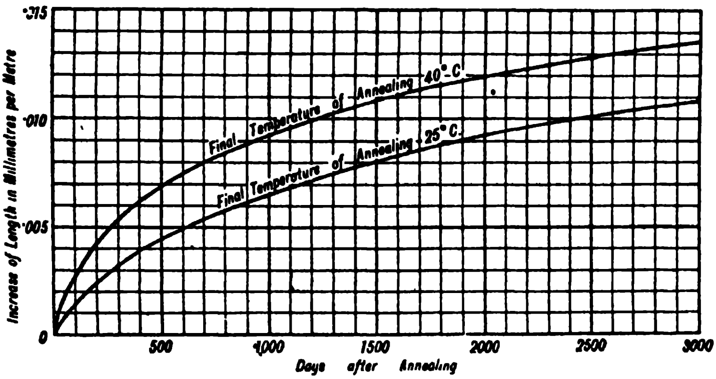


FIG. 71.

slow residual contraction or creep is exhibited. When, on the other hand, the alloy is subjected to a fall of temperature, the residual change is an expansion. For temperatures between 0° and 100° C. the creep on a length L is given by Dr. Guillaume as

$$\delta L = 0{\cdot}00325 \times 10^{-6} L(T_1^2 - T_2^2),$$

where T_1 is the original, and T_2 the new temperature in degrees centigrade. At atmospheric temperatures creep develops very slowly, and several months may elapse before the alloy assumes equilibrium under the new temperature. At high temperatures the movement is much more rapid.

The coefficient of expansion of invar tapes may also show slight variations with time and it is advisable to have the coefficients of the tapes determined both before and after an important base measurement. For this reason there is a growing tendency not to seek coefficients so nearly equal to zero, but to use a more stable material at the expense of a slightly increased coefficient of expansion. Invar stable, so called, appears to go too far in this direction.

In general, the precision with which allowance can be made for these variations is sufficient for geodetic purposes. Reference tapes or wires are standardised before and after use, and, if a considerable period elapses between the standardisations, the lengths at the time of base measurement are interpolated.

Other properties of invar are favourable to its employment in geodetic measurements. The tensile strength is 100,000 to 125,000 lbs. per sq. in., the elastic limit about 75% of the ultimate strength, and the modulus of elasticity 21 to 23,000,000 lb. per sq. in. In tests made at the

Bureau of Standards, Washington, no permanent set was detected under loads of at least 1/8th of the ultimate strength.

Care of Invar Tapes and Wires. Invar is much softer than steel and needs very careful handling if kinks are to be avoided. Particular care should therefore be taken when unrolling a tape or wire from its drum and when rolling it back again. Sudden jars should be avoided and great care taken when the unrolled tape or wire is being carried forward from one bay to another. Tension, whether by weight or by spring balance, should be slowly and carefully applied. If weights are used, they should be lowered very gradually.

The drums on which the tapes or wires are wound should be of as large diameter as possible. Generally, a drum of about 10-in. interior radius is provided by the makers, but Hotine considers that in most cases rolling on such a small drum may cause undesirable stresses due to the bending of the tape, and, for a tape $\frac{1}{4}$ in. by $\frac{1}{50}$ in. cross section, he recommends a drum of about 4 ft. diameter.*

After use, tapes and wires should be well greased with a non-corrosive grease or vaseline and they should be well cleaned before use so that their weight is not altered in any way by oil or grease that may be adhering to them. They should never be tied up at the ends with string, as the effect of the moisture in the string may be to cause corrosion in the body of the metal, and it may be very difficult to detect this corrosion from the appearance of the surface. Accordingly, the end of the tape should be fastened to the drum by a stout leather strap or thong.

Measurement by Wires or Tapes in Catenary. In this method, introduced

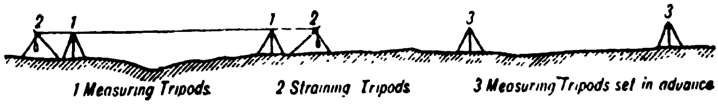


FIG. 72.

by Jäderin, the successive intervals between a series of movable marks are measured by a wire or tape freely suspended and subjected by means of straining tripods to a constant tension (Fig. 72). The movable marks are carried on tripods, which are set in alignment at approximately correct intervals in advance of the measurement, and the elevation of each is recorded for reduction of the measurement of each span to the horizontal.

As the employment of invar wires or tapes on this system yields the best results which can be attained by flexible apparatus, the following description relates to their use.

Apparatus. *Wires and Tapes.* Wires are employed in the apparatus, designed by Dr. Guillaume and M. Carpentier, which was much used in the years immediately following the introduction of invar. The wires are of 24 m. nominal length by 1·65 mm. diameter, and weigh 17·32 gm. per metre. They have attached at each end a *réglette* or scale, graduated to millimetres, the reading edge of which lies in the prolongation of the

* See "Theory of Tape Suspension in Base Measurement," by Major M. Hotine, R.E., in the *Empire Survey Review*, Vol. V, No. 31, January, 1939.

axis of the wire (Fig. 73). Both scales read in one direction. When not in use, the wires are wound on a drum, of 50 cm. diameter, mounted in a box.

Invar tapes for simple suspension are usually either 24 m. or 100 ft. long with a cross section of 3 mm. by 0·5 mm. In recent years, tapes 300 ft. long have been successfully used in some of the British Colonies but these have been supported at one or more equally spaced intervals. The metric tape is usually divided to millimetres for a length of 10 cm. at each end, and in the case of the 100 ft. tape the end scales are usually divided either to 1/25 in. or 1/100 ft. In the measurement of the bases used in the survey of the East African Arc of Meridian in 1931 Hotine used 100 ft. tapes in which 0·4 ft. at each end was graduated to 0·002 ft.,

FIG. 73. *Réglette.*

readings being estimated and booked to tenths of divisions, or to 0·0002 ft.; these readings were taken with the aid of a magnifying glass attached to the measuring tripod. The lengths were 99·6 ft. between zeros and the scales at both ends read outwards, to obviate errors of sign.

On some forms of measuring tripods makers supply small micrometers which clamp on to the small upright pillar or cylinder that forms the head of the tripod. The small difference in length between the index mark on the tape and the mark on the top of the cylinder can then be read by means of the micrometer. The use of these instruments, however, necessitates very exact setting out so that the distance between tripods is very approximately equal to the length of the chord joining the end marks on the tape. Micrometers of this kind are specially useful for field standardisations, but, apart from this, careful visual readings of end differences on the scales engraved on the tapes are more convenient than micrometer readings and are sufficiently accurate for all practical purposes, since errors in end readings tend to cancel out while those of standardisation are cumulative in their effect.

To allow for the possibility of injury, as well as to afford checks on the work, two or three of these wires or tapes are required for use in the actual measurement. As many more are reserved as reference tapes exclusively for periodical standardisation of the field tapes during the measurement. Whether wires or tapes are employed, a short, fully divided tape is required for measurement of the distance between the last tripod and the base terminal. It may also be necessary to provide one or more wires or tapes of twice or three times the normal length for spanning gorges, etc. but abnormally long unsupported spans should be avoided wherever possible.

Measuring Tripods. The Guillaume tripods (Fig. 74), usually ten in number, carry a small upright cylinder upon which is engraved the line to which measurement is made. The cylinder is set vertical by means of levels and foot screws, and is capable of fine adjustment, relatively to the tripod, for alignment. Fig. 75 illustrates the best form of cylinder head, half being cut away so that the surface of the tape or the face of

the *réglette* may lie in the plane of the reference mark. To prevent fouling and to allow for the slope of the tape when hanging in its natural curve, it is well to have the upper surface of the cut-away portion chamfered off slightly away from the centre and in the same direction as the cut. When the day's work is finished, or when it is otherwise necessary to transfer the tripod mark to the ground, a plumb bob may be used for the purpose, but it is preferable to employ two theodolites at right angles, the sights being taken on a fine needle held on the mark. Some forms of apparatus, such as the "Macca" base line measuring equipment manufactured by Messrs. Cooke, Troughton & Simms, are provided with special "transferring heads" which can be used for transferring the tripod mark to the ground. In the Zeiss apparatus, an "optical plummet," which fits on the tribrach of the measuring tripod and replaces the cylinder carrying the mark, is provided for this purpose.

Another useful accessory to the measuring tripod is an aligning and

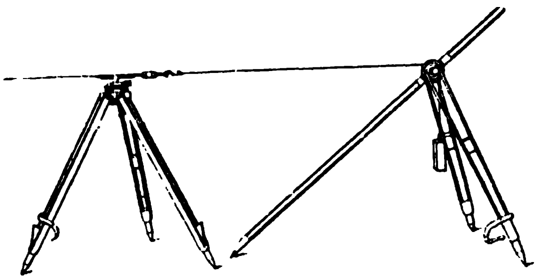


FIG. 74. MEASURING AND STRAINING TRIPODS.

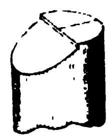


FIG. 75. REFERENCE MARK.

levelling telescope which can be slipped over the vertical cylinder, the latter thus becoming the vertical axis of the telescope. The telescope can then be used for aligning the tripods, and, if fitted with a good bubble and a short graduated vertical arc, together with a vernier arm, it can also be employed for observing slopes for the determination of slope correction and heights. When a telescope of this kind is provided for measuring vertical angles, one or two small sighting vanes are also provided. These slip over the vertical cylinder on the next tripod that is being sighted, and the height of the sighting mark above the vertical cylinder is the same as that of the levelling telescope. Consequently, the use of this target avoids any height of instrument or height of signal correction when reducing the vertical angles.

Straining Apparatus. Straining of the wire or tape is effected by weights which are carried at each end by a cord attached through a swivel hook to the ring of the wire or tape. They are suspended from two straining tripods (Fig. 74), each of which carries a pulley on ball-bearings. The tension should be not less than 20 times the weight of the wire or tape. The Guillaume-Carpentier 24-m. wires are strained by weights of 10 kgm. In the measurement of the Lossiemouth base line by the Ordnance Survey the 100-ft. tapes were given a tension of 20 lb. and the same weight was used in the measurement of the base on the East African Arc of Meridian. In West Africa, tapes 300 ft. long, suitably

supported and of section $\frac{1}{4}$ in. by $\frac{1}{25}$ in., have been used with a tension of 30 lb. It is important that the same straining apparatus should be used in field standardisations as in the actual measurement because of uncertainty arising from friction in the pulleys. It is also a great convenience if the straining trestles are provided with means of altering the height of the pulley relative to the tripod which carries it and of adjusting it laterally in a direction at right angles to the line of the tape.

Miscellaneous. If no special aligning telescope is provided with the equipment, a theodolite will be required for the alignment of the measuring heads, and, if no other means exist for transferring to the ground the mark on the measuring tripod, a second theodolite should be carried so that these transfers can be made independently of the plumb line.

The levelling observations are made either by a level and a light staff, fitted with a rubber pad for contact with the tripod heads, or by the levelling telescope already described.

Other items of equipment include a spacing tape or wire, usually of steel, for setting out the tripods, thermometers, reading glasses and iron pickets with movable heads for marking section terminals. About six thermometers in all should be provided, at least two of these having been standardised in a properly equipped standardising laboratory. These standardised thermometers should be kept for use as standards only, and the remainder, which are to be employed in the field measurements, should be compared with them at frequent intervals and their errors noted. These comparisons can best be made in water at different temperatures.

Field Work. *Preliminary Operations.* The ends of the base are defined in the usual manner by a sub-surface and a surface mark, as well as by reference points: the surface mark for base terminals is sometimes elevated on a pillar of concrete about 3 ft. above the ground. The line of the base must be cleared of trees and bushes, and a series of line marks is accurately established at intervals of about half a mile to control the alignment of the tripods.

Organization of Field Parties. The personnel is divided into two parties:

(1) A setting-out party, whose duty is to place the measuring tripods in alignment in advance of the measurement, and at as nearly as possible the correct intervals. This party consists of two surveyors with a number of porters for carrying forward tripods.

(2) A measuring party of two observers, recorder, leveller, and staff-man, with porters for the transport of apparatus. The duties of recorder and leveller are sometimes combined.

Setting Tripods. If the base terminal is not elevated above the ground, the first tripod must be set with its fiducial mark vertically over the station mark. Thereafter the tripods are aligned by a theodolite centered over one of the line marks previously established along the base, or by the aligning telescope. The setting out of the approximately correct intervals is performed by means of the spacing wire or tape, which is used at the tension adopted for the measuring wires or tapes, the pull being measured by a spring balance. Each tripod when placed at the proper distance has the head set level and is finally adjusted into alignment.

The setting of the tripods proceeds at about the same rate as the

measurement. Fresh tripods are carried forward from the rear, but one, at least, is always left behind the measuring party for reference in case of disturbance of the tripods under measurement.

Measurement. The series of operations involved in measuring the distance between each pair of tripods must be executed on a uniform system to avoid delays. The first measurement in each bay is that of the difference of level between the tripod heads. If it is made by level, the instrument is set up equidistant from the tripods, and readings are estimated to 0·001 ft. With the aligning and levelling telescope, the angle of slope is read to the nearest graduation on the vernier, and is obtained as the mean of a forward and back reading.

While the levels are being observed, the wire or tape, weights, and straining tripods are brought forward from the preceding span. The latter are placed at the proper distance from the measuring tripods and as nearly as possible in the correct line. The swivel hooks fastened to the cords over the pulleys are then handed to the observers, who attach them to the wire, and at the word of command the men carrying the weights simultaneously hook them to the cords as gently as possible. To avoid lateral friction of the cords in the pulleys, the latter must lie in the vertical plane of the base. The straining trestles are finally adjusted for line and level to bring the scales into gentle contact with, and in the plane of, the reference marks on the measuring tripods. Before reading, an examination is made for twist of the wire or tape.

The scale distance to be applied to the known length between the zeros is obtained as the mean of an even number of pairs of readings, usually six to eight, taken by the observers in simultaneous pairs, the wire being displaced longitudinally between each pair, first in one direction and then in the reverse direction. In order to avoid possible errors due to eccentricity of the pulleys of the straining trestles, it is advisable to rotate the pulley through approximately equal angles between each set of readings, so that the cord supporting the weights lies in turn along different parts of the rim of the pulley. Readings are taken with the aid of magnifying glasses to one-tenth of a scale division, and are noted by the recorder, who at the same time tabulates the differences. The maximum range in the differences should not exceed 0·3 mm., otherwise further readings are required. The record may be arranged as follows :

Span No.	Wire No.	Readings in mm. Rear.	Readings in mm. Forward.	Difference.	Mean.	Temp. Fahr
23	2	16·5	11·4	− 5·1	− 5·18	53°·5
		28·2	23·0	− 5·2		
		42·4	37·3	− 5·1		
		58·7	53·5	− 5·2		
		65·1	59·8	− 5·3		
		72·9	67·7	− 5·2		

The temperature during the measurement of each bay is read by the recorder. If two or more field wires are used, the same procedure

is repeated with another wire before lifting the straining tripods. When the measurement of a bay is completed, the weights are simultaneously unhooked, and the apparatus is carried forward for the next span. The wires must, of course, not be allowed to drag on the ground, and should be carefully guarded against accidental overstrain, so that the observers should themselves take part in their transport.

To reduce errors of personal equation, the observers change ends after every tenth span, or oftener. Each day's work forms a section, the end of which is marked by transferring the last tripod mark to a picket set in the ground. Each section must be measured at least once in each direction, the magnitude of the discrepancies indicating whether further measurements are required. Work must be suspended when wind causes appreciable vibration of the scales.

Sometimes, when wind is likely to be troublesome, it is useful to employ a long low canvas or grass screen which can be erected alongside the tape to protect it from the effects of the wind. In the Sudan, wind rendered long tapes useless.

Standardisation of Field Wires and Tapes. The reference or standard wires or tapes which are provided for comparison with the field wires or tapes should be standardised before they go on service and again on their return. The expansion coefficients of the field wires must also have been determined, but the lengths of the latter are determined in the field before and after the measurement of the base and at intervals during its progress. As a medium of comparison, a short reference base may be laid out in the field in a convenient position. The length may be one or several wire lengths, and can be marked by fine lines on metal plugs firmly bedded in concrete. Generally, however, it is more convenient to use two measuring tripods as a base, and to measure the distance between the marks on them several times with the reference wire, and then to compare this distance with that obtained from the field wires. If tripods are used in this way, comparison between standard and field wires may be made before the beginning and at the end of each day's work.

Use of Long Tapes. In the course of a measurement by the system just described, if roughness of the ground necessitates the use of a long wire or tape, a greater pull is required, and the consequent increase of friction in the straining pulleys is a source of uncertainty affecting the value of the actual tension. If possible, one or more intermediate supports should be provided.

The system of using long tapes supported at intervals has been greatly used in American base measurements. The tapes are 50 m. and 100 m., or 300 ft., long by about 6 mm. by 0·5 mm., and supports are provided at regular intervals of 10 m. to 25 m. A series of trestles or, more usually, firmly driven posts, about 4 in. by 4 in. are set out in the line of the base approximately a tape length apart. On the top of each is nailed a plate of copper or zinc on which is to be marked the position reached by the forward end graduation of the tape. Lighter pickets serve as the intermediate supports, the tape resting either on a nail in the side of each or in suspended wire loops. The points of support are set on a uniform gradient between the mark posts: when this proves inconvenient, their

respective elevations are recorded, and must be such that the tape when stretched will not lift clear of any support.

The rear end of the tape is connected to a straining stake behind the mark post, and the tension is applied at the forward end by means of a tape stretcher. A common and simple form consists of a wooden base to which an upright lever is hinged by a universal joint (see Fig. 105). A spring balance, graduated to ounces, is connected to the lever through a gimbal, and to it the tape is attached. Provision is made for adjusting the position of the spring balance and tape on the lever to suit the height of the mark post. The operator, standing on the base plate, applies the tension by pulling the lever, and sees by the indication of the spring balance that it is maintained. When the tape is strained, the position of the forward end graduation is engraved on the plate by means of a sharp awl. After carrying forward the tape, the rear end is adjusted to coincide with the mark previously made, or its position is also marked on the plate, and the gaps or overlaps are measured by a finely divided scale.

The rate of measurement on this system is roughly twice that with short, freely suspended wires or tapes, speeds of over a mile an hour being occasionally reached. Although excellent results have been obtained, the error of marking the plates is naturally greater than can occur in the mean of repeated readings of scales against tripod marks, while the effect of friction at the supports is an additional source of uncertainty.

In certain of the British Colonies a number of bases have been measured with tapes 300 ft. in length. These tapes have usually been suspended in three equal spans of 100 ft. each, the tension of 25 to 30 lb. being applied by means of weights hung over the pulleys of the two straining trestles, and the end measurements taken to marks on ordinary measuring tripods. The supports at the 100 ft. and 200 ft. marks were the hubs of bicycle wheels mounted on brackets which could be slid up and down and clamped on poles held by labourers (see Fig. 108). These hubs were carefully aligned, both horizontally and vertically, between the tops of the measuring tripods. It was found that there was very little friction in the system and the results obtained appeared to be of the same standard of accuracy as those obtained with a tape 100 ft. long used in a single span.

If fairly heavy slopes are encountered, and the tension is applied by weights hung over straining trestles, it may be necessary to prevent longitudinal movement of the tape by putting small extra weights at the upper end of the latter. Otherwise, one end of the tape may be anchored or fastened to a straining trestle placed at that end and the weight applied at the other. In this case, care should be taken to note in the field book whether the tension was applied at the upper or the lower end of the tape. If extra weights are applied, their exact amount should also be carefully noted.

Temperature Measurement. The measurement of the actual temperature of a wire or tape under field conditions is one of considerable difficulty. A close determination is essential when steel tapes are employed, but in the case of invar the influence of small errors of thermometry is considerably reduced by the smallness of the coefficient of expansion.

The coefficient of expansion of steel tapes is about ·0000063 per 1° F., or thirty times the mean value of the coefficient for invar, and an uncer-

tainty of 1° Fahr. in the tape temperature therefore corresponds to an uncertainty in distance of 1/160,000. In sunshine the temperature indicated by a mercurial thermometer may differ from that of the tape by several degrees, and to attain an accuracy of 1/500,000 or over, measurement by steel tape must be executed only when the air and the ground are at the same temperature, *i.e.* in densely cloudy weather or, preferably, at night.

The thermometers used should be read to 0·1° C. or 0·2° F. Probably the best type to use is one enclosed in a metal tube or casing in which there is a narrow slot running down one side through which the graduations can be seen and read. The casing should, if possible, be of very similar colour, polish, brightness and capacity for heat to the tape itself. Three thermometers are placed at intervals along the tape and they should be held close to the tape in an almost horizontal position, but with the bulb very slightly lower than the top of the stem.* These thermometers should be read before and after the measurement of each tape length. The measurement of each section of the base should be divided equally between rising and falling temperatures to reduce the influence of lag in the thermometers caused by volume change in glass lagging behind temperature change.

All thermometers should be examined at frequent intervals to see that there are no broken threads of mercury.

Errors arising from uncertainty in temperature measurement may, in future, be greatly reduced by standardising wires and tapes in terms of their electrical resistance instead of at a stated temperature, and by observing the resistance during the field measurement. One difficulty about this appears to be that of maintaining the constancy of the resistances of the standard resistance coils under the varying conditions met with in the field. In Australia this method is being used with steel tapes for precise base measurement.

Measurement by Steel and Brass Wires. In Jäderin's application of the principle of the bimetallic thermometer to flexible apparatus, the measurement is made by a steel wire and by one of brass. These are 25 m. long by 1·5 to 2·6 mm. in diameter, and have end scales 10 cm. long divided to millimetres. The wires used in combination are of similar diameter, and are nickel plated to ensure the same temperature conditions for both. They are freely suspended, one after the other, between movable tripods carrying a fine needle against which the scales are read. Tension is measured by a spring balance graduated to tenths of a kilogramme and read to hundredths by estimation. From the total lengths given by the steel and brass wires respectively the temperature effect is eliminated by the method of page 182.

Base measurement on this system can be executed in sunshine with as great precision as can be attained by the use of steel tapes in densely cloudy weather or at night. The method has, however, been superseded by the employment of invar.

* Some surveyors prefer to use thermometers suspended in a vertical position, but experiment appears to indicate that they are more likely to give the correct temperature of the tape if suspended in the horizontal position. See *Report of Proceedings, Conference of Empire Survey Officers*, 1935, pages 295 to 325.

Calculation of Length of Base. The final value of a base measurement is arrived at from the immediate results of the field work by the application of a series of corrections. Most of these corrections are exactly the same as those applicable to measurements with steel tapes and have already been described in some detail in Vol. I, pages 158–163.

A number of corrections refer to the apparatus, and aim at the elimination of the effects produced by differences between its absolute and nominal length and between the field conditions and those of standardisation. In measurement by rigid bars, corrections of this class are required for absolute length and temperature. In the case of flexible apparatus, additional corrections are necessitated by any deviation from the standard conditions as to suspension and pull, which govern the form of the curve assumed by the wire or tape. The application of these corrections yields the distance along the gradients in which the measurement has been made.

The remaining corrections depend upon the vertical and horizontal alignment and elevation of the base. The length of each slope must be reduced to its horizontal equivalent. A correction for horizontal alignment is required in cases where the location of the base necessitates an alignment in plan which deviates from the straight line between the terminals. Lastly the length of the base is reduced to the value it would have if projected upon the mean sea level surface. For uniformity, all stations in a geodetic survey are considered as projected normally upon this surface, and all linear distances are arcs upon it. To obtain the projected lengths of all the triangle sides it is sufficient to reduce those of the measured bases.

Corrections are positive or negative according as the uncorrected distance is to be increased or decreased. Each section of the base is separately corrected, as the degree of consistency obtained in repeated measures can be gauged only from the final reduced lengths.

Correction for Absolute Length. The absolute length of a base measuring instrument as obtained in standardising is usually expressed as its nominal length, l, plus or minus a small quantity, c, the value of which is given for the temperature of standardisation. If, therefore, the instrument is laid down n times, the nominal distance, nl, is subject to a correction, nc, of the same sign as that of c in the expression for the absolute length of the instrument.

Correction for Temperature. It is usually sufficient to deduce from the thermometer readings the average temperature prevailing during the measurement of each section and to apply a temperature correction for the whole section. If, however, the value of the coefficient of expansion of the apparatus varies under change of temperature, it is necessary to correct each bar or wire length. Thermometer readings are first corrected for scale errors, and it is important in precise work that the coefficient of expansion should be known for the particular instruments used, instead of adopting average values.

Let T_m = mean temperature during measurement,

T_s = temperature of standardisation,

a = coefficient of expansion,

L = measured distance.

Correction for temperature difference $= + a(T_m - T_s)L$.

Temperature Correction for Bimetallic Apparatus.

Let L_s, L_b = distances as computed from the absolute lengths of the steel and brass components at standard temperature,

a_s, a_b = coefficients of expansion of the components,

D = corrected distance,

$T = T_m - T_s$.

Then $D = L_s(1 + a_sT) = L_b(1 + a_bT)$,

$$\text{whence } T = \frac{L_s - L_b}{a_bL_b - a_sL_s},$$

$$\text{and } D = \frac{L_sL_b(a_b - a_s)}{a_bL_b - a_sL_s},$$

so that correction to result given by steel component $= D - L_s$

$$= + \frac{a_sL_s(L_s - L_b)}{a_bL_b - a_sL_s}, \text{ or, with ample accuracy, } + \frac{a_s(L_s - L_b)}{a_b - a_s}.$$

$$\text{Similarly, correction for brass component} = + \frac{a_b(L_s - L_b)}{a_b - a_s}.$$

Correction for Change of Tension. This correction is required when flexible apparatus is subjected in the measurement to a different tension from that of standardisation.

Let F_m = tension applied during measurement,

F_s = tension applied during standardisation,

A = cross-sectional area of wire or tape,

E = modulus of elasticity of do.,

L = measured distance.

$$\text{Correction for change of tension} = + \frac{(F_m - F_s)L}{AE}.$$

The cross-sectional area is most accurately obtained by weighing the wire or tape and dividing the weight by the length times the density. For geodetic work, in place of adopting a general value for the modulus of elasticity, it should be determined by subjecting the wire or tape to widely different tensions and measuring the change of length with change of tension.

It must be remembered that, if a wire or tape is standardised in catenary, an alteration in tension will alter the sag correction. In this case, the sag correction for the tension used during standardisation should be added to the standardised chord length to give the equivalent length on the flat. The sag correction for the tension applied during measurement may then be subtracted from this result to give the new chord length. Alternatively, when $(F_m - F_s)$ is small, the new sag correction C_m will be given by :—

$$C_m = C_s - 2C_s \times \frac{(F_m - F_s)}{F_s}.$$

Correction for Sag. This correction represents the difference between the actual length of a suspended wire or tape and the chord distance between the end graduations when these are at the same level. The correction is applicable when the wire or tape is standardised on the flat

and suspended during measurement, in which case the value of the correction must be known with a precision equivalent to that of standardisation. Wires and tapes to be used in single catenary are standardised in terms of the chord distance obtained at standard tension, so that no correction for sag is necessary.

Let w = weight of wire per unit length,
F = applied tension in same unit,
S = chord length between zeros of wire when suspended,
L = actual length of wire.

$$\text{Sag correction} = (S - L) = -\frac{S^3w^2}{24F^2}, \text{ or } -\frac{L^3w^2}{24F^2}$$

with sufficient approximation if F is not less than $20\ wL$.*

If the wire or tape is supported at intermediate points, forming n equal bights, the correction becomes $\frac{1}{n^2}$ times the above or n times the sag correction for a single span of length S/n.

Correction for Index Error of Spring Balance. When tension is measured by spring balance, the same instrument should be employed in the field measurement as is used in standardising. If not, it is necessary to apply an index correction the value of which for horizontal tension differs from that for a vertical position of the balance. A balance adjusted to read correctly in the vertical position indicates less than the true tension when used horizontally.

Let I = index error when balance is vertical, *i.e.* reading when suspended hook downwards, without load,
I' = reading with balance inverted and suspended from hook,
W = total weight of balance, found by weighing,

then index error for horizontal position $= \frac{W - I - I'}{2}$,

I being taken as negative when less than zero.

The value of the correction for readings taken on a slope is investigated by Young in a paper to the Institution of Civil Engineers (*Min. Proc.*, Vol. CLXI).

The indications of spring balances are also influenced to some extent by temperature changes. Balances should therefore be subjected at the standard tension to different temperatures, and a table of corrections prepared.

Correction for Change of Gravity. When a wire or tape measurement is made in a different latitude or at a different elevation from that of the place of standardisation, allowance is made in geodetic work for the change in the value of g, the acceleration due to gravity. When tension is applied by means of suspended weights, the pull they exert and the weight of the wire are proportional to g. The form of the catenary remains constant, but the wire suffers a small extension or shortening by the resulting increase or decrease from the standard tension. If the tension is measured by spring balance, it is unaffected by change of g.

* See Appendix II for proof of this formula.

but the weight of the wire is altered, so that in this case increase of g causes increase of sag.

The various formulæ which have been proposed to express the variation of g over the earth give sufficiently accordant results that any of them can be used for the present purpose. The formula of Helmert is

$$g = g_0(1 + 0{\cdot}005302 \sin^2\phi),$$

where g = the acceleration due to gravity at sea level in latitude ϕ,
g_0 = the value at sea level at the equator.*

The above value of g is subject to various corrections depending upon the elevation of the place, the density of the underlying strata, and the conformation of the surrounding country. The elevation corrections have the values :

(a) For height above mean sea level, $-g\dfrac{2h}{R}$.

(b) For the mass between sea level and the station, $+ g\dfrac{3}{4}\dfrac{h}{R}$

where h = elevation of station above mean sea level in ft.,
R = mean radius of the earth = 21×10^6 ft.

In the case of straining by weights, let g_1 and g_2 be the respective values of gravity, in terms of g_0, at the places of standardisation and of measurement.

$$\text{Then } \frac{g_2 - g_1}{g_1} = \frac{\delta F}{F},$$

where δF is the change produced on the nominal tension F. The correction is then evaluated by the change of tension formula, and is positive or negative according as g_2 is larger or smaller than g_1.

Correction for Slope.

Let l_1, l_2, etc. = lengths of successive uniform gradients,
h_1, h_2, etc. = differences of elevation between the ends of each.

For any gradient of length l and difference of elevation h,

$$\text{the slope correction} = -(l - \sqrt{l^2 - h^2}),$$

$$= -\left(\frac{h^2}{2l} + \frac{h^4}{8l^3} + \text{etc.}\right).$$

The second term may safely be neglected for slopes flatter than about 1 in 25. With its omission,

$$\text{total correction} = -\tfrac{1}{2}\left(\frac{h_1^2}{l_1} + \frac{h_2^2}{l_2} + \ldots\right),$$

$$= -\frac{\Sigma h^2}{2l}, \text{ when the gradients are of uniform length } l.$$

* Recent work by Jeffreys gives for any latitude

$$g = 978{\cdot}049\ (1 + 0{\cdot}0052895 \sin^2\phi - 0{\cdot}0000059 \sin^2 2\phi),$$

where g is in cm./sec.² For g in ft./sec.² the constant 978·049 outside the bracket becomes 32·088.

(See *Monthly Notices Royal Astronomical Society* 101, No. 1, 1941 ; and *Nature*, No. 3733, Vol. 147, page 613.)

If the slope is measured in terms of θ, the angle of elevation or depression,

$$\text{correction} = -l(1 - \cos\theta) = -2l\sin^2\tfrac{1}{2}\theta = -l \text{ versine } \theta.$$

Wires and Tapes in Catenary.—The above slope corrections are strictly applicable only to measurements by rigid apparatus, but are also usually employed in the case of tapes with frequent supports. For a wire or tape suspended in catenary, account should be taken in precise work of the deformation of the catenary when the ends are not at the same elevation. The development of formulæ for this is too lengthy for inclusion, and the reader is referred to Prof. Henrici's "Theory of Tapes in Catenary" in Ordnance Survey Professional Paper No. 1. The formula obtained by Henrici for measured differences of elevation is as follows.

Let S = nominal length of tape or wire, along chord between zeros,
h = difference of elevation between measuring tripods,
l = length along catenary between tripod marks,
a_0 = standardisation correction to nominal length,
a_1 = algebraic sum of scale readings,
a = coefficient of expansion,
t = excess of field temperature over standard temperature,
$A = a_0 + a_1 + Sat$,
λ = apparent shortening due to sag $= \dfrac{X_0^3 w^2}{24F^2}$, as before,
X = required horizontal distance between tripod marks,
X_0 = value of X when $h = 0$.

Then $X_0 = l - \lambda = S + A$,

$$\text{and } X = X_0\left(1 - \frac{h^2}{l^2}\right)^{\frac{1}{2}},$$

$$= (S + A)\left(1 - \frac{h^2}{(S + A + \lambda)^2}\right)^{\frac{1}{2}}.$$

Expanding, and retaining only the first powers of A and λ, this becomes

$$X = S - P + A + Q + R,$$

$$\text{where } P = S\left(\tfrac{1}{2}\frac{h^2}{S^2} + \tfrac{1}{8}\frac{h^4}{S^4} + \tfrac{1}{16}\frac{h^6}{S^6} + \tfrac{5}{128}\frac{h^8}{S^8} + \ldots\right),$$

$$Q = A\left(\tfrac{1}{2}\frac{h^2}{S^2} + \tfrac{3}{8}\frac{h^4}{S^4} + \tfrac{5}{16}\frac{h^6}{S^6} + \ldots\right)$$

$$R = \lambda\left(\frac{h^2}{S^2} + \tfrac{1}{2}\frac{h^4}{S^4} + \tfrac{3}{8}\frac{h^6}{S^6} + \ldots\right).$$

This formula for X gives a result accurate to 1/10,000,000 if $\dfrac{h}{S}$ is less than $\frac{1}{10}$, A is less than S/1000, and the tension applied exceeds 20 times the weight of the tape, and to 1/2,000,000 if $\dfrac{h}{S}$ is less than $\frac{1}{5}$, and A is less than S/1,000, provided the mean tension is constant.

If angles of slope instead of differences of elevation are measured, the sag correction is given by $C = -kl$, where k is a constant the value of

which depends on whether the tension used in computing it is that applied at the upper or at the lower end of the tape. If the tension F is applied at the upper end,

$$k = B^2 \cos^2 \theta(1 + B \sin \theta)/24,$$

where $B = \frac{wl}{F}$ and θ is the angle of slope. If the tension F is that applied at the lower end :—

$$k = B^2 \cos^2 \theta(1 - B \sin \theta)/24.$$

It will be noted that, if $\theta = 0°$, the sag correction assumes the ordinary value already given for a tape horizontally suspended.

The two tables on the opposite page, taken from the Gold Coast Survey Publication *Tables for Use in the Department*, give the values of k for different values of B and for different slopes. In both cases they are for a tape standardised on the flat. If the tape has been standardised in catenary, the k to be used is obtained by subtracting the value tabulated for $\theta = 0°$ from the value tabulated for the observed slope. Thus, for $B = \cdot 12$, with tension applied at the upper end and $\theta = 25°$, $k = -0{\cdot}000082$ and the correction is $+ 0{\cdot}000082l$ instead of $- 0{\cdot}000518l$.

All of the above formulæ for sag correction are derived on the assumption that the tape is perfectly flexible ; in other words, there can be no shear or bending moment at any part of it. This assumption is not strictly true for any tape or wire, as all tapes and wires have a certain amount of rigidity. Hotine, however, in his paper on the "Theory of Tape Suspension in Base Measurement" in the *Empire Survey Review*, Vol. V, No. 31, January 1939, has shown that the rigidity term is given by $\frac{w^2EIl \cos \theta}{F^3}$ where θ is the angle of slope, E is Young's Modulus of Elasticity and I is the moment of inertia of the tape about its neutral axis.* For all ordinary tapes or wires, such as are used in base measurement, this term is negligible.

It has also been found that, with the ordinary tape used in base measurement, the total sag correction for a tape supported in n equal spans is equal to n times the sag correction for a single span as computed by the ordinary formula.

Correction for Inclination of End Readings. If the graduations are etched on the tape itself, the end scales will be inclined to the horizontal when the tape is suspended in catenary since the end of the tape itself is inclined to the horizontal. This inclination is given by :

$$\tan \phi = \frac{wl}{2F}$$

or, since ϕ is small,

$$\phi = \frac{wl}{2F}$$

* This paper gives the complete theory of base measurement with wires or tapes and it also contains many hints of practical value. For a good account of the measurement of a modern base line and of modern base line apparatus see the same author's paper on "The East African Arc" in the *Empire Survey Review*, Vol. III, No. 18.

SAG CORRECTION ON A SLOPE

If $B = \frac{wl}{T}$, then correction $= -kl$.

TABLE GIVING VALUES OF k.

(A) Tension Applied at Upper End of Tape

	0°	5°	10°	15°	20°	25°	30°	35°	40°
0·01	0·000 004	004	004	004	004	003	003	003	002
·02	017	017	016	016	015	014	013	012	010
·03	038	038	037	036	033	032	029	026	022
·04	067	067	065	063	060	056	051	046	040
·05	104	104	102	098	094	087	080	072	063
·06	150	150	147	142	135	126	116	104	091
·07	204	204	200	194	185	172	158	142	125
·08	267	267	262	254	242	227	208	187	165
·09	338	338	332	323	307	288	265	239	210
·10	417	417	411	399	381	357	328	296	260
·11	504	505	498	484	462	433	399	359	317
·12	600	602	594	577	552	518	477	430	379
·13	704	707	698	679	649	610	562	508	448
·14	817	821	811	790	756	711	655	592	522
0·15	0·000 938	943	933	909	870	819	756	684	603

(B) Tension Applied at Lower End of Tape

	0°	5°	10°	15°	20°	25°	30°	35°	40°
0·01	0·000 004	004	004	004	004	003	[illegible]	003	002
·02	017	017	016	016	015	014	012	011	010
·03	038	038	036	035	033	031	028	025	022
·04	067	066	064	062	058	054	049	044	038
·05	104	103	100	096	090	084	076	068	059
·06	150	148	144	138	130	120	109	097	085
·07	204	201	196	187	176	163	148	131	114
·08	267	263	255	244	229	212	192	171	148
·09	338	333	322	308	289	267	242	215	187
·10	417	410	397	379	355	328	297	264	229
·11	504	495	480	457	428	395	357	317	275
·12	600	589	570	542	508	468	423	375	325
·13	704	691	668	635	594	546	494	437	379
·14	817	801	773	735	687	631	570	504	436
0·15	0·000 938	919	886	841	785	722	650	575	497

Note. The length obtained by using the above table or the original formula is the length of the inclined chord between the end marks of the tape. Slope correction must then be applied to this chord length in the ordinary way.

Hence, if Σ denotes summation so that Σr is the algebraic sum of the end readings, the correction to be applied to these readings is :—

$$
\begin{aligned}
-\Sigma r(1 - \cos\phi) &= -\Sigma r\left(1 - 1 + \frac{\phi^2}{2} \ldots\right) \\
&= -\Sigma r \,.\, \frac{\phi^2}{2} \\
&= -\Sigma r \,\frac{w^2 l^2}{8F^2} \\
&= -3\Sigma r \,.\, \frac{C}{l}.
\end{aligned}
$$

Taken over a whole section or a whole base, the quantity Σr is usually very small and this correction is often neglected. It should, however, be taken into account when tapes are being standardised in catenary, and it is just as well to compute it, and, if necessary, apply it in other cases as well.

Correction for Horizontal Alignment. In the exceptional case where the base must be aligned as a series of two or more courses of different bearings, the distance between the terminals falls to be computed as for a traverse survey. The deviation from straight alignment is usually small, and the formulæ for slope correction are applicable if l represents the length of any course, θ, the angle it makes with the straight line between terminals, or h, the difference between the offsets from that line to the ends of the course.

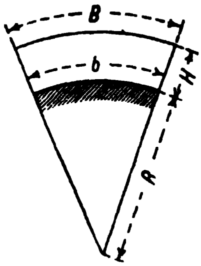


Fig. 76.

Reduction to Mean Sea Level. The profile of the measurement must be referred to mean sea level, and the mean elevation of the base deduced.

Let B = measured length of base,
b = length reduced to mean sea level,
H = average elevation of base,
R = radius of earth in latitude and azimuth of base.

From Fig. 76, $b = B\,\dfrac{R}{H + R}$,

or correction $(b - B) = -B\,\dfrac{H}{H + R}$ or $= -B\,\dfrac{H}{R}$, since H is small compared with R.

R is commonly taken as the mean radius of the earth for the mean latitude of the base, *i.e.*, $\sqrt{RN}$, where R is the radius of curvature of the meridian and N the radius of curvature of the prime vertical section, taken out for the mean latitude (page 318). This quantity is best obtained from special geodetic tables.

Note. Although it is usual to reduce the length of a geodetic base to its value at mean sea level, the reduction should strictly be made to the surface of the spheroid of reference, *i.e.*, the ideal geometrical surface on which the triangulation is computed (page 317). Owing to variations in the direction and intensity of the earth's gravitational field, mean sea level all over the oceans does not lie on a perfect spheroid, as it would do if the gravitational field were entirely regular, but on an irregular

equipotential surface – the geoid – which is everywhere normal to the direction of gravity and approximates closely to a spheroidal surface, but with relatively small local surface undulations caused by the gravity variations. Accordingly, mean sea level at any place chosen as a datum for elevations will lie on the geoid, and a small height correction is necessary, in theory at any rate, to transfer mean sea level to the surface of the spheroid. The correction is not of importance in the case of small countries where the separation between the geoid and the spheroid of reference is unlikely to be large, but, when a triangulation is carried over large continental areas, or when triangulations in smaller countries are joined together, it cannot be ignored if consistent and reliable results are to be obtained. Thus, to take what is possibly an extreme case, the Mergui base in Burma is only 10 ft. above sea level but 347 ft. above the spheroid, with the result that the correction would amount to about 1/60,000 of the length of the base.* Unfortunately, the true shape and size of the geoid is seldom known at the beginning of a triangulation and accordingly it is generally necessary to assume that mean sea level is on the surface of the spheroid, so that lengths of base lines are usually reduced to the adopted mean sea level at some particular place in or near the area covered by the triangulation.

Probable Error of Base Measurement. The probable error of measurements with a long steel tape has been fully discussed in Vol. I, pages 169–174, and the methods and arguments there described are very similar to those that are applicable to the calculation of the probable error of the measurement of a base line. The principal difference, perhaps, is that, in a base measurement with invar tapes or wires, the effects of constant and cumulative errors are far more thoroughly eliminated than they are in the case of ordinary measurements with a long steel tape. Hence, in base measurement the principal errors affecting the observations are almost entirely of the accidental type and we can therefore assume that the probable error of the final result is the square root of the sum of the squares of the probable errors of the individual operations which can appreciably influence the result. To take the case of a wire or tape measurement, the following errors require recognition and investigation.

Errors in Standardisation: (*a*) Probable error of prototype bar, as compared with the original international metre standard.

(*b*) Probable error of mural base at standards laboratory, as derived from prototype.

(*c*) Probable error of standardising reference wires on mural base.

(*d*) Probable error of values of coefficient of expansion of reference and field wires.

(*e*) Probable error in estimation of length of reference wires at dates of comparison with field wires arising, particularly in the case of invar, from molecular change.

(*f*) Probable error of comparison of field wires with reference wires.

Errors in Measurement: (*g*) Probable error of measurement of differences in level of supports.

(*h*) Probable error in reading scales.

(*i*) Probable errors of tension or of change in weight of wire due to adhesion of dirt, making the catenary assumed during measurement differ from that at comparison.

* Provisional General Report by Brigadier G. Bomford to the Geodetic Astronomy Section of the International Association of Geodesy in 1948. (*Bulletin Géodésique* No. 13 (New Series), September, 1949, page 257.)

(j) Probable error of temperature measurement.

(k) Probable error in value of mean elevation of base, affecting reduction to mean sea level.

To estimate the final probable error of a measurement, that contributed by each of the above sources is separately worked out for the whole base. The probable errors of standardisation operate as constant errors, and are proportional to the length of the base. The data required for the evaluation of the total probable errors from a, b, c, and d are obtained from the standards office. The effect of d for the reference wires is proportional to the difference between the standard temperature and the mean temperature of the field comparisons, and in the case of the field wires depends upon the difference between the latter temperature and that during the base measurement. The value of e can only be roughly estimated, while that of f is obtained by the usual method of evaluating probable error from the results of repeated observations (page 277).

The combined effect of all purely accidental errors of measurement, such as g and h, is obtained in the same manner for each section by its repeated measurement, and the probable error for the whole base from this source is the square root of the sum of the squares of the probable errors of the several sections. Error i is non-compensating, but allowance for it is unlikely to be required if the same straining apparatus is used for standardisations as in the field measurement. The value of j cannot be exactly assessed. Error k is obtained from the probable error of the value taken for the elevation of the reference datum and of the levelling between the bench mark and the base.

As indicating the relative magnitudes of the errors, those of the Semliki base in Uganda may be quoted.* This base was measured in 1908 with the Guillaume-Carpentier apparatus, and the reduced length is 16,532·37614 m. The individual probable errors for the whole base are in millimetres: $a = \pm 1{\cdot}65$; $b = \pm 6{\cdot}89$; $c = \pm 3{\cdot}59$; $d = \pm 8{\cdot}06$; $e = \pm 6{\cdot}365$; $f = \pm 1{\cdot}878$; $g + h = \pm 4{\cdot}511$; $j = \pm 3{\cdot}746$; $k = \pm 4{\cdot}03$. Errors c, d, and e are for the mean of three reference wires, and f is for three field wires used in combinations of two. The mean temperature during measurement happened to be the same as that of the comparisons, so that no account had to be taken in d of errors in the coefficient of expansion of the field wires.

The probable error of the result is therefore $= \pm\sqrt{1{\cdot}65^2 + 6{\cdot}89^2 + \text{etc.}}$

$$= \pm 14{\cdot}92 \text{ mm. or } \frac{1}{1{,}108{,}000}.$$

An example of a somewhat similar calculation—that of the estimated probable error of the length of a traverse line as measured with an invar tape—is given on page 250.

Accurate Length Measurements by Combined Optical and Electronic Means. A new method of accurate distance measurement by means of modulated light waves and a special electronic receiver, which promises to have many useful applications in surveying, has recently been devised by E. Bergstrand, of the Geographical Survey, Stockholm. In time this method may replace the use of invar wires and tapes for the measurement

* "Report of the Measurement of an Arc of Meridian in Uganda," Vol. I. Colonial Survey Committee, 1913.

of trigonometrical bases, but, for first-order work at any rate, this depends on a more accurate knowledge of the velocity of light and its changes with atmospheric conditions than we now possess being available,* and on more tests over measured lines in different parts of the world. Even as it is, the possibility of introducing check measurements of the lengths of sides at much more frequent intervals than are practicable at present reduces to some extent the need for measuring bases with the extreme accuracy now considered essential, and in such circumstances it is possible that observations with the new apparatus could replace those with ordinary tapes or wires. Another possible use is to enable the lengths of *all* the sides of a triangulation to be measured direct, so avoiding the rapid growth of linear error that now occurs in ordinary triangulation. The apparatus also seems to offer promise as regards the measurement of the lengths of the legs of precise traverses, particularly in enabling a precise traverse to be carried forward from hill to hill, or from rise to rise, over very rough country where *ordinary taping is almost impossible or where it* would necessitate heavy clearing or damage to standing crops. The method, which is described in Appendix IV, is of great interest. It is not yet in general use, but enough work has been done on it to show that it has considerable possibilities and the necessary apparatus is being put on the market by a Swedish firm.

ANGLE MEASUREMENT

Instruments. Considerable variety exists in the forms of theodolites for precise work, but they do not differ materially in essentials from the patterns used in ordinary surveying. In the early days of geodetic surveying the required refinement of reading was secured by the use of large circles, the great theodolite of the Ordnance Survey having a diameter of 36 in. Modern improvements in graduating engines enable as good work to be done with much smaller instruments. For first-order triangulation the usual sizes for theodolites of the ordinary micrometer type with silver circles are now 10 in. and 12 in., with 5 in. to 8 in. instruments for second-order work, but more recently the tendency has been to replace this type of theodolite by others of the double reading type, such as the Wild and Tavistock. The horizontal circles of the large geodetic types of Wild and Tavistock theodolites are made of glass and are only $5\frac{1}{2}$ in. and 5 in. diameter respectively; the corresponding vertical circles, also made of glass, are $3\frac{8}{11}$ in. and $2\frac{3}{4}$ in.

A high standard of workmanship is required throughout primary instruments. The telescope must be of the best quality, particularly as regards definition, and should be of sufficient magnifying power that the refinement of pointing may not be inferior to that with which the circle can be read. Several eyepieces are provided, and the magnifying power

* Bergstrand has recently used his apparatus to determine the velocity of light and has obtained a value (*in vacuo*) of 299 793·1 km./sec. with a mean square error of $\pm$ 0·25 km./sec. This m.s.e. works out at 1/1,200,000, which, so far as the consistency of the results in themselves is concerned, is of full geodetic accuracy. (See "A Determination of the Velocity of Light," by E. Bergstrand. *Arkiv for Fysik*, Vol. 2, No. 15, 1950.)

usually ranges from about 30 to 80, the aperture of the objective being $2\frac{1}{2}$ in. to 3 in. An eyepiece micrometer is commonly fitted for use in making signal bisections by means of the movable vertical hair. By turning the micrometer into the vertical position, it is made available for the measurement of small vertical angles. To illuminate the field for night work, a small electric light, the intensity of which is under control, is attached to one of the standards, and the rays are projected through a lens in the hollow trunnion axis on to a very small mirror which reflects them to the hairs. The electrical equipment of modern instruments also provides for the electrical illumination of the graduations in the field of view of the micrometers on the horizontal and vertical circles, and, when an instrument is so equipped, it is advisable to use this illumination for daylight as well as for night observations.

In the pages which follow we shall first of all describe the ordinary micrometer type of geodetic theodolite and its adjustments and then the double reading type of which the large Wild and the geodetic Tavistock are prototypes.

Geodetic Theodolites of the Ordinary Micrometer Type. In this type of theodolite the circle is usually divided to 5 min., and is read by equidistant micrometers to 0·1 sec. by estimation. For reading the large circles formerly used, five micrometers were fitted, but two or three is the usual number on 8 in. to 12-in. theodolites, and two on smaller instruments. A pointer microscope is provided for reading the figures. The vertical circle may be of the same diameter as the horizontal circle, or may be much smaller and intended merely for use in finding signals or for setting approximately on a known vertical angle. In some foreign patterns it is omitted altogether. When the vertical circle is read by micrometers with a refinement similar to that for horizontal angles, the instrument is adapted for astronomical observation and may be distinguished by the name, altazimuth instrument.

Plate levels are fitted as in smaller theodolites, but the final leveliing is performed by means of a striding level placed on the horizontal axis. The sensitiveness of this level is not less than 2 sec. per division, and should be ascertained either on a level trier (Vol. I, page 37) or by placing the level longitudinally on the telescope and measuring a small vertical angle or by the "Wisconsin method" (page 78). Centering is performed either by plumb bob or by means of a nadiral or look-down telescope (optical plummet), which can be screwed into the horizontal plate. Such a telescope is particularly useful on elevated scaffolds. The weight of large theodolites necessitates the provision of a stout lifting ring by which they can be handled without danger of overstrain.

The instrument illustrated in Fig. 77 is by Messrs. E. R. Watts and Son, Ltd., London, and was constructed for the Ordnance Survey for use on the test triangulation in Scotland.

The horizontal circle is of 12 in. diameter, and is entirely protected from dust. It is graduated to 5 min., and is read by three micrometers to single seconds directly and to 0·1 sec. by estimation. The micrometers and a pointer microscope are carried by a single casting. The vertical circle is of 6 in. diameter, reading to 1 min., and is used only for setting the line of sight approximately to a known vertical angle.

The telescope, which transits both ways, has an objective of 24 in. cal length and 3 in. aperture. It is fitted with an eyepiece micrometer,

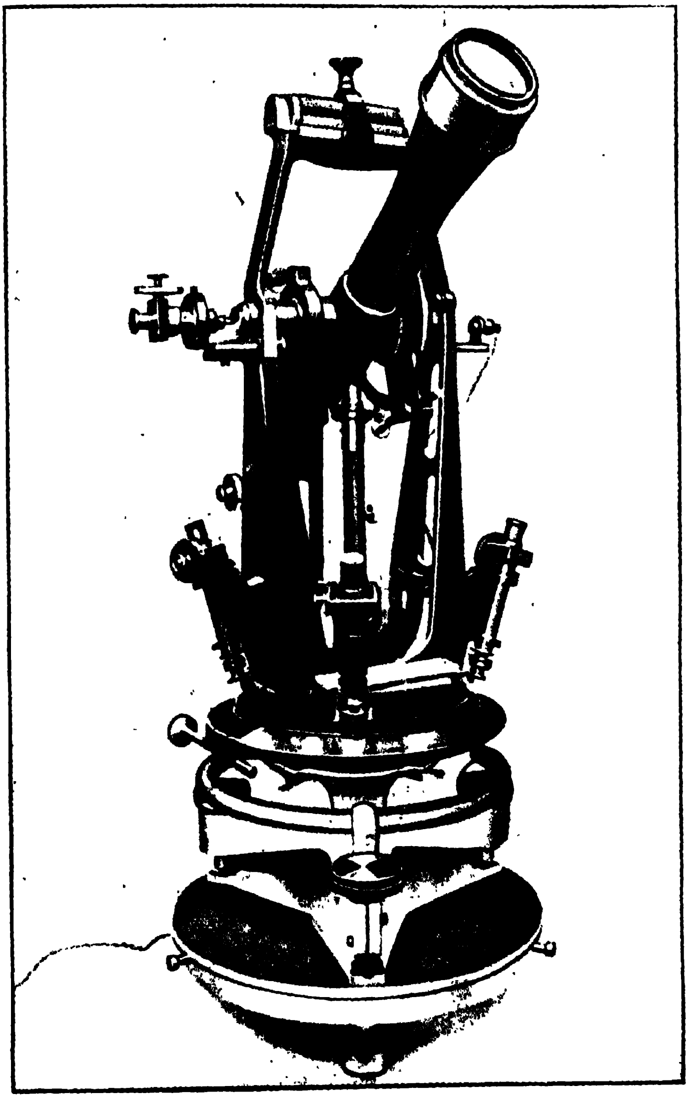


FIG. 77. ORDNANCE SURVEY GEODETIC THEODOLITE.

and the powers of the eyepieces range from 40 to 80. The field of view is illuminated electrically from either end of the trunnion axis. The attachment shown at the left side is designed to verify the cylindrical form of

the pivots. The striding level has a sensitiveness of 1·25 sec. per 0·1 in. division.

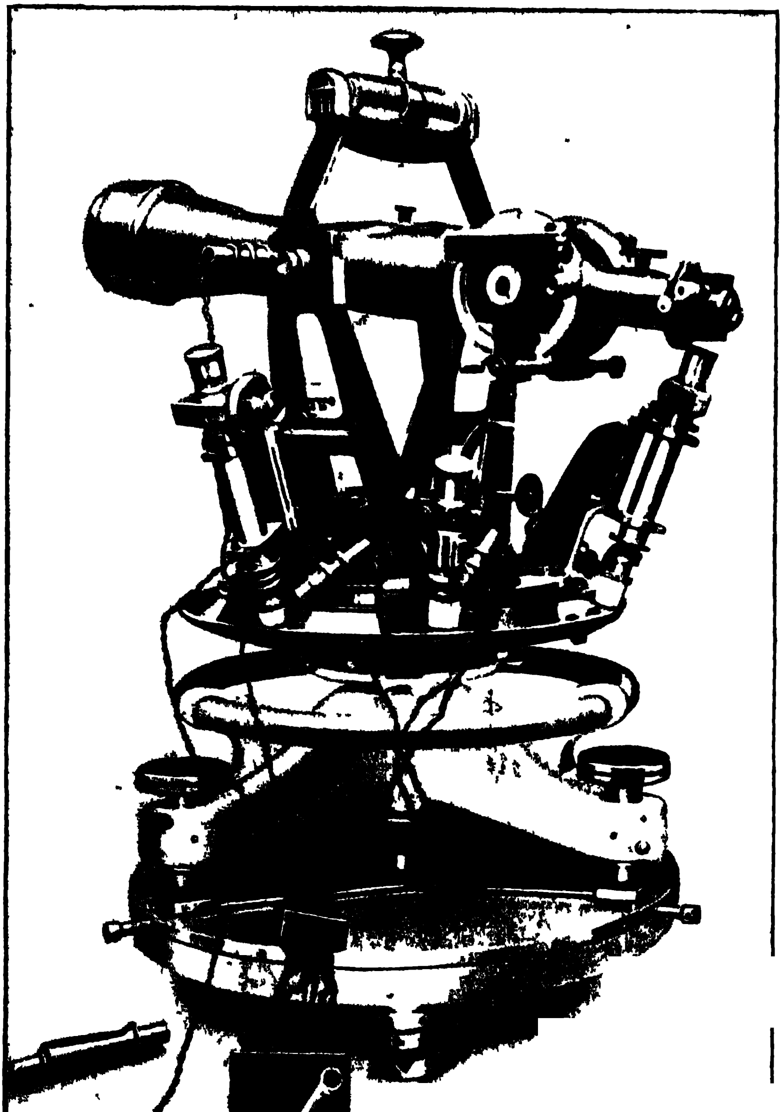


FIG 78. CANADIAN SUF

For the accurate centering of the instrument over a station, a nadiral telescope, having an objective of about 8 in. focal length, is screwed into the upper horizontal plate. Centering is effected by bringing the

line of sight of this telescope upon the station mark by means of three radial screws operating upon the lower base plate. A strong circular leather-covered lifting handle is fitted to the tribrach plate.

Fig. 78 illustrates the instrument adopted by the Canadian Government Geodetic Survey and manufactured by Messrs. Watts. The horizontal circle is of 12 in. diameter, and is graduated and read in the same manner as in the foregoing example. The setting circles are of 4 in. diameter and read to 1 min. The telescope is non-transiting. The objective has a focal length of $18\frac{1}{2}$ in. and an aperture of $3\frac{1}{4}$ in. An eyepiece micrometer is fitted, and the powers of the eyepieces range from 36 to 72.

Adjustments of the Theodolite. The adjustment of a precise theodolite does not differ materially from that of smaller instruments, but advantage is taken of the striding level for horizontal axis adjustment. The adjustments pertaining to horizontal angle measurement are :

1. Plate level adjustment.
2. Striding level adjustment.
3. Horizontal axis adjustment.
4. Collimation adjustment.
5. Micrometer adjustments.

Adjustment of the Plate Levels. Unless the sights are nearly horizontal, these levels should be employed only in the approximate levelling of the instrument, and are subordinate to the striding level. Their adjustment is performed as described in Vol. I, page 92.

Adjustment of the Striding Level. *Object.* To make the level axis parallel to its supports. The V-shaped foot of either leg of the level affords two points of support, and the level axis is required to be parallel to the line joining the points midway between these two points in each V. Two steps are necessary : *A*, to make the level axis coplanar with this line ; *B*, to place them parallel.

Necessity. To enable the striding level to indicate the horizontality or otherwise of the horizontal axis.

A : Test. 1. Set the striding level upon the horizontal axis, and level the instrument approximately.

2. Incline the level a little to one side or the other of the vertical plane containing it. If the bubble remains in a constant position, the level is in lateral adjustment.

Adjustment. If not, by means of the lateral controlling screw adjust the level until the test is fulfilled.

B : Test. 1. Level the instrument by reference to the plate levels.

2. Place the striding level on the horizontal axis, and centre the bubble exactly by the levelling screws.

3. Remove the striding level, and carefully replace it end for end. If the bubble remains central, the level is in longitudinal adjustment.

Adjustment. 1. If not, bring the bubble half-way back by means of the adjusting screws controlling the tube vertically.

2. Relevel by the foot screws, and repeat until the test is satisfied.

Adjustment of the Horizontal Axis. *Object and Necessity.* As for the engineer's transit theodolite (Vol. I, page 96).

Test. 1. Adjust the foot screws until the striding level maintains a

constant position while the instrument is swung through 180° in azimuth. The vertical axis is now truly vertical.

2. If the bubble is central, the horizontal axis is correct, since the striding level is in adjustment.

Adjustment. 1. If not, bring the bubble back to its central position by means of the screws controlling the trunnion support in one standard.

2. Relevel, and repeat until the test is fulfilled.

Note. If the error is small, it may be left, as its effect is eliminated by change of face, and in observations unbalanced as regards change of face the error can be corrected out from the reading of the striding level. Provision for making the adjustment is sometimes omitted in precise theodolites as in smaller instruments.

Adjustment of the Collimation Line. This adjustment is performed as for the engineer's transit theodolite, except that the test sights should be as long as possible. If the telescope does not transit, reversal must be made by removing the axis from the standards and replacing with the telescope end for end.

Micrometers. The essential features of the micrometer microscope have been described in Vol. I, page 72. Those fitted on precise theodolites are required to carry the subdivision to single seconds, and the magnifying power of the microscope must be sufficient to justify estimation of the readings to 0·1 sec. The circle is usually divided to 5 min., and the pitch of the micrometer screw is such that five turns are required to move the hairs from one graduation to the next. The micrometer drum is divided into sixty parts, which therefore represent single seconds. For convenience in keeping count of the number of complete turns, there is provided a fixed comb scale, the teeth of which have the same pitch as the screw, so that one turn moves the hairs from the centre of one notch to that of the next. The centre notch is distinguished from the others, by greater depth or otherwise, and corresponds to the single notch in small instruments in approximately marking the position of the zero line.

The appearance presented through the microscope is shown in Fig. 79. The hairs are shown centered over graduation 144° 10′, which is the approximate reading since the centre notch lies between 144° 10′ and 144° 15′. Assuming the hairs have been moved from the centre notch to this position, they have passed two notches, but have not reached the centre of the third. The micrometer reading is therefore 2 min. plus the number of seconds recorded on the drum, the complete reading being

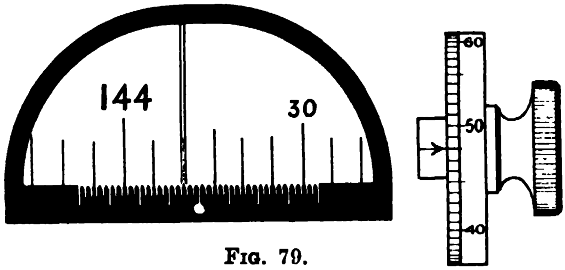


FIG. 79.

$$144° \, 10' + 2' + 48''{\cdot}0 = 144° \, 12' \, 48''{\cdot}0.$$

Notes. (1) In the reading of the comb scale, single minutes are reckoned by the number of spaces from hollow to hollow, and not from crest to crest of the notches. The number of spaces to be counted is always that between the centre notch and the preceding graduation, representing the approximate reading, irre-

spective of the fact that the hairs may be centered over the following graduation. If the hairs appear to lie centrally in a notch, the drum reading shows whether they have reached the centre or passed it.

(2) The utmost delicacy is required in manipulation, and strict attention must be paid to the precautions given in Vol. I, page 74, notes 3, 4, and 5.

Run of the Micrometer. The condition, hitherto assumed, that an exact number of turns of the screw should carry the hairs precisely from one graduation to the next, can be only approximately realised. If, on moving the hairs across a circle space, the drum registers a different reading after the movement from that at starting, the discrepancy between the micrometer measurement of the space and its nominal value is termed the run of the micrometer. When run is present, the value of a fractional part of the circle division is not correctly given either by the back or the forward reading, *viz.* the respective readings obtained by setting the hairs on the graduation last passed by the zero line and on the graduation in advance of it. Since the drum reading decreases as the hairs are moved from left to right, the forward reading is less than the back reading when more than five turns are required to move the hairs across the space. The micrometer is then said to overrun, or the run is positive: otherwise it is negative.

Apart from errors of observation and manipulation, many factors contribute to the existence and variation of run. The circle spaces are not themselves equal on account of imperfect graduation, unequal temperature, or mechanical strain. Again, while the distances between the circle and the microscope objective and between the objective and the hairs should be such as to make the length of the real image of a circle division exactly five times the pitch of the screw, these distances are subject to variation under change of temperature, and the former is also affected by eccentricity. Means are provided for adjusting the microscope to eliminate run, but with high power microscopes the perfecting of the adjustment is a delicate operation. Since the run does not preserve a constant value, all that can be done is to limit the error to a few seconds. The influence of this residual error on the measurement of a fractional part of a circle division is then eliminated by applying a correction.

Correction for Run. If for each observation both the back and forward micrometer readings are noted, the value of the run is known for the particular division used and under the prevailing temperature conditions. The amount of correction corresponding to the position of the zero line between the graduations can then be obtained as follows.

Let D = the nominal value of the circle division, *i.e.* its mean value for the whole circle, = 300″ in the usual case;

b = the back reading, *i.e.* the micrometer measurement of the arc between the zero line and the graduation last passed by it, as obtained by setting the hairs on that graduation;

f = the forward reading, *i.e.* the micrometer measurement of the same arc, as obtained by setting the hairs on the graduation in front of the zero line;

$m = \dfrac{b+f}{2}$;

r = the run developed in D, = $(b - f)$

D_1 = the micrometer measurement of D, $= (D + r)$;
M = the corrected micrometer reading to be applied to the approximate reading.

It is to be assumed that the effect of all errors causing run is to displace the back and forward graduations by equal amounts and in contrary directions. For positive run the image of a division is too long by $\frac{r}{2}$ on either side of its centre line, and the corrected micrometer reading must be less than b and more than f. Negative run produces the opposite effect. Distances on the actual image bear the same ratio to D_1 as their correct equivalents should bear to D, so that

$$\frac{M}{b} = \frac{D}{D_1},$$

$$\text{whence } M = \frac{bD}{D_1},$$

$$\text{or correction to } b = -\frac{br}{D_1}.$$

M is commonly obtained from m,

$$\text{and since } m = \frac{b+f}{2} = b - \frac{r}{2},$$

$$M = \left(m + \frac{r}{2}\right)\frac{D}{D_1},$$

$$= m + \frac{r}{2} - \frac{mr}{D} \text{ with ample precision,}$$

$$\text{or correction to } m = \frac{r}{2} - \frac{mr}{D}.$$

The correction has the same sign as the run for values of m up to 2′ 30″ and the opposite sign for values between 2′ 30″ and 5′. Its value may be tabulated * for various values of r and m. In making the reductions, it is unnecessary to correct each micrometer reading. The back and forward readings of all micrometers are noted and the mean back reading and the mean forward reading obtained. From the mean of these means and the mean run the correction is evaluated for each pointing.

Adjustments of the Micrometer. These are:

1. Adjustment of the entire microscope to place the image of the graduations across the middle of the field of view.
2. Adjustment of the zero line.
3. Adjustment for run.

There may be no means of making the first adjustment, but if the bracket carrying the microscope is controlled by adjusting screws, the method will be apparent.

Adjustment of the Zero Line. *Object.* To place the zero lines of the several microscopes at the desired intervals, and to make the hairs lie centrally in the middle notch when in the zero position.

* See Davidson, "The Run of the Micrometer," United States Coast and Geodetic Survey Report, 1884, Appendix No. 8 or "Geodetic Survey of South Africa," Vol. I, Cape Colony and Natal, 1882.

Necessity. It is not essential that the intervals between the microscopes should be exactly equal, but it is convenient in recording that the discrepancies should be small. The adjustment is usually only necessary when new hairs have been fixed. The comb scale may then require adjustment to make the reference notch coincide with the hairs when they are brought to the zero position.

Test. 1. For equidistance, set the hairs of one micrometer in the zero position, and bring a circle graduation centrally between them.

2. See if the corresponding graduations fall on the zeros of the other micrometers.

Adjustment. 1. If not, by means of the screw at the side of the box opposite the drum (Vol. I, Fig. 64) move the comb scale until the centre of the index notch coincides with the image of the graduation.

2. Centre the hairs on the graduation.

3. Slacken the screw holding the milled head at the drum, and turn the drum until it reads zero, without rotating the micrometer screw. Finally, tighten the head.

Note. In some instruments there is provision for moving bodily all but one of the micrometers in a circumferential direction. The greater part of an error of spacing may be thus removed, and the remainder is eliminated as above.

Adjustment for Run. *Object.* To eliminate run as far as possible.

Necessity. As the effect of run can be corrected out in the reduction, it is sufficient for the most refined work that its value for a 5′ space should not exceed 2″ or 3″, so that no appreciable error can be introduced in the correction by assuming that the error is developed at a uniform rate. It is, however, convenient to have the run adjusted down so that the mean of the back and forward readings, or, in low grade work, the back or forward reading alone, may be used without correction.

Test. 1. Focus the eyepiece for distinct vision of the hairs.

2. Move the hairs across a division, and note the amount and sign of the run.

Adjustment. 1. If the run is positive (negative), the image is too large (small). Release the clamping ring of the objective cell, and move the objective towards (away from) the hairs.

2. The image is now decreased (increased), but does not lie in the plane of the hairs. Eliminate the resulting parallax by moving the whole microscope in its collar away from (towards) the circle.

3. Again take the run, and repeat the adjustment as often as necessary.

Notes. (1) Since the micrometer reading for each pointing is obtained as the mean of the readings of all micrometers, it is usual to adjust for the mean run. Instead of adjusting each micrometer, the run of one of them is made equal and opposite to the sum of the runs of the others.

(2) The elimination of parallax must be regarded as an essential feature of the adjustment, as its presence makes accurate reading impossible.

(3) The hairs must be left parallel to the graduation lines.

Double Reading Geodetic Theodolites. The "double reading theodolite," with optical micrometer, was originally the invention of the late H. Wild when he was still an employee of the German firm of Zeiss. The first instruments were made by Zeiss to his design, but, in 1921, he left Zeiss and moved to Heerbrugg in Switzerland, where he started the manufacture of the instrument that now bears his name.

At present two instruments of the double reading type which are suitable for primary geodetic work are on the market, the one being the large Wild, manufactured by Wild in his works at Heerbrugg, and the other the large Tavistock, manufactured by Messrs. Cooke, Troughton & Simms at York. In both these instruments the graduated circles are made of glass with the graduations etched on the glass. The great advantage of these glass circles as compared with ordinary silvered ones is that the graduations are very much finer than any that can be made on silver and this makes it possible to use a higher magnification in the micrometers. They are also less affected by temperature changes, but, for use in the tropics, they have the disadvantage that they are liable to attack by "fungus" (Vol. I, page 34). In addition, Canadian experience indicates that it is most desirable to have a thorough overhaul of these instruments by an expert about once a year. In this respect, the older type of ordinary micrometer theodolite has the advantage, as it is not nearly so liable to get out of order.

Other advantages of the direct reading as compared with the older types of theodolites are :—

1. They are much smaller and lighter.
2. All readings can be made from the same side of the instrument facing the eye end of the telescope and there is no need for the observer to move his position or to walk round the instrument to read the different micrometers and levels.
3. The two diameters of a circle, 180° apart, can be brought together or slightly separated and viewed simultaneously in a single eyepiece.
4. The micrometers are so designed that a single reading gives the mean of the readings at diametrically opposite parts of the circle.
5. The illumination is brilliantly and evenly spread over the field of view of the micrometers, and illumination for night observation is easily provided.
6. The instrument can be made watertight and dustproof more easily than the ordinary instrument.

The Optical Micrometer. The characteristic feature of the double reading theodolite is that, by means of a system of prisms or mirrors, the graduations of diametrically opposite sides of the circle are brought into the same field of view, and can be read in a single microscope, so that it is unnecessary for the surveyor to move from one side of the instrument to the other when taking his observations. The observed reading is then the arithmetical mean of the readings which would otherwise be obtained by using two micrometers 180° apart. In this way, errors due to eccentricity of the circle are automatically eliminated.

In Fig. 80 the numbers 64, 65, 66 and 67 represent graduations on the upper part of the circle, as seen through a small window above the circle.* Above these are the images of the graduations of the lower and diametrically opposite part of the circle when these have been reflected to appear in the same field of view as the others. The vertical line with

* Note that the figure is drawn for the case of the Zeiss and Wild theodolites where the opposite graduations appear in the reflected images to run or increase in *opposite* directions. In the Tavistock theodolite the graduations on the reflected images appear to run or increase in the *same* direction.

arrows is an index line with reference to which the observations are made.

In the upper part of the figure the graduation 65° falls short of the index line by an amount x, and, in the lower part of the figure, 245° falls short of the index line by an amount y. If the lower reading were taken by means of a second micrometer, the observed mean angle would be taken as $65° + \frac{x + y}{2d}$, where d is the distance between successive graduations on the circle. As the outer graduations and figures in the upper part of the figure are mirror images of those in the lower, the 245° graduation in the upper part also falls short of the index line by the amount y. Hence, the reading is still $65° + \frac{x + y}{2d} = 65° + \frac{z}{2d}$ where $z = x + y$ is the distance between the 65° and 245° graduations as seen through the window in the upper part of the figure.

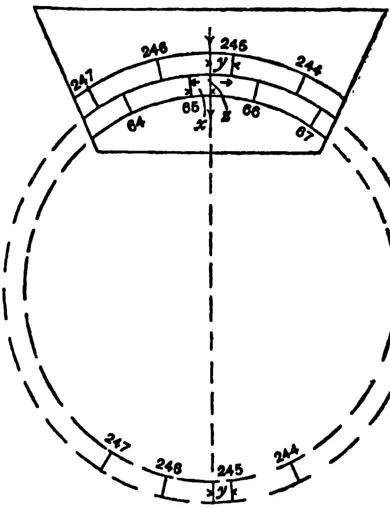


Fig. 80.

It will be noted that, if we imagine the outer graduations to move to the left, the 246° and 66°, and the 247° and 67°, graduations have passed each other but the 244° and 245° have not passed the 64° and 65° marks.

Fig. 81 (a) shows a circle graduated to 20′. The distance between the 30° 40′ and 210° 40′ graduations is seen to be 0·8 of the length of a 20′ interval. Hence, reading $= 30°\ 40' + \frac{0{\cdot}8 \times 20}{2} = 30°\ 48'$. Alternatively, the distance between the 30° and the 210° graduations is seen

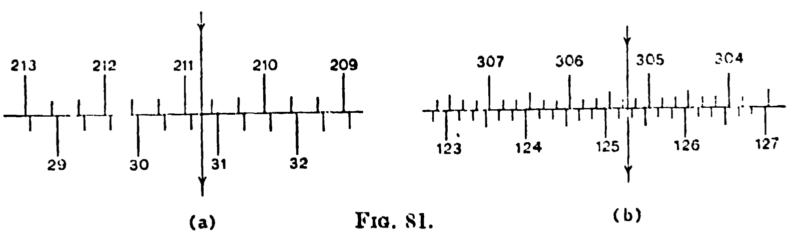


Fig. 81.

to be 1·6 times the distance between two successive degree divisions. Hence, the reading is $30° + \frac{1{\cdot}6 \times 60}{2} = 30°\ 48'$. Similarly, in Fig. 81 (b) the distance between the 125° 10′ and 305° 10′ graduations is seen to be 1·3 times a 10′ interval. The reading is consequently $125°\ 10' + \frac{1{\cdot}3 \times 10}{2} = 125°\ 16{\cdot}5'$.

It will also be noticed that the index line, shown in the figure by a vertical line with double arrows, acts as little more than a guide to obtain the coarser reading. The finer measurements depend solely on the variable distance between two graduations on the opposing scales and not on any measurement made to the index line. Various types of optical micrometers differ mainly in the means adopted to measure the distance between the relevant graduation marks. In the case of the Zeiss Universal and the Wild smaller and larger theodolites this distance is measured by the amount of movement necessary to bring the images of the diametrically opposite graduations into coincidence, and this method is also used in the Casella double reading theodolite (Vol. I, pages 78 and 76).* In the case of the Tavistock theodolite, both small and large models, the images of opposing graduations are not brought into coincidence but so that they lie on either side of, and equidistant from, an index line which appears in a small window in the field of view of the microscope. In all these cases the finer readings are measured on the drum of the micrometer which imparts the necessary motion to the images of the graduations, an image of part of the circumference of this drum also being seen in the field of view of the microscope.

In the Zeiss Universal and Wild theodolites the two images of the circle graduations are made to coincide by means of equal and opposite displacements of the rays from opposite sides of the circle as they pass through two parallel plate micrometers (page 399). The rotations of the micrometers in opposing directions are combined in a single motion which is read on a drum, an image of which appears in the field of view of the microscope, either above the image of the graduations in the case of the Wild or below it in the case of the Zeiss. Fig. 82 shows the appearance of the images in the microscope of the Zeiss instrument, the reading in this case being 358° 21′ 55″·7. The appearance of the images in the large Wild instrument is shown in Fig. 85, page 206

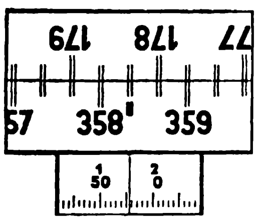


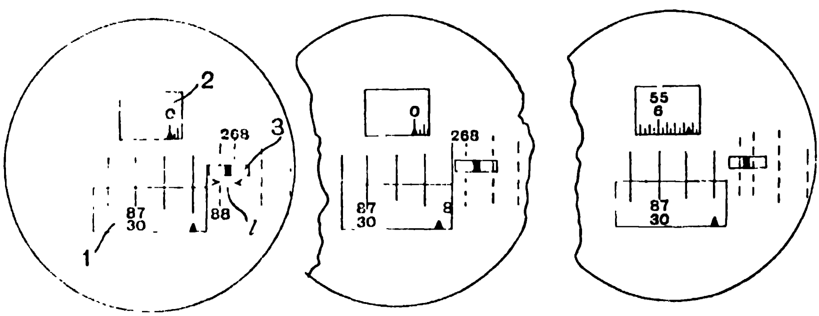
FIG. 82.

In the Tavistock theodolite, the line of separation between the opposite graduations is parallel to the graduations themselves, instead of being at right angles to them, as in the case of the Wild and Zeiss instruments. Also, the images of the diametrically opposite graduations are such that the latter appear to run or increase, and when the milled head of the micrometer is turned, to move *together*, in the *same* direction and the reading is taken when these graduations appear to be equally spaced on either side of a fixed index line. The motion is controlled by two travelling prisms which are mounted on a single frame, the motion of the frame being measured by the graduations on a glass drum, an image of which appears in a small window in the field of view of the microscope.

* The Casella and Zeiss Universal as well as the smaller models of the Wild and Tavistock theodolites are only suitable for use on minor triangulation or on precise traversing. For primary triangulation for geodetic purposes the instruments now generally used are the larger Wild or the geodetic Tavistock.

Figs. 89 and 90 show the optical arrangements of the optical micrometers for reading the circles of the Tavistock theodolite.

The three windows which appear in the field of view of the microscope for reading the horizontal circle of the Tavistock geodetic theodolite are shown in Fig. 83 (a). The circles in this instrument are graduated to 10 minutes of arc. Coarse readings, to the nearest 10′, are taken in window 1, and the finer readings, direct to 0″·5, in window 2, the graduations which appear in this window being those on the drum of the micrometer which controls the movement of the images of the circle graduations. In the centre of window 3 there is a coarse line which acts as an index line. This line is really the line of separation between two adjoining reflecting prisms, the image of the graduations from one part of the circle being reflected in the left-hand prism and that of the diametrically opposite graduations being reflected in the right-hand one. (The two prisms concerned are shown as JJ in Fig. 89.) Consequently, graduations from one part of the circle only are seen on one side of the index line and



(a) (b) (c)

FIG. 83.

those from the opposite part on the other. The reading is taken when the index line appears to lie equally spaced between the images of the two graduations that appear on either side of it.

In Fig. 83 (a) the graduations 88° and 268° are equally spaced on either side of the index mark in window 3. The reading on the centre of the index line for the 88° graduation is $88° + \frac{l}{2}$, where l is the distance, in seconds of arc, between the two graduation marks, and, for the 268° graduation, it will be $268° - \frac{l}{2}$. Hence the mean reading

$$= \frac{88° + \frac{l}{2} + 88° - \frac{l}{2}}{2} = 88°.$$

In window 1, however, the graduation 87° 50′ appears against the pointer in that window and the reading on the scale in window 2 is 0. Consequently, the reading will be taken as 87° 50′.

Let the upper part of the theodolite now be turned slightly to the right so that the pointer in window 1 comes somewhere between the 87° 50′ and

88° marks. As the screw which operates the micrometer has not been touched, the reading in window 2 will still be zero. In window 3 either no graduations will appear at all or else they will be unequally disposed about the index mark as in Fig. 83 (b).

Let the milled head which works the micrometer be rotated until the index line in window 3 appears to come in the middle of the space between two graduations. The appearance will be as shown in Fig. 83 (c). The pointer in window 2 now reads 6′ 58″·5, while the pointer in window 1 reads 87° 50′. The complete reading is therefore 87° 56′ 58″·5.

From the system of reading adopted it will be seen that the reading of the 88° graduation in Fig. 83 (b) may be considered to be referred to an imaginary index $\frac{1}{2}l$ graduations to the left of the centre of the black index line in window 3, and the 268° reading to an imaginary index $\frac{1}{2}l$ graduations to the right of the centre of the black index line. Hence, if x be the distance, in scale graduations, of the 88° mark to its imaginary index, and y the distance, in scale graduations, of the 268° mark to its imaginary index, the mean reading referred to the centre of the black index line in the window will be :—

$$\tfrac{1}{2}\left(88^\circ + x + \frac{l}{2} + 88^\circ + y - \frac{l}{2}\right) = 88^\circ + \frac{(x+y)}{2},$$

where x and y are very nearly equal in value. The function of the micrometer is therefore to measure the interval $\frac{1}{2}(x + y)$. As the readings of the graduations on the circle itself are referred to the pointer in window 1, the reading to be taken is $87^\circ\ 50' + \frac{1}{2}(x + y)$.

The relative positions of the windows and circle and micrometer divisions are also shown in Fig. 88 (a), and, although this particular diagram has been drawn to illustrate the arrangement on the small Tavistock theodolite, the principle is the same for the geodetic model, the only difference being a 10′ instead of a 20′ interval between the graduations on the horizontal circle, and the micrometer shown on the larger instrument is graduated to half instead of one second intervals.

The Wild Precise Theodolite for Primary Triangulation. This instrument, considering the accuracy of its readings, is remarkable for its lightness, the theodolite itself weighing only 22¾ lbs. and the special metal carrying case 12½ lbs. Both circles are made of glass, the horizontal circle being 5½ in. diameter and the vertical circle $3\frac{8}{11}$ in. The graduation interval of the horizontal circle is 4′ and of the vertical circle 8′, readings being made on the optical micrometer direct to 0″·2 and by estimation to 0″·02. The reading microscope for reading both circles is placed alongside and parallel to the telescope. This telescope has an effective aperture of 2·4 in., is 10¼ in. long, and is of the internal focusing type. Three eyepieces, giving magnifications of 24, 30 and 40 diameters, are provided with the instrument, the magnification ordinarily used being 30 diameters. Fig. 84 shows the general appearance of the instrument.

Illumination of the circles is provided by light reflected through two small prisms, one for illuminating the horizontal and the other for illuminating the vertical circle. Electrical illumination is also obtainable if desired. This includes illumination of the horizontal and vertical circles, the diaphragm of the telescope and the altitude level. This level

is read through one of two prisms, one of which faces the observer in either the face right or face left positions.

The optical arrangements of this model are very similar to those of the Wild Universal Theodolite which are described in Vol. I, pages 77–78.

The micrometers for reading the horizontal and vertical circles are both viewed in the same eyepiece which lies at the side of the telescope and is clearly seen in the illustration. To view the horizontal circle reading, an "inverter" knob, which is to be found on one standard, is turned in a clockwise direction; to view the vertical circle reading, this knob is turned in the reverse direction. Two windows will be seen in the field of view of the microscope. In the top window, the graduations of opposite parts of the circle are seen separated by a horizontal line, a vertical line in the bottom half of the window serving as an index from which the "coarse" readings are taken. As a general rule, the two sets of graduations do not coincide. To read the micrometer, the top knurled knob on the standard which does not carry the altitude level is turned so that the two sets of graduations appear to approach one another. Movement is continued until equivalent graduations in the top window are brought into coincidence. The seconds readings will then be given by the scale and pointer in the lower window.

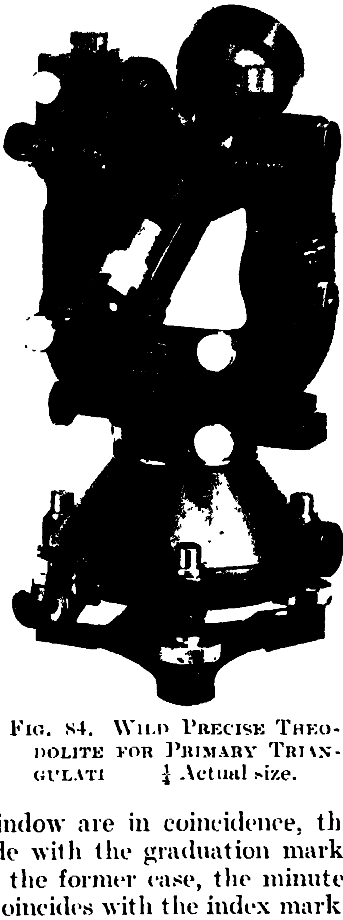
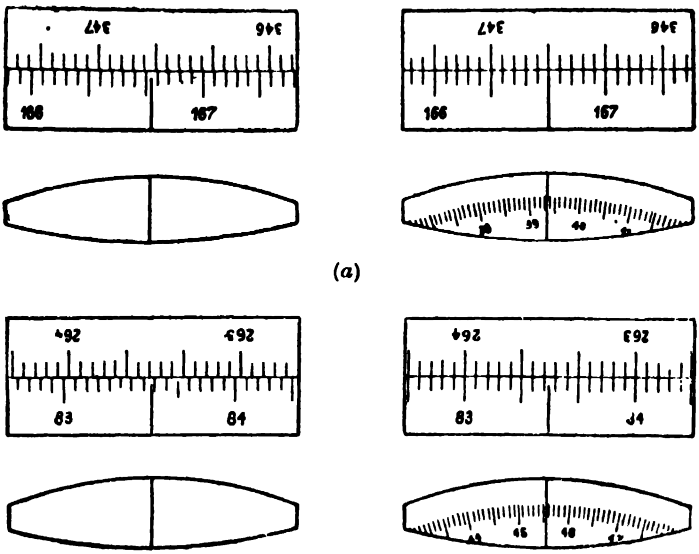


FIG. 84. WILD PRECISE THEODOLITE FOR PRIMARY TRIANGULATI $\frac{1}{4}$ Actual size.

When the graduations in the upper window are in coincidence, the pointer in that window will either coincide with the graduation marks or will come half way between them. In the former case, the minutes are read direct from the 4′ division which coincides with the index mark; in the latter case, 2′ must be added to the nearest 4′ division lying to the left of the index mark. The reading on the seconds scale in the bottom window is one half of the proper reading. Accordingly, the number of seconds which are read on this scale must be doubled, or, better still, opposite graduations in the upper window should be brought into coincidence twice and the two readings on the seconds scale added together.

The method of reading the micrometer of the horizontal circle will be clearly understood from Fig. 85. In (a) the index in the upper window in the right-hand figure coincides with the graduation 166° 40′ and this figure therefore represents the coarse reading. The opposite graduations were twice brought into coincidence, the first reading on the seconds scale in the lower window being 39″·34 and the second reading 39″·39. Hence, the reading to be taken is 166° 40′ + 39″·34 + 39″·39 =

166° 41′ 18″·73. In Fig. 85 (*b*) the index mark comes half way between the 83° 28′ and 83° 32′ divisions so that the coarse reading is 83° 30′. The two readings on the seconds scale were 45″·56 and 45″·50. Hence, the reading on the circle is 83° 31′ 31″·06.



(*a*)

(*b*)

Fig. 85.

The vertical circle, owing to its smaller size, is graduated into 180° instead of 360° and the interval between graduations is 8′ instead of 4′. Hence, the readings on this circle are taken as follows :—

1. Having brought opposite graduations into coincidence in the upper window, book degrees and minutes.
2. Take two separate readings of seconds. (Two intersections of the distant mark.)
3. Add the two readings of the seconds and add the result to the degrees and minutes.
4. Subtract the total from 90°.
5. Double the result.

To make this clear, the work may be set out as in the following example ·

Reading.	Sum − 90°.	Angle doubled.
90° 02′ 47″·74 47″·68	0° 03′ 35″·42	0° 07′ 10″·88

If the horizontal circle is to be set on a given reading, the seconds of the reading should be halved and the seconds drum set to read the result. This is done by means of the slow-motion micrometer seconds drum screw on one standard. Next set the circle, by means of the fine motion screw on the horizontal plate (covered by a hood), so that the required degrees and minutes are in coincidence with the opposite graduations in the upper window. After the instrument has been set, care must be

taken to close the hood over the fine motion screw so as to obviate an accidental disturbance of the setting.

The ordinary adjustments of the Wild Theodolite are similar to those of an ordinary theodolite. Adjustments of the optical micrometers should be left to the makers.

For setting on a pillar, the instrument is provided with a solid base plate which can easily be centered by means of a special centering pin with circular level.

The Wild "Astronomical" Theodolite. Messrs. Wild have very recently (1941) brought out a new theodolite, Model T4, which is larger than the ordinary geodetic model and which is specially designed for

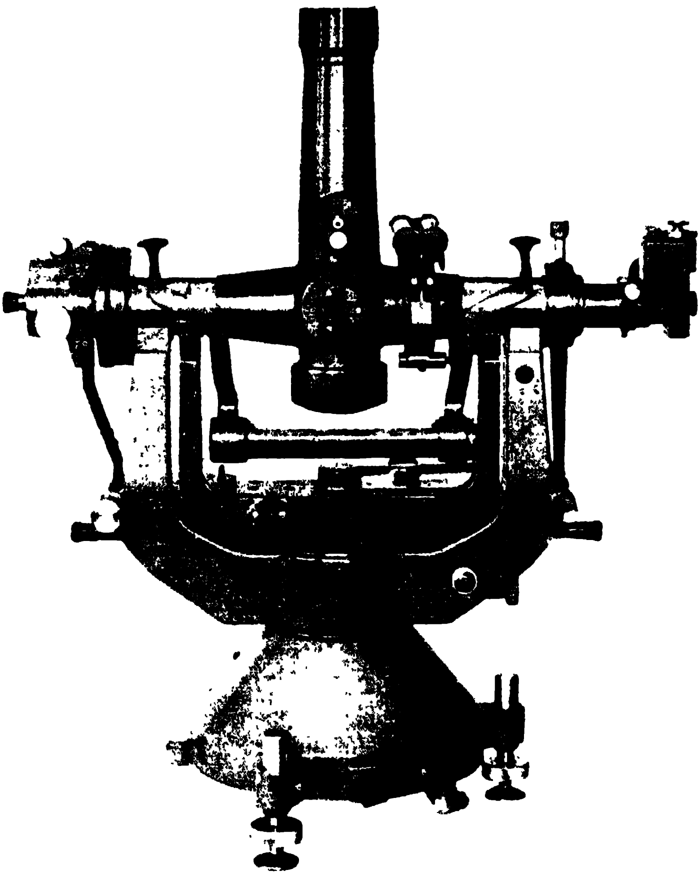


FIG. 86. WILD ASTRONOMICAL THEODOLITE.

precise astronomical observations. This instrument is provided with a horizontal circle almost double the diameter of that on the geodetic model, so that not only can horizontal angles be measured but the observations of these angles should also be considerably more accurate

than similar ones taken with the smaller instrument. The theodolite is of the "broken telescope" type; that is, the image formed in the telescope is viewed through an eyepiece placed at one end of the trunnion axis, the latter being made hollow so that rays of light proceeding from the objective are reflected through it by means of a right-angled prism placed at the end of the short telescope barrel. The general appearance of the instrument is shown in Fig. 86.

One particularly interesting feature of this theodolite is that reversal of the horizontal axis and telescope is carried out by a special hydraulic arrangement which ensures freedom from vibration. Readings of both horizontal and vertical circles are taken with an optical coincidence micrometer, a single reading giving, as in the case of the ordinary model, the mean of the readings on opposite sides of the circle.

An impersonal eyepiece micrometer, which can be rotated through 90° between stops, so that it can be used as either a horizontal o. vertical micrometer, is fitted to the instrument and this, of course, makes the latter peculiarly suitable for high-class astronomical work. Electrical lighting, to illuminate both circles and field, is built into the body, the connection to the battery being made by means of a plug at the base.

The principal dimensions of the instrument are as follows: Diameter of horizontal circle 10 in. Graduation interval 2′ with direct reading to 0″·1. Diameter of vertical circle 3·5 in. Graduation interval 20′ with direct reading to 1″. Clear aperture of object glass 2·4 in. and magnification of telescope 65 diameters. Sensitivity of suspending level, and of the two Horrebow-Talcott levels which are supplied with the instrument, 1″ per 2 mm. Sensitivity of level for vertical circle 5″ per 2 mm.

The Geodetic Tavistock Theodolite. This instrument, manufactured by Messrs. Cooke, Troughton & Simms, now weighs 24 lbs. and the case 18 lbs. Both circles are of glass, the diameter of the horizontal circle being 5 in. and of the vertical circle $2\frac{3}{4}$ in. The horizontal circle graduations are at 10′ intervals and the readings on the optical micrometer are made direct to 0″·5. The aperture of the telescope, which is of 10·1 in. focal length and an overall length of $11\frac{1}{2}$ in., is 2·375 in., the telescope being of the self-focusing type. Two eyepieces are provided and these give to the telescope magnifications of 20 and 30 diameters. Illumination is by daylight, but electrical illumination can also be supplied as an extra. This electrical equipment gives illumination to the diaphragm of the telescope, the optical micrometers and the altitude level. Other fittings which can be obtained as extras are a striding level, eye piece micrometer, diagonal eyepiece and a canvas outer case. A tripod is included with the ordinary equipment, but, for main triangulation, the instrument is best used set on a concrete pillar.

Fig. 87 shows the instrument complete with striding level, eye-piece micrometer and diagonal eyepiece. The eyepiece micrometer is mounted in a conical bearing so that it can be rotated through an angle of 90° against stops and clamped in either position.

It will be noticed that there are separate microscopes or eyepieces for reading the micrometers of the horizontal and vertical circles. These eyepieces are pivoted and can be swung into any convenient reading position, so that the micrometers can be read by the observer when

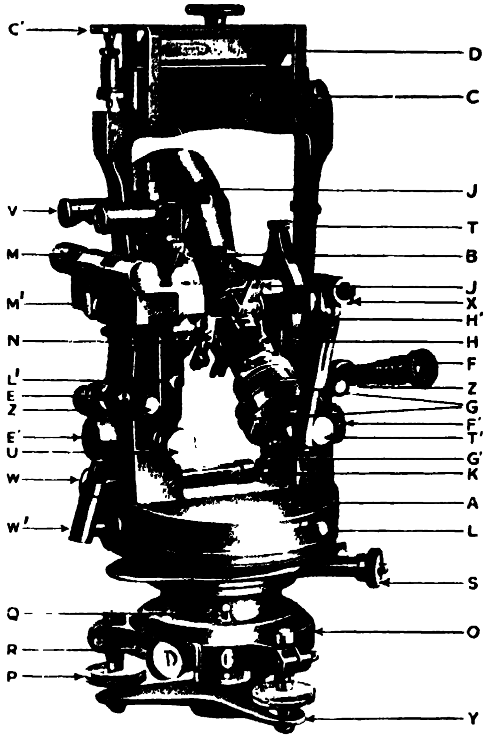


FIG. 87. THE GEODETIC TAVISTOCK THEODOLITE FOR PRIMARY TRIANGULATION.

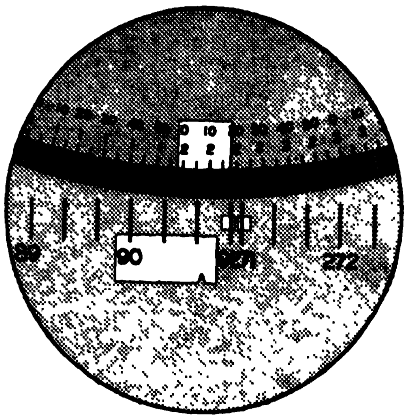
(By permission of Messrs. Cooke, Troughton & Simms.)

LETTER REFERENCES

- **A** Coverplate containing optical system of horizontal circle.
- **B** Box containing optical system for vertical circle.
- **C** Striding level casing.
- **C'** Striding level adjusting screw.
- **D** Striding level mirror.
- **E** Reading eyepiece for vertical circle.
- **E'** Micrometer milled head for vertical circle.
- **F** Reading eyepiece for horizontal circle.
- **F'** Micrometer milled head for horizontal circle.
- **G** Adjusting screws of reticule or diaphragm.
- **G'** Clamping screws for reticule adjustment.
- **H** Focusing ring for telescope.
- **H'** Diagonal eyepiece.
- **J** Luminous sights.
- **K** Plate spirit level.
- **L** Screwed cap covering optical adjustment in A.
- **L'** Screwed cap covering optical adjustment in B.
- **M** Altitude spirit level.
- **M'** Altitude spirit level prism reader.
- **N** Zero setting adjustment for vertical circle.
- **O** Tribrach.
- **P** Footscrews.
- **Q** Weight relieving adjusting screw.
- **R** Milled head for rotating horizontal circle.
- **S** Clamp for upper plate.
- **S'** Slow motion screw for upper plate (not seen).
- **T** Clamp for telescope.
- **T'** Slow motion screw for telescope.
- **U** Setting screw for altitude spirit level.
- **V** Electric illumination for vertical circle and altitude spirit level.
- **W** Daylight illumination reflector for horizontal circle
- **W'** Electric illumination for horizontal circle.
- **X** Electric illumination to telescope graticule.
- **Y** Plate securing theodolite to tripod head.
- **Z** Spring plungers locking circle reading eyepieces.

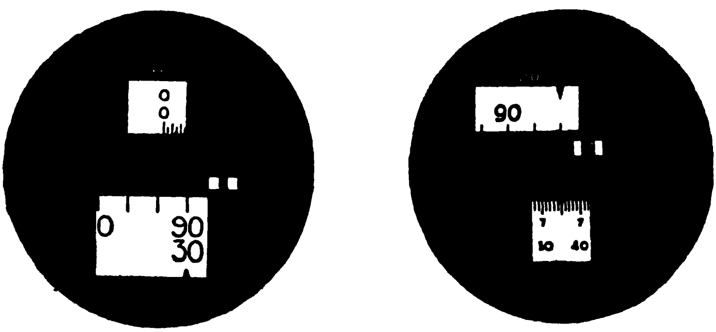
looking at the telescope in either the face right or face left position. Hence, it is not necessary to move round the instrument when observations are in progress as all the neccsssay observations can be made from the one position.

The principles involved in reading the optical micrometers of the Tavistock theodolite have already been described in pages 200–204 and,



(*a*)

Large and Square Windows and Circle and Micrometer Divisions.



(*b*) (c)

FIG. 88. EXAMPLES OF CIRCLE READINGS.

(*By permission of Messrs. Cooke, Troughton & Simms.*)

from this description, the methods of reading should easily be understood. After the instrument has been sighted and clamped to view the object whose direction is to be observed, look into the horizontal circle reading eyepiece (F in Fig. 87) and assume that the circle reading in the largest window, with reference to the small index tooth, is approximately 90° 30′ (*see* Fig. 88 (b)). It is possible that no circle graduations will appear in the smallest aperture, but, if they do, they will generally not be equally spaced on either side of the index mark. In either event, the odd minutes and seconds must be found by rotating the micrometer milled head F′ until two graduations are equally spaced on either side of the index line

in the smallest window, when the minutes and seconds are read on the image of the micrometer drum that appears in the top square-shaped window. Fig. 88 (b) shows the appearance in the field of view of the eyepiece of the horizontal circle and Fig. 88 (c) the appearance in the field of view of the eyepiece of the vertical circle. In the first case, the reading in the degrees window is 90° 30′ and in the seconds window it is 0′ 1″·5. Hence, the reading on the horizontal circle is 90° 30′ 01″·5. Similarly, the reading on the vertical circle is 90° 40′ plus 7′ 35″ = 90° 47′ 35″.

Fig. 88 (a) shows the relative positions of the windows and of the graduations underneath them. The graduations actually shown in this diagram relate to the small Tavistock theodolite, where the divisions on the main horizontal circle are at 20′ instead of 10′ intervals, but, apart from this, the arrangement of the relative positions of windows and circles and micrometer divisions in the geodetic model is exactly the same as in the smaller instrument.

The vertical circle readings are viewed in the eyepiece E, and the setting of the graduations on either side of the index line in the small window is controlled by the milled head E′ in Fig. 87.

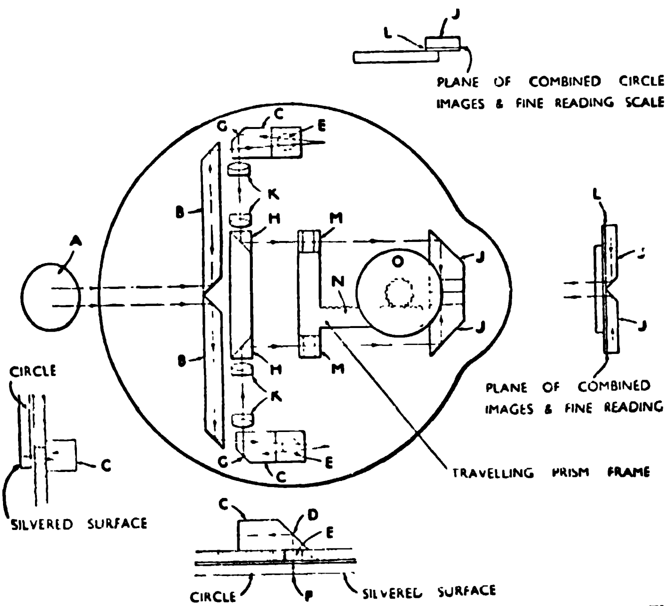


FIG. 89. DIAGRAM SHOWING OPTICAL SYSTEM IN MICROMETER OF TAVISTOCK THEODOLITE.

(By permission of Messrs Cooke, Troughton & Simms.)

The paths of the rays of light in the micrometer are shown diagrammatically in Fig. 89. A beam of light proceeding from A is split up into two parts by the central reflecting surfaces of the prisms B in such a way that one half of the beam travels in one direction and the other half in the opposite direction. These halves travel along the paths indicated by the arrows until they enter the prisms C. Here they are deflected by

the surface D (see small figure at bottom) from a plane parallel to that of the circle to a direction at right angles to it, and so enter the window E which lies in front of the silvered graduated arc. Before leaving B and entering C they are slightly deflected, and, after being reflected in the graduated circle, they undergo a second reflection at the surfaces D, so that they again travel parallel to the plane of the circle to the surface G, whence they are directed to the reflecting surfaces of the prisms H. From H they proceed in parallel paths to the prism J, where they are reflected towards each other, and finally emerge together in a direction at right angles to the plane of the circle.

Two objectives K are placed between G and H and the combination is such that an image of the figures and graduations appearing under the window E is formed in the focal plane of the prisms J. The parts of the micrometer are so arranged that the images from opposite diameters of the circle are brought together and appear continuous on either side of the index line as shown in Fig. 88 (a), the index line in the small window being the line of junction of the edges of the outward reflecting surfaces of the prisms J.

Two deflecting prisms M are mounted on a travelling frame so that they intercept the beams on their passage from H to J. The frame carries a rack N which gears with a pinion attached to the glass drum O. Movement of the prisms, which is effected by rotating the milled head of the micrometer, displaces the images of the graduations formed in the focal plane L of the prism J by an amount which is proportional to the distance of the prisms from L, and the amount of this displacement is measured by reference to the divisions on the drum O, these divisions being viewed in the fine reading window of the micrometer.

The actual appearance of the optical parts of the two micrometers is shown in Fig. 90, the top figure referring to the micrometer of the horizontal circle and the bottom figure to the micrometer of the vertical circle.

Adjustments of the Geodetic Tavistock Theodolite. The ordinary adjustments for levelling the instrument and for collimation, etc., are similar to those for an ordinary theodolite, but the following two are special to this particular instrument.

1. Adjustment of width of light gap in micrometer on each side of the setting index.

2. Adjustment of focus of the circle graduations and for micrometer run.

The first of these adjustments consists in spacing the images of the graduations in the smallest window in the micrometer so that the light gap between them and the central or setting index is of a convenient width. This width is about 30 seconds of arc, as read in the micrometer by bringing the circle graduations into coincidence with the index and noting the difference in micrometer reading.

To carry out this adjustment for the horizontal circle, tighten the clamp S, Fig. 87, and, by means of the milled head F′, set the micrometer so that the narrow gaps on each side of the setting index are equal. If these gaps are of a convenient width no further adjustment is necessary, but, if they are too narrow or too wide, remove the screwed

cap L in the horizontal cover plate A. An adjustment spindle will be seen in the aperture and this spindle can be engaged with the key provided. On turning the key, the width of one gap will be altered. Now use milled head F′ to equalise the two gaps. If they are still too narrow or too wide, the above operations should be repeated until the desired width is obtained, when the cap L should be replaced.

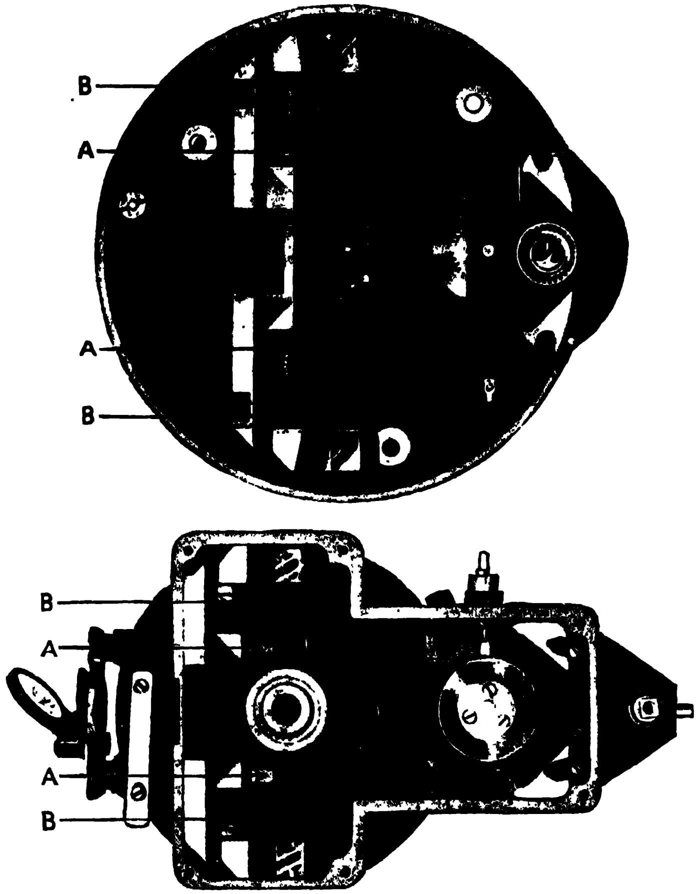


FIG. 90. VIEW OF OPTICAL SYSTEM OF GEODETIC TAVISTOCK THEODOLITE.

When setting the light gap for the vertical circle, a screwdriver must be used to turn the setting screw at L′, which is normally covered with a clip. In this circle, movement of the spindle alters the width of both gaps, so that the adjustment can be made at once and there is no need to equalise the gaps a second time. This adjustment does not disturb the index setting of the vertical circle.

The second adjustment is for focus of the circle graduations and for micrometer run, and it involves movements of the lenses K in Fig. 89. These two lenses are shown as B and A in Fig. 90, which shows the

appearance inside the micrometer boxes when the covers are removed according to the instructions provided by the makers. The lens A controls the magnification but has little effect on the position of the image, this position being controlled mainly by that of the lens B. A movement of A towards B increases the magnification, and a movement of B towards A causes the plane of the image of the graduations to move inwards and fall short of the plane of the setting index and the graduations on the fine reading circle. The adjustment is made by removing the small grub screw clamping the mount of the lens A and then inserting a pointed tool through the screw hole in order to move A away from B. If the movement required is rather large, it may be necessary to move cell B in order to focus the circle divisions properly. This movement of B may upset the adjustment of A, in which case the operations must be repeated. The test for run is to set the fine reading scale to read zero, and then to use the fine motion screw to bring the nearest divisions into a position where the light gaps are equalised. The circle being kept clamped, the micrometer drum is then rotated until the gaps are equalised over the next division, when the fine reading scale should read 10′ very approximately.

Correction for Vertical Circle Index Setting Error. On account of the method which has to be adopted for figuring the circle, the ordinary method of eliminating index setting error by taking the mean of face right and face left observations does not apply when vertical angles are observed with a double reading theodolite. Hence it becomes necessary to determine the amount of the error and to apply it to the observed readings. The procedure is as follows :—

Take face right and face left pointings and readings and add the results together. If the sum is exactly 180° there is no index error, but, if it is not exactly equal to 180°, half the excess or deficit represents the index error. If the sum of the readings is more than 180°, the circle readings are too large and the error is subtractive. If the sum of the readings is less than 180°, the index error is additive to each reading.

Example. Let the two readings be :—

Face Left	=	30° 20′ 28″
Face Right	=	149 39 36
Sum	=	180° 00′ 04″
Hence, Index Error	=	−2″·0

and corrected readings are :—

Face Left	=	30° 20′ 26″
Face Right	=	149 39 34
Sum	=	180° 00′ 00″

The operation should be repeated a number of times and the mean value of the index error taken.

Watts "Microptic" Theodolite No. 2. Besides being suitable for main triangulation up to and including secondary work, the Watts "Microptic" Theodolite No. 2, which enables angles to be read direct to 1″ of arc and estimated to 0″·2, has the advantage that a special traversing model can be obtained with special equipment for traversing, thus enabling the instrument to be used with the "three-tripod" system of observing (page 236, and Vol. I, page 84).

The circles in this instrument are both made of glass, the horizontal circle being 3·875 in. (98 mm.) and the vertical circle 3 in. (76 mm.) in diameter, and readings from opposite ends of a diameter are brought together in the optical micrometer, the eyepiece of which is fixed alongside the eyepiece of the telescope. Both circles are divided to 10′ of arc, and the optical micrometer enables readings to be taken direct to single seconds and estimated to fifths of a second. The telescope has an aperture

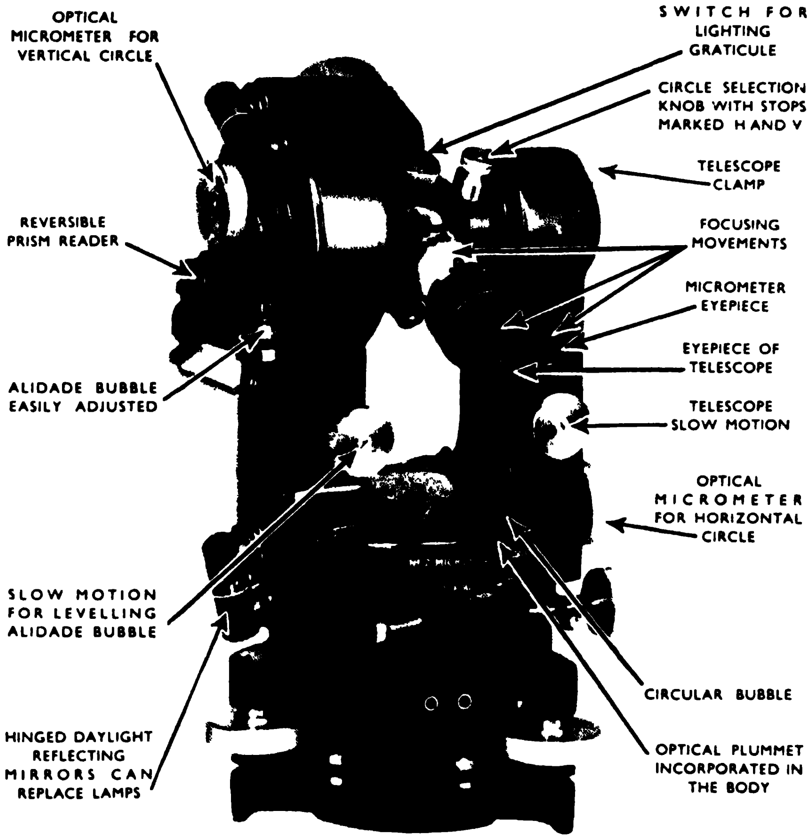


FIG. 91. WATTS "MICROPTIC" THEODOLITE No. 2.

of $1\frac{5}{8}$ in. (41 mm.), a total length of 7 in. (178 mm.), and a magnification of 27 diameters. The sensitivity of the plate and altitude bubbles is 20″ per millimetre division. In addition to the ordinary plate levels, a circular level for preliminary levelling is mounted on the tribrach and, if necessary, a striding level and a Talcott bubble for Talcott latitude observations can be obtained as extras, the sensitivity of each of the last two levels being 10″ per 2 mm. division. An optical plummet is incorporated in the body of the instrument; and built-in illumination for circles, telescope graticule, and plate and altitude bubbles, is provided, hinged

daylight reflecting mirrors replacing artificial lighting when conditions are favourable. The instrument, which is $12\frac{1}{2}$ in. high, weighs $12\frac{1}{2}$ lbs., or $17\frac{1}{2}$ lbs. with the water-proof case, which measures $8 \times 7 \times 14$ in. This water-proof case cannot be closed until the instrument is secured in position.

The standard model is mounted on an ordinary tribrach, but the upper part of the special model (No. B.61) for using with the traverse equipment is mounted on a circular bevelled base which fits into the tops of the special tribrachs that support either instrument or targets. Index heads for catenary taping, and a special micrometer index head for standardisations or fine measurements, which also fit into the special tribrachs, are obtainable as extras. Other extras are diagonal eyepieces for telescope

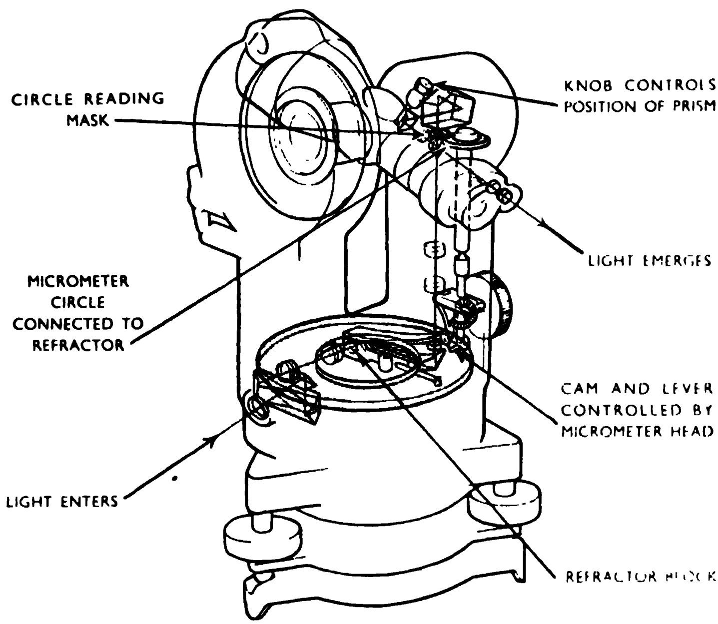


Fig. 92. The Horizontal Circle Optical System.

and circle reader, an optical plummet for centering tribrachs, a prism eyecap for putting over both telescope and micrometer eyepieces, etc. The targets for the traversing equipment are mounted circular discs of 4 in. diameter, with a special sighting mark consisting of a large, dark Maltese cross and a small, fine, light cross in its centre. These targets, as well as the index heads and optical plummet, are fitted with two plate levels. The targets can be used either by day or by night, electrical illumination, obtainable as an extra, being used for night work.

The readings on both horizontal and vertical circles are viewed in the same micrometer eyepiece at the side of the telescope, though not at the same time. Rotation of a small knob on top of the telescope rotates a prism in the optical system, thus enabling the observer to select the readings of the circle which he wants to observe. This knob has stops

marked H and V to indicate which circle is required. The passage of the rays through the horizontal circle to the micrometer eyepiece is shown in Fig. 92.

The method of reading the circles will be understood from Fig. 93. The field of view contains three windows. An image of the graduations of the circle is seen in the large window in the centre and in the lower window appears an image of the micrometer scale graduated to single seconds and figured every ten seconds. The small window at the top shows images of graduations taken from the diametrically opposite divisions of the circle. From one side single lines at 20′ intervals appear

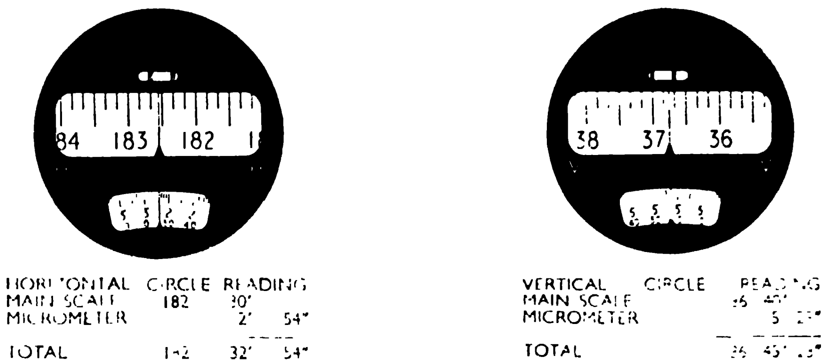


Fig. 93

and from the other side double lines at the same intervals. As the circle is rotated, these appear to move in opposite directions, but coincidence is obtained every 10′. For any setting of the circle, the degrees and nearest 10′ are read against an index line in the central window. The drum of the micrometer for the circle from which readings are to be taken is then rotated until the double lines in the top window are set symmetrically about the single lines, when the odd minutes and seconds can be read on the scale in the bottom window. The letters H and V below the large window indicate whether the micrometer is set for reading the horizontal or vertical circle.

The displacement of the images of the scales relative to one another is caused by the bending of a ray of light as it passes through a parallel plate reflector on its way from one side of the circle to the other. This refractor can be tilted by means of a cam and lever operated by the micrometer, so that the amount of relative displacement of the images to secure coincidence can be read directly on the micrometer circle seen in the bottom window.

Axis Strain in Light-Weight Geodetic Theodolites. As a result of several years' experience of the use of a number of early models of the Wild Geodetic Theodolite, the Canadian Geodetic Survey came to the conclusion that the results obtained with some of them were disappointing when compared with the results obtained from others or from older and larger types of theodolite. The discrepancies, although small enough to be

inappreciable in ordinary classes of work, were definitely noticeable when it was attempted to use the instruments for first-order observing. Accordingly, an elaborate investigation was made in the Physics Laboratory of the National Research Council of Canada and it was established that the main sources of trouble were due to phenomena known as "creep" and "axis strain," which appeared to be caused largely by defective design of the bearings and by the tolerances between the parts of the bearing surfaces being too close. A slight modification in the design of the axis system was proposed and tried, and, when this modification had been made, the instrument yielded satisfactory results. A full account of this investigation, and of the conclusions reached, which are of importance from the point of view of instrument design, will be found in a paper by J. L. Rannie and W. M. Dennis on "Improving the Performance of Primary Triangulation Theodolites as a Result of Laboratory Tests," which is published in *The Canadian Journal of Research*, Vol. 10, No. 3, March 1934.* In this paper, the authors recommend the adoption of the following precautions during observing with a view to eliminating the possible effects of creep and axis strain :—

1. The footscrews of light theodolites should be kept much tighter than those of heavier instruments.

2. The position of the footscrews should be changed several times during observations. Thus, a programme for observations might be divided into three parts and the direction of the footscrews altered through 120° between each part.

3. Whenever the elevation of pointing is altered the top centre of the telescope should be tapped to avoid error due to climb of the horizontal axis.

An account of some tests for axis strain carried out on Tavistock theodolites will be found in Hotine's paper on the "Re-triangulation of Great Britain" in *The Empire Survey Review*, Vol. IV, No. 29, pages 391–395. These tests showed that axis strain in these instruments was virtually non-existent. The axes of the Wild instruments have been re-designed in accordance with Rannie and Dennis's recommendations, and it is claimed that axis strain is absent in the later models.

FIELD OBSERVATIONS

Stability of Instrument. The instrument is supported on a scaffold, a masonry pier, or upon its own tripod set on the ground. In the latter case the tripod legs are supported on firmly driven stakes, the tops of which are level and at such a height that the observer can use the telescope without stooping.

In first- and second-order triangulation some form of observatory hut or tent must be erected round the instrument to protect it against air currents and to shade it from the sun. On observing scaffolds the tent must be entirely supported by the outer or observer's tower. It should be provided with means for lowering the walls sufficiently for sighting, or have removable flaps on each wall at the level of the telescope. For tertiary work an umbrella gives sufficient protection.

* See also *Empire Survey Review*, Vol. II, No. 13, July, 1934, pages 424–428, and Vol. III, No. 15, January, 1935, pages 2–5.

An instrument mounted on an elevated scaffold is exposed to the vibration of the instrument tower in wind, and the observations are also subject to error arising from twist of the tower under lateral heating. These effects are sometimes guarded against by wrapping the outer scaffold with canvas to shelter the inner. Observations from a movable tower are never of the same precision as those taken at ground level.

Conditions Favourable for Observation. Irregular atmospheric refraction forms a difficult source of error, particularly in the case of rays which are not greatly elevated above the ground. No uncertainty need be occasioned by visible phenomena, since work must be suspended when irregular refraction causes apparent trembling of the signals or when shimmer manifests itself by expanding a luminous signal into a wide vibrating sheet of light impossible of accurate bisection. Lateral refraction, however, may exercise the more dangerous effect of causing a slow swing of the image of the signal to one side of the vertical hair, although the light may appear of normal size and free from vibration.

Observations should therefore be made only under favourable atmospheric conditions. In densely cloudy weather observing on mast signals can be carried on all day. The best results with heliotropes are obtained from about 4 p.m. till sunset, but an hour or two at sunrise usually permit of satisfactory work. Observation on night signals is generally confined to the period between sunset and midnight. To minimise error due to lateral refraction, a rule followed in some surveys is that observations at each first-order station should be distributed over at least two days. Care, however, must be taken not to exalt lateral refraction into a fetish or a cover for all ills.

Relative Merits of Day and Night Observations. Except for short lines on which opaque signals can be satisfactorily employed, the usual practice is to observe either on heliotropes or lamps as the weather allows. The chief advantage gained by the addition of night work consists in its doubling the number of hours a day available for good observation, but in some respects observing can be carried on better at night than by day. With heliotrope signals serious delays are occasioned by cloudy weather. Night observation can be conducted with less interruption and sometimes with greater accuracy. Further, refraction is much greater at night than by day, and, in consequence, the line of sight is farther removed above intervening elevated ground, so that towers and signals erected exclusively for night observation may be rather lower than would be necessary for day work—a point of some importance sometimes.

In the tropics, trouble may be experienced in getting natives employed as signalmen to remain on isolated hills at night, as many of these hills are regarded with superstitious awe by the local inhabitants, who regard them as "juju."

General Methods of Observation. In precise angle measurement the routine of observation must be specially arranged to reduce to a minimum the effect of instrumental and observational errors. Half of the observations for each angle are made with the telescope direct and half with it reversed, to eliminate errors of collimation and horizontal axis. If the telescope does not transit, it must be removed from the standards

and replaced end for end with the trunnion pivots placed in the same supports as before reversal. Half of the observations are taken from left to right and half from right to left to eliminate errors arising in manipulation or due to twist of the instrument and its support caused by lateral heating. Accidental errors of signal bisection and of reading are reduced to any extent by increasing the number of observations. Errors of eccentricity are rendered negligible by reading all the micrometers at each observation, and the effect of graduation errors is sufficiently reduced by using a different part of the circle for each measurement.

Observational programmes belong either to the *Direction* or the *Repetition* system. In the direction, or reiteration, method the several angles at a station are measured in terms of the directions of their sides from that of an initial station. The signals are bisected successively, and a value is obtained for each direction at each of several rounds of observations. The initial or reference station should be that one of the triangulation stations which is most likely to be always visible. When all the sights are so long that one of the stations cannot be preferred for the purpose, a referring signal, of a form suitable for accurate bisection, may be established not less than a mile and a half away. This practice, however, is not advisable on precise work, and it is better to measure individual angles, or combinations of angles, using as R.O. any convenient station that happens to be showing even if this means loss of time and a station adjustment by least squares (page 281).

The distinguishing feature of the repetition method consists in measuring each angle independently by multiplying it mechanically on the circle, the result being obtained by dividing the multiple angle by the number of repetitions. Theoretically, any desired refinement of reading can be obtained by sufficiently increasing the number of repetitions, the effect being to reduce the least count correspondingly. Practically, however, owing chiefly to errors introduced in clamping, there exists for any instrument a limit beyond which the accuracy is not improved by increase of the number of repetitions. The system is designed for vernier instruments, and may be adopted when these are employed for fine angle work.

Vernier instruments, when they are used on main triangulation for the observation of angles by the method of repetition, are sometimes called "repeating instruments." The essential feature of a repeating instrument is that it must have a slow-motion screw for the lower plate. The larger instruments used in geodetic surveying do not always have this fitting and are thus only suitable for measurements by the direction or reiteration method. Hence, they are called "direction instruments."

Although the repetition method was formerly applied to primary observations, it is generally confined to secondary and tertiary work. The method of reiteration should be employed in primary triangulation. It is designed for micrometer instruments, and in British geodesy these have always been used for refined work to the exclusion of vernier instruments. Owing to the modern application of micrometer reading to all sizes of theodolites, the reiteration method is equally suitable for all grades of triangulation.

Programme of Measurement by Reiteration. To measure the angles AOB, BOC, and COD (Fig. 94), A being adopted as the initial station, first set one of the micrometers to about 0°, point on A with telescope direct, and read all micrometers. Swing on to B, C, and D successively, booking the micrometer readings after each pointing. Overshoot D, *i.e.* move the line of sight a little beyond D to the right. Again point on D, and read the micrometers. Swing on to C, B, and A successively, taking the readings at each. Overshoot A, reverse the telescope, setting to about 180° the microscope originally at 0°, and repeat the same observations to D and back to A. This constitutes one series, and yields four measures of each angle. The microscope originally at 0° is now brought to a new reading, and a second series is observed in the same manner on a different part of the circle. The required number of series depends upon the quality of the graduation and the number of equidistant micrometers as well as the grade of the work. Six or eight series as above, giving twenty-four to thirty-two measures, each derived from the mean of the micrometer readings, are sufficient for geodetic triangulation with modern instruments.

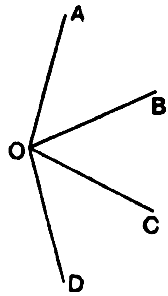


Fig. 94.

Instead of each series consisting of a swing right and a swing left on each face, as described, it is a common practice to use one face for the swing right and the other for the swing left. The number of pointings in a series is thus halved, so that to afford the same number of measures as before, twice as many zeroes are required, and graduation error is likely to be reduced. A second modification consists in closing the horizon by continuing each swing to finish on the initial station. This ensures detection of any disturbance or twist of the instrument during the round, but it is generally considered that the alternation of right and left swings is sufficient precaution against error from this source.

First-order Angle Observations. The average probable error of first-order angles in modern triangulation is less than 0″·4, so that, to attain the necessary consistency between the several measures, the utmost delicacy is required in manipulating the instrument and, particularly, the micrometers. Clamping must be lightly performed to avoid stress: for the same reason the vertical circle is better left free. In making pointings, the line of sight should be brought on to all the signals in the direction of swing to avoid possible error due to friction. Care should therefore be exercised not to overshoot signals except in order to reverse the direction of swing at the end of a set. When a horizontal eyepiece micrometer is fitted, the signals are bisected by movement of the vertical hair and not by the tangent screw. Several bisections are made before and after reading the circle micrometers, the mean reading of the eyepiece micrometer being applied to that of the circle. In making bisections, the observer should examine the image long enough to make sure that it is not swinging under the influence of lateral refraction.

For the elimination of periodic error of graduation, the change of zero between series must be such that the microscope readings for each signal are made on uniformly spaced points round the circle. For a three-

micrometer instrument the interval between these points should be about $\frac{60°}{n}$, where n is the number of series. When one microscope is set at 0°, the others are at 120° and 240° respectively, and, after reversal, the readings are 180°, 300°, and 60°. By reversal of the telescope one zero therefore gives, for each signal, micrometer readings at 60° intervals round the circle. For a two-micrometer instrument the successive difference should be about $\frac{180°}{n}$. The zero shift should not be an exact number of degrees in order better to distribute the microscope settings over the smallest division of the circle. The following are examples of the settings adopted in different surveys for primary triangulation.

Survey of India. Three-micrometer theodolite: six series, each of three swings right and three swings left to each face.

Telescope Direct:

0° 0′	70° 1′	140° 2′	210° 3′	280° 4′	350° 5′

Telescope Reversed:

180° 0′	250° 1′	320° 2′	30° 2′	100° 4′	170° 5′

Ordnance Survey, Test Triangulation in N.E. Scotland, 1910-12. Three-micrometer theodolite: eight series, partly of swing right and swing left on alternate faces, and partly of swing right and swing left to each face.

Telescope Direct:

0° 0′	45° 1′	90° 2′	135° 3′	200° 4′	245° 5′	290° 6′	335° 7′

Telescope Reversed:

180° 0′	225° 1′	270° 2′	315° 3′	20° 4′	65° 5′	110° 6′	155° 7′

Ordnance Survey. Re-triangulation of Great Britain. Geodetic Tavistock theodolite: double-face reiterations in two series, each on eight zeroes given by the following face left readings on the R.O.

First series:

0° 01′ 05″,	90° 08′ 55″,	45° 02′ 10″,	135° 07′ 50″,
22° 33′ 20″,	112° 36′ 40″,	67° 34′ 30″,	157° 35′ 30″.

Second series:

11° 15′ 05″,	101° 23′ 55″,	56° 17′ 10″,	146° 22′ 50″,
33° 48′ 20″,	123° 51′ 40″,	78° 49′ 30″,	168° 50′ 30″.

United States Coast and Geodetic Survey. Three-micrometer theodolite (one division of circle = 5 minutes): sixteen series, each of swing right on one face and swing left on the other.

Telescope Direct:

0° 0′ 40″ 195° 1′ 50″ 30° 3′ 10″ 225° 4′ 20″ 64° 0′ 40″...... 57° 4′ 20″

Telescope Reversed:

180° 0′ 40″ 15° 1′ 50″ 210° 3′ 10″ 45° 4′ 20″ 244° 0′ 40″...... 237° 4′ 20″,

the same intervals being repeated in groups of four, with a change of 18° 56′ 20″ between the groups.

Two micrometer theodolite (one division of circle = 5 minutes): sixteen series, each of swing right on one face and swing left on the other.

Telescope Direct:

0° 00′ 40″,	191° 01′ 50″,	22° 03′ 10″,	213° 04′ 20″,
45° 00′ 40″,	236° 01′ 50″	67° 03′ 10″,	258° 04′ 20″.

Telescope Reversed:

90° 00′ 40″,	281° 01′ 50″,	112° 03′ 10″,	303° 04′ 20″,
135° 00′ 40″,	326° 01′ 50″,	157° 03′ 10″,	348° 04′ 20″.

Note. In the first example it will be seen that, neglecting minutes, readings for each signal are made by the different micrometers at 10° intervals round the circle. In the second the intervals are 5° and 10°, in the third 11¼°, in the fourth 3° and 4° and in the fifth 11° and 12°.

In these settings the odd minutes and seconds are used to eliminate the effects of errors of run of the micrometers.

None of the above systems of settings makes any provision for altering the positions of the footscrews during observation, but the recent researches of Rannie and Dennis on axis strain in theodolites indicate that, unless the instrument is known to be free from axis strain, it is advisable to divide the observations into series and to alter the position of the foot-screws in relation to the R.O. between each series (page 217).

On completion of a series, the consistency between the measures may be found unsatisfactory, and additional swings are made if the discrepancy between any two measures in the series exceeds about 4″.* Discrepant measures are not to be omitted from the record, but the results of observations which are known to be not entirely satisfactory are noted as doubtful. A decision as to whether they will be retained or cancelled is made before adjusting the angles (page 280).

It sometimes happens in the course of a swing that one or more signals are invisible and must be omitted. When the opportunity occurs, such swings are completed, with the appropriate zeroes, by sighting the omitted stations in conjunction with the initial station. If the latter is obscured, reference may be made to any other one station previously included.

Second- and Third-order Reiteration Observations. Although the routine is much less elaborate for observations other than first-order, measures should in all cases be equally divided between the two faces and directions of swing. While the programme to be adopted should depend largely upon the character of the instrument, it is generally sufficient in second-order triangulation to use two or three zeroes with two faces to each and two swings on each face, or, alternatively, about five zeroes each of one swing on each face. For third-order work two zeroes with swing right on one face and swing left on the other are all that are necessary. Angles taken merely to fix points by intersection require only one zero with a swing on each face.

Programme of Measurement by Repetition. To measure angle AOB (Fig. 94), set one of the verniers to about 0°, point on A with telescope direct, and book the readings of all the verniers. Release the upper clamp, swing clockwise, and bisect B. Read a vernier to ascertain the approximate value of the angle. Slacken the lower clamp, turn *clockwise*, and again set on A. Loosen the upper clamp, and bisect B. Release the lower clamp, again turn clockwise, and point on A. Swing on to B as before. This constitutes three repetitions with telescope direct. Reverse the telescope, and, leaving the verniers unchanged, swing clockwise on to A. After making another three repetitions exactly as before, book the vernier readings. With the verniers unchanged, and the telescope still reversed, point on B, and, maintaining the clockwise direction of swing, make three repetitions on the exterior angle BOA, followed by three more with telescope direct. Finally, note the readings of the verniers. These six repetitions of the angle with six on its explement

* This applies only to the latest instruments in which the error of division does not exceed about 1″. In older primary instruments the *range* in a series of measures amounted to as much as 8″ in an angle of 90°.

constitute one set, and additional sets to the number required are taken in the same manner from different initial readings.

A programme for repetition observation must provide for the elimination of error introduced by repeated clamping. This may be accomplished either by making half the repetitions from right to left and half from left to right or by the above method of measuring the exterior angle in exactly the same manner as the interior. By the latter routine the sign of the error is made the same for both angles, and, as its magnitude may be taken to be independent of their size, the mean of the measured value of the required angle and that given by subtracting the measured value of its explement from 360° is free from clamp error.

When the best possible results are required, all the angles at a station, including that required to close the horizon, are observed according to the above programme. Usually, however, the measurement of the explement of each angle is omitted, since, in closing the horizon, the explement of the sum of the required angles is measured. The amount by which the sum differs from 360° is equally divided among all the angles.

Angles, beside being measured individually, are sometimes observed in various combinations, *e.g.* AOC, AOD, BOD (Fig. 94). While "all combinations" might lead to a better determination of the required angles, the labour of station adjustment is somewhat increased, and, where weather permits, it is economical rather to amplify the programme for the measurement only of the angles to be used in calculating the triangulation system. When cloud persists there are strong practical objections to measurement in all combinations; and experience does not show any increment of accuracy.

In applying the repetition method to first-order work, six sets of six repetitions each, as described, have been used with 8-in. to 12-in. theodolites. For second-order triangulation with 7-in. to 10-in. instruments, two to four sets are sufficient in ordinary cases, and the same for third-order work with 6-in. to 8-in. instruments.

The Angle Book. Readings must be registered in a permanent manner as soon as they are announced. The recorder should apply the necessary corrections and enter up the individual measures of the angles before the observations are completed, so that it may be decided whether further measures are required.

The tabular arrangement of the angle book may take various forms. That shown (Fig. 95) is suitable for observations with a three-micrometer instrument without eyepiece readings, and is arranged for run correction. The corrected directions or angles are transferred to an abstract in which the final average results are shown.

Miscellaneous Corrections. Further corrections may have to be applied to give the final measured values which are to be subsequently adjusted. These include corrections for:

1. Horizontal Axis Dislevelment.
2. Bisections by Eyepiece Micrometer.
3. Phase of Signal.
4. Eccentricity of Instrument.
5. Eccentricity of Signal.
6. Reduction of Directions to Mean Sea Level.

Station.......... *Date*.......... *Instrument*.......... *Observer*.......... *Recorder*..........

Series.	Time.	Station Observed.	Face.	° ′	Micr.	″ Back.	″ Forward.	″ Mean.	Run.	Run Correction.	Direction.	Angle.	Remarks.
1	5.10 p.m.	P	R.	00 00	A	15·2	16·4						
					B	18·3	16·6						Calm Cloudy
					C	17·8	17·0						
						17·10	16·67	16·88	0·43	+ 0·19	00° 00′ 17″·07		
		Q	R.	47 17	A	57·9	58·5						
					B	62·2	59·9						
					C	62·0	60·8						
						60·70	59·73	60·22	+ 0·97	− 0·10	47° 18′ 00″·12	47° 17′ 43″·05	

FIG. 95.

1. Horizontal Axis Dislevelment. When the horizontal axis of the telescope is not level, and inclined sights are taken, a correction may have to be applied to each observed horizontal direction when its inclination to the horizontal is appreciable. The amount of this correction can be deduced from observations with the striding level.

Let e = dislevelment, in seconds, of the horizontal axis, as given by the readings of the striding level and the known value of its division (page 77).

α = angle of elevation, or depression, of the signal.

Then correction to observed direction in seconds = $e \tan \alpha$.*

The error causes an apparent displacement of signals towards the side of the higher pivot for angles of elevation and *vice versa* for angles of depression. The signs of the corrections to observed directions are therefore

	Elevation.	Depression.
Right pivot higher	—	+
Left pivot higher	+	—

2. Eyepiece Micrometer. The mean of the micrometer readings for several bisections made by the movable hair is multiplied by the angular value of one division of the eyepiece micrometer, and the result is applied to the mean reading of the circle micrometers. The telescope pointings are made sufficiently closely that the amount of correction need seldom exceed one or two seconds.

The value of one division of the eyepiece micrometer must be determined from time to time. It is most simply obtained by measuring a small angle both on the circle and with the micrometer. Bisect a well-defined distant signal with the movable hair, and read the eyepiece micrometer and the circle micrometers. Move the hair from the signal by giving the micrometer screw, say, n turns. Again bisect the signal by means of the upper tangent screw. Read the circle micrometers, thus obtaining the angle through which the line of sight was turned, $1/n$ of which is the value of one turn of the eyepiece micrometer screw. A mean value is obtained by repeating the same process several times for each of several different parts of the screw. Another method, which is independent of the divisions on the horizontal circle, is to use the movable hair in the horizontal position to take observations to a circumpolar star near elongation, and to note the times on a sidereal chronometer as the star travels through intervals corresponding to different complete revolutions of the micrometer drum. If m is the value of a micrometer division in seconds of arc, and the interval between successive transits is 100 divisions on the micrometer drum and t seconds on the chronometer, then :

$$100m = \frac{\sin (15t) \cos \delta}{\sin 1''}.$$

Great care must be taken while observations are in progress to maintain the bubble in a constant position, or to apply the proper bubble correction. The instrument should be set with one footscrew approximately in the meridian plane, and a number of different observations made.

* See Vol. 1, page 225, for proof of this formula.

the mean of the values deduced from each observation being taken as the accepted value of m.

3. Phase of Signal. It has been explained (page 155) that the error of pointing caused by phase is a definite quantity for signals of cylindrical form. When the surface is sufficiently smooth to reflect sunlight in a bright line, as occurs with metal cylinders or wet surfaces, the pointing is made upon that line. With canvas-covered or whitewashed signals, the portion of the illuminated surface as seen from the instrument will be bisected. The correction is applied to convert the observed direction to that of the centre of the signal.

In Fig. 96, let A be the position of the observer, and C the centre of the signal, and let the direction of the sun make an angle a with AC. The

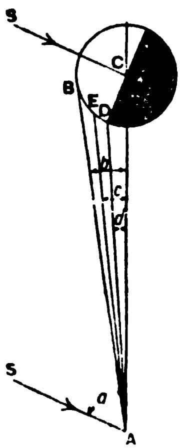


FIG. 96.

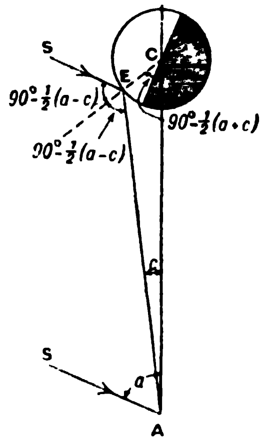


FIG. 97.

visible part of the illuminated surface extends from B to D. If the line of sight is directed along AE to bisect its projection, the value of the phase correction is

$$EAC = c = \tfrac{1}{2}(b + d).$$

But, denoting the radius of the cylinder by r and the length of sight by D, since b and d are very small, we may put

$$b = \frac{r}{D} \text{ and } d = \frac{r\cos a}{D}, \text{ in circular measure,}$$

$$\text{so that } c = \frac{r(1 + \cos a)}{2D} = \frac{r\cos^2 \frac{1}{2}a}{D}, \text{ or}$$

$$\text{correction, in seconds, to bisection of illuminated portion} = \frac{r\cos^2 \frac{1}{2}a}{D\sin 1''}.$$

If the observation is made on the bright line formed by the reflected rays, then, in Fig. 97, let SEA represent their path. The values of the marked angles are readily deduced, and, with the approximations allowable since c is small, it follows that

$$\text{correction, in seconds, to pointing on bright line} = \frac{r\cos \frac{1}{2}a}{D\sin 1''}.$$

The correction must be applied positively or negatively, according to the relative position of the sun and the signals.

4. Eccentricity of Instrument. When existing features, such as steeples, are adopted as triangulation points on account of their visibility, it frequently happens that the instrument cannot be centered over them. In such a case, a subsidiary instrument station, called a *satellite station*, is selected near the true station, and the values of the angles measured there are *reduced to centre*, *i.e.* corrected to the values they would have if measured at the triangulation point.

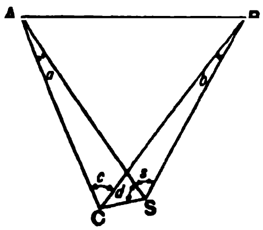


Fig. 98.

In Fig. 98, C represents the true station to which observations have been made from stations A and B, and S is the satellite station, at which angle s is measured with the same precision as if S were a triangulation point. The further measurements required for the reduction are the distance SC and the angle ASC $= d$. The unadjusted values of angles ABC and BAC are already known, and by solving triangle ABC the distances AC and BC are obtained with sufficient precision for the present purpose.

Now, in triangle ACS, $\sin a = \frac{\text{CS} \sin d}{\text{AC}}$, or, since a is usually very small, we may write

$$a\text{, in seconds, } = \frac{\text{CS} \sin d}{\text{AC} \sin 1''}.$$

Similarly, from triangle BCS,

$$b\text{, in seconds, } = \frac{\text{CS} \sin (d + s)}{\text{BC} \sin 1''}.$$

But a represents the difference in direction between CA and SA, and b that between CB and SB, so that, by applying a and b with the appropriate signs, the included angle c is readily deduced from s.

To ascertain the signs of the corrections, it is convenient to regard SC as an arbitrary meridian for the observations made at S. Bearing SA, situated in the first quadrant, would require the addition of the correction a to yield the bearing CA. Similarly, SB in the second quadrant is converted to CB by addition of b, but lines in the third and fourth quadrants would have the corrections applied negatively. In the case illustrated, $(s - c) = (a - b)$, but when A and B are on opposite sides of CS, $(s - c) = (a + b)$.

5. Eccentricity of Signal. Correction is required when observations are made upon a signal which is found to be out of centre or is for special reasons placed so. The value of the correction is obtained as in the previous case, the distance and direction between the signal and the centre being measured. If, in Fig. 98, the signal for station C is situated at S, the observed angles BAS and ABS are to be corrected by a and b respectively.

6. Reduction of Directions to Mean Sea Level. This correction is made only in the most refined first-order triangulation in districts at a

considerable elevation above sea level and particularly in low latitudes. Owing to the spheroidal form of the earth the vertical through an observation-station A is not coplanar with that through a signal B, except when A and B are in the same meridian or of the same latitude. When, therefore, A and B are projected normally to A_1 and B_1 upon the mean sea level surface, the plane AA_1B, containing the observed direction, does not coincide with the plane AA_1B_1, containing its projection, and the direction AB must be corrected to yield that of A_1B_1.

The value of the correction, applicable to the observed direction, as given by Clarke in *Geodesy* is, in seconds,

$$c = \frac{e^2 h}{2a} \sin 2A \cos^2 \phi \operatorname{cosec} 1'',$$

where e = eccentricity of the earth (page 317),
a = major semi-axis of the earth,
h = elevation above mean sea level of the signal station B,
A = azimuth of AB, reckoned from north by east,
ϕ = mean latitude of AB.

The sign of the correction is given by that of $\sin 2A$, *viz.* positive for values of $2A$ in the first and second quadrants and negative for the third and fourth. Its maximum value occurs in azimuth 45° at the equator, when it amounts to 0·″0335 per 1,000 ft. elevation of the observed station.

PRECISE TRAVERSING

In the British Colonies, precise traversing, in the sense now understood by that term, appears first to have been undertaken by the Malaya States Survey Department before the Great War of 1914–18, when it was decided to try and replace triangulation, in parts of the country that were unsuitable for it, by standard traverses in which the standard of accuracy aimed at would be of the order of accuracy of about 1/10,000. In this work, the closing errors actually obtained, as measured by the errors of closure on fixed trigonometrical points, turned out on the average to be of the order of about 1/30,000. Since then, the method has been developed in other Colonies, notably in the Gold Coast and in Nigeria, where closing errors of the order of 1/70,000 to 1/100,000 are now regularly expected and attained. Meantime, in America, the United States Coast and Geodetic Survey had commenced work of this kind in 1916 and the work done by it shows an average closing error of somewhere between 1/70,000 to 1/100,000. Precise traversing is also used in flat forest country in Australia and in Canada, the Canadian methods being very similar to those adopted in the United States.

The considerations that govern the choice between triangulation and precise traversing have been described at length on pages 143 to 145. Briefly, they may be summarised by saying that, unless the highest possible degree of accuracy is required, a good rule to adopt is the American one which lays down the principle that precise traversing should only be adopted when triangulation would be likely to cost more than twice as much as traversing. The conditions to which this rule would ordinarily apply are :—

1. Very flat country which would involve the use of very high observa-

tion towers for triangulation, although the invention of the portable Bilby tower (page 153) has done much to overcome the difficulties of triangulation in flat country.

2. Flat country combined with heavy forest.

3. Climatic conditions unfavourable for the long sights required in triangulation.

In addition, in small triangulation schemes, where the lengths of the sides of the triangles would be very short, a carefully executed scheme of precise traverses may yield more accurate and more satisfactory results than triangulation.

Classification of Traverses. The United States Coast and Geodetic Survey divide their traverses, like their triangulation, into three classes—first-order, second-order and third-order—according to their standard of accuracy. The following table gives the minimum requirements for each class :—

	First-Order	Second-Order	Third-Order
Closing error in position, not to exceed .	1 : 25,000	1 : 10,000	1 : 5,000
Probable error of main scheme angles . .	1·5 sec.	3·0 sec.	6·0 sec.
Number of stations between astronomical azimuths	10 to 15	15 to 25	20 to 35
Astronomical azimuth, discrepancy per main angle not to exceed	1·0 sec.	2·0 sec.	5·0 sec.
Astronomical azimuth, probable error of result	0·5 sec.	2·0 sec.	5·0 sec.

In the pages which follow we shall concern ourselves principally with the description of the methods used in what, under the above classification, would be called traverses of the first-order, and we shall then describe the simplifications usually adopted for the survey of traverses of the second-order. Traverses of the third-order are of the kind that are described in Vol. I.

Differences in Methods of Survey of Precise and Ordinary Traverses. The differences between the methods to be used in the survey of precise and ordinary traverses are, of course, such as to secure the highest possible standard of accuracy in the survey of precise traverses and a less accurate standard, and hence a decrease in the cost, in the survey of ordinary traverses. Broadly speaking, however, the following represents the essential differences between methods of precise and of ordinary traversing :

1. The standard of angular observation is much higher in precise traversing. This means the use of larger theodolites, more sets of observations at each station and possibly all horizontal angular measurements being made at night.

2. The use of much longer lines for the main angular observations.

3. The last requirement generally means an extensive use of "deviations" or "loops"—that is, cases where the line of the taping follows a slightly different route to the line followed by the main angles, the actual distances between the main angular stations being obtained by means of subsidiary traverses or other measurements.

4. More frequent and more accurate azimuth observations are taken on precise traverses.

5. All linear measurements on precise traverses are made with invar instead of with steel tapes, such as are used on ordinary traverses.

6. More frequent and more accurate standardisations of field tapes.

7. Greater care and accuracy in reading tapes and end differences, in applying tension and in observing temperature, heights and slopes.

8. Lines on precise traverses are often laid out so that they may be as closely as possible an exact number of tape lengths long, so avoiding the use of odd unstandardised portions of the tape.

9. Greater care in check taping and in preventing the occurrence of gross errors.

10. Application of more corrections to measured lengths.

11. Modifications in methods of computation, including, if necessary, the use of formulæ which take into account the spheroidal shape of the earth, and the application in certain cases of the method of least squares.

12. Use of "Laplace Stations" (page 48) where considered necessary or advisable.

Differences between British Colonial and American Methods of Precise Traversing. The main difference between British Colonial and American methods of precise traversing lies in the taping and in the methods of computing the results. In American practice the traverse very often follows a railway and the tape is stretched along one rail, the positions of the end marks being scratched on the rail. In other cases the line may follow highways or city streets, when the tape may be either supported throughout its length on the pavement or sidewalk or it may be supported on stakes driven alongside the road or on portable tripods. Thus, the American methods often involve a considerable amount of "surface taping." In the British Colonies, on the other hand, long "straights" and flat curves on railways and roads are seldom available and the taping very often has to be taken over broken ground or along lines cut through the bush. Hence, in nearly all cases, taping is done in "catenary" in much the same way as base lines are measured with long invar tapes.

Layout of Precise Traverses. Before actual measurements begin it is well to make a preliminary reconnaissance of the route to be followed, and to select the stations which will be occupied as main angular stations and those which will be used as azimuth stations. In selecting the main angular stations, the object to be achieved is to obtain as long lines as possible, at the same time avoiding "grazing" rays and keeping the lines as high above ground surface as may be practicable. To do this, the lines which the taping will follow need not be the same as the line followed by the main angular line, and "deviations" and "loops" should be freely used when desirable.

In Fig. 99, A and B are main angular stations which are intervisible, and the taping line follows the line A*abcdef*B, with changes of direction at *a*, *b*, *c*, *d*, *e* and *f*. The line A*abcdef*B is measured as a subsidiary traverse and the distance AB computed, the direction AB being taken as a line of zero bearing when computing the subsidiary traverse.

When laying out the subsidiary traverse it is well, if this is possible, to select the different lines up to the point f so that each measures an almost

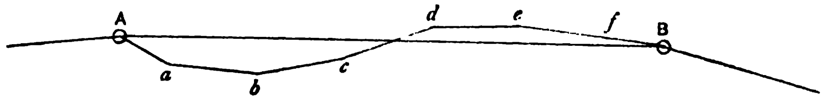


FIG. 99.

exact number of tape lengths. This avoids using unstandardised graduations of the tape except on the last leg. This point, however, is not of great importance.

Figs. 100 (a) and (b) show examples taken from American practice. A and B are main angular stations but the linear measurements are made along the line ab which coincides with the surface of one rail, the distances

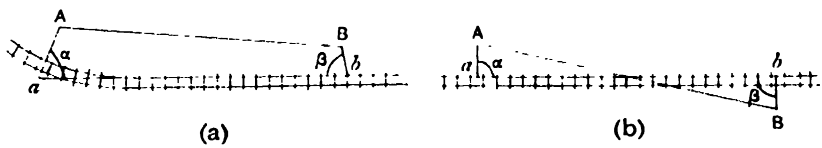


FIG. 100.

aA and bB being very short when compared with the distances AB and ab. The angles α and β at a and b, together with the short distances aA and bB, are measured, and, from these observations and the measured length ab, the length of AB may be computed.

The above are only typical examples, but much ingenuity may be employed in planning suitable diversions which will ensure long and high lines for the main angular line and at the same time suitable lines for the taping.

In every case where a deviation is used, a sketch should be made in the field book and this sketch should indicate very clearly the particular angles and distances that are measured.

Joining Traverse to Triangulation. As a general rule, the principal triangulation points will be on high hills which are usually inaccessible or inconvenient for direct traversing. Hence, it becomes necessary to devise some means of joining a point on the traverse to the main triangulation point without having to measure a traverse directly between them. This can usually be done by using a short length of the traverse as a traverse base. If necessary, a special length of traverse running in a direction differing from the main direction may have to be measured for the purpose.

Figs. 101 (a) and (b) show two typical connections. In (a) the main line of the traverse is AB, but, from B, two subsidiary traverses BC and BD are measured to the points C and D, the trigometrical point T being visible from each of these points. It is not essential that C and D should be intervisible, though it is desirable that they should. The length and bearing of the line CD are computed from the traverse CBD, and these

computed quantities, together with the measured angles at C, D and T, give all the essential data for computing the triangle CDT.

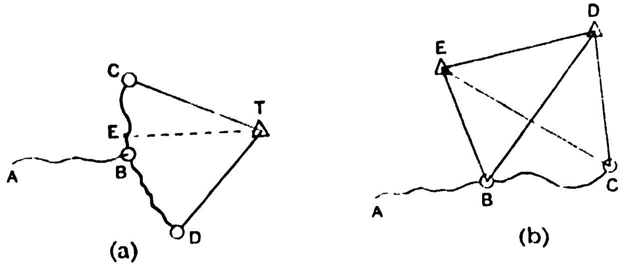


FIG. 101.

Sights to T from one or more intermediate points such as E will strengthen the connection.

In Fig. 101 (b) the connection is made to the two trigonometrical points E and D, the traverse length BC serving as a base from which to compute the quadrilateral.

Combined Traverse and Triangulation. Chances sometimes occur of combining triangulation and traverse in cases where triangulation or

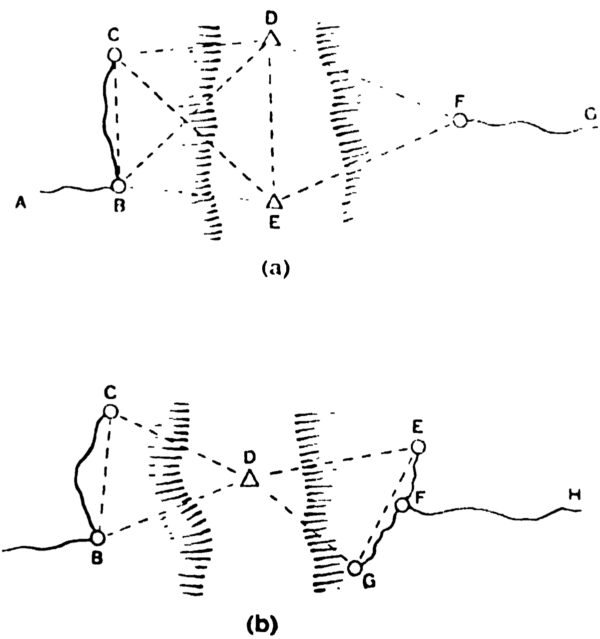


FIG. 102.

traverse would be difficult by themselves. Many combinations are possible, but Fig. 102 (a) and (b) show two simple examples. In all cases where triangulation is combined with traverse, and the legs of the triangles

are short, great care should be taken with the centering of both instrument and signals, but more particularly of the latter.

Setting Out. Before distances or angles are measured, a small party is generally sent ahead to erect pillars, and towers when these are used, to put in station marks, clear lines, and, when the tape is used in catenary, to drive pegs on line at every point where a measuring tripod will be erected. Lining out is best done with a small theodolite, and care has to be taken in placing at the proper intervals the pegs over which the measuring tripods are to be erected. Otherwise the main taping party will have trouble with the taping, especially when the interval between pegs is too large.

The setting out is best done with a steel tape graduated in a different unit to the unit of length used in the main taping, as the chainage of the setting out party can be used to serve as a check against gross error in the main taping. For example, if the invar tape is 300 ft. long, the setting out can be done with a graduated steel tape 100 metres long. In this case, the distances along the setting out tape which correspond to multiples of the 300-ft. mark when the invar tape is used in catenary must be carefully tabulated. It is, of course, inconvenient to set out bays approximately 300 ft. long with a 100-metre tape, but this inconvenience is balanced by the advantage of using the setting out chainage as a check on the main taping, and the chances of undetected mistakes are lessened if the two measurements are made in different units of length.

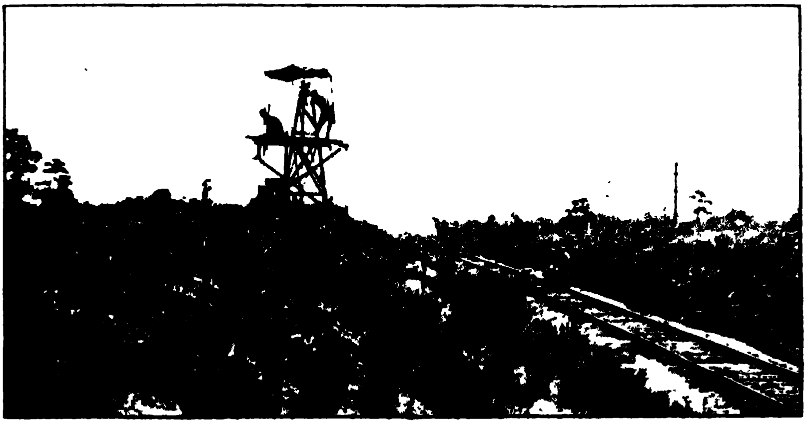
The setting out party sometimes forms part of the reconnaissance party. It is also responsible for preparing descriptions of all permanent beacons and station marks, and it notes the chainage where permanent detail, such as roads, rivers or railways, cuts the traverse lines, as this information is often useful for mapping or topographical purposes, or for finding a station when the detail near it has altered with the passage of time.

ANGULAR MEASUREMENTS

Signals and Station Marks. In all first-order work it is best to observe the main angles at night or on very cloudy days, and this is now generally done, the signals used for both night and day observations preferably being electrical lamps worked from dry batteries or accumulators. If the ground is very flat, the lamps may be supported on low wooden tripods or quadripods something like the inner tripod illustrated in Fig. 59, but on a much smaller scale. These tripods also serve to support the instrument, and consequently, if they are more than about 5 ft. high, a separate tower or scaffold must be built around them to carry an observing platform. Tripods or towers of this kind must, of course, be very stiffly constructed, and they should be of such a height as to insure that there is a good clearance between the line of sight and all objects on the ground, a minimum clearance of about 4 ft. being desirable. In the United States, special towers are almost invariably erected at the main angular stations, the ground mark being a mark in a rock, or a low concrete post or pillar built in the shape of the frustrum of a cone. Fig. 103 (*a*) shows an observing tower in use, and Fig. 103 (*b*) linear measurements

being made from the rail to the ground station underneath a tower used as a signal.

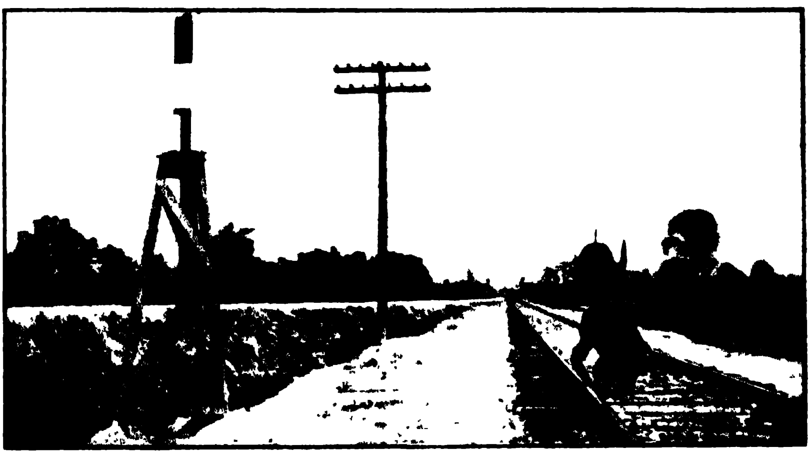
If special towers are not used, the instrument can be set on its own



(a)

First order Traverse Work in U.S.A. Tripod and Scaffold for Elevating the Instrument and Observer.

(Photo by U.S. Coast and Geodetic Survey)



(b)

Tripod with Target in Place. Measuring the Offset Distance.

(Photo by U.S. Coast and Geodetic Survey)

Fig. 103.

tripod, provided this is stiff enough, with the signals set on similar tripods. A better plan, however, is to build a good concrete pillar, about 4 ft. high above ground surface, at every main angular station, the pillar being sited on the top of a ridge or on any bump or elevation that may be

available, so as to avoid grazing rays and to raise the line of sight as high above the ground as possible. Both instrument and signal can then be set on each pillar in turn and carefully centered over a nail or brass plug let into the top of the pillar to indicate the exact station mark.

If a very steep sight of elevation or depression has to be observed, readings on the striding level should always be taken and the correction $c = e \tan \alpha$ (page 226) applied. Very steep slopes will not normally occur on the main angular lines but they may sometimes occur on the subsidiary taping lines.

The three-tripod system of observing (Vol. I, page 84) is particularly suitable for use in connection with the measurement of subsidiary traverses or deviations in which the legs are comparatively short, and the standard of accuracy need not be so high as that for the measurement of the main angles. In this system, of course, the signals are small targets of a type similar to that illustrated in Vol. I, Fig. 74, which can be mounted on tripods that will take either signal or theodolite.

Main Angular Observations. The main angles of a precise traverse should be measured with a good micrometer theodolite which should enable readings to be taken direct to a second or two. In recent years, much work has been done in the Colonies with the ordinary Tavistock, Zeiss Universal or Wild instruments, although these are on the small side for the measurement of the main angles of a first-order traverse and it would be better to use the larger geodetic models of the same types such as are used for the observation of first-order triangulation. With instruments of this kind at least twelve measurements of the angle, each on a different zero and with face right and face left on each zero, will normally be required, and it may be an advantage to measure six of these zeroes on one night and the remaining six on another night. In this case, the discrepancy between the observed azimuth at the end of a section and the azimuth calculated through the traverse should not be allowed to exceed $6\sqrt{n}$ seconds, where n is the number of instrument stations, including the two azimuth stations. Normally, the discrepancy should be very much less than this value.

In the United States, the observations are taken at night with a "high-order" micrometer theodolite, using eight zeroes, but, if a smaller instrument is used, the number of zeroes is increased to twelve. For each zero, a pointing is made on every station in a clockwise direction with face right and then in an anti-clockwise direction with face left. Here the rule is that an observation on a single zero should be rejected, and the observation repeated, if it differs from the mean of all the observations by more than 5 seconds, and, for any set of observations, the arithmetical mean should have a probable error not exceeding 1·5 seconds. In addition, the discrepancy between the azimuth carried forward through the traverse should not differ from the observed azimuth by more than one second times the number of stations between azimuth stations.

Some considerations regarding the standard of accuracy to be aimed at in the angular work are stated on pages 248 to 254, and these considerations afford a useful guide to the number of observations that should be taken with any particular instrument to reach a desired standard of accuracy for the traverse as a whole.

Measurement of Subsidiary Angles. The subsidiary angles are the angles measured on deviations or loops to enable the distance between the main angular stations to be computed. These angles need not be observed with the same accuracy as the main angles, and, if a large geodetic type of instrument is used for the measurement of the main angles, a smaller type can, if desired, be used for the observation of the subsidiary angles. For all ordinary cases, a theodolite of the ordinary Tavistock type, or one of similar type and performance, is suitable, one set of six zeroes being measured, preferably at night, at each station. With an instrument of this kind, the angular misclosure on a deviation should not exceed $12\sqrt{n}$ seconds, where n is the number of subsidiary angles in the deviation.

In the United States, the angles of small loops are sometimes measured with a 7-in. repeating theodolite, one set of six repetitions, each repetition on face right and face left with alternate swings being measured on the interior angle and a similar set on the exterior angle. Otherwise, they are observed with a small direction instrument, four zeroes being used. When, however, loops are long and the intermediate stations are permanently marked, the angles are measured with a direction instrument to the same degree of accuracy as the main angles.

In all cases where deviations or loops occur, the angles should be "closed" and the closing error in bearing or azimuth obtained and adjusted among the various angles

Azimuth Observations. Azimuth observations must be taken at frequent intervals along a traverse in order to control the bearings. The number of angular stations between azimuth stations will depend mainly on the average length of the legs, but this interval should never exceed about 50 miles or 20 traverse legs, whichever is the shorter, and more frequent azimuth stations—say at every 10 to 15 instrument stations—are desirable when circumstances permit. If possible, and more particularly in higher latitudes, longitude observations should also be taken at intervals of about 100 to 150 miles, and a Laplace correction (page 48) applied when necessary.

In American practice, the observations are taken to Polaris at any hour angle. At ordinary azimuth stations the number of zeroes observed is 16, but at Laplace stations this number is doubled, the observations being spread over two or three nights. As the exact hour angle of Polaris is required, the chronometer correction has to be obtained, the method used for this being observed altitudes of east and west stars near the prime vertical.

The method of observing hour angles of Polaris for azimuths is not suitable for use in the tropics, and here the one commonly used on traverse work consists of observing altitudes of east and west stars at or near elongation, not necessarily at or near the prime vertical. The stars chosen must be carefully paired for both altitude and declination and a programme made out before observing commences. If azimuths are not more than 20 miles apart, twelve pairs of stars, equally spaced around the circle, should suffice, but, if the azimuth stations are more than 20 miles apart, it is advisable to increase the number of pairs to eighteen. In either case, it is well to observe in sets of six pairs, each pair on a different night.

The correction for dislevelment of the horizontal axis of the instrument (page 226) must be applied to all azimuths observed to control the main angular observations of a precise traverse.

LINEAR MEASUREMENTS

The lengths of the legs of a precise traverse are now almost invariably measured with invar tapes, generally from 100 ft. to 300 ft. long. The most common practice is to use the tape "in catenary" in a manner very similar to that employed in base measurement, but, in certain parts of the United States and of Canada where the traverse follows a railway or a flat highway the measurements are made with the tape laid flat along the surface of one rail or on the surface of the road. This method of surface taping is practicable when the line of the railway or road consists of long straights, but, in most undeveloped or partially developed countries the railways are not suitable for surface taping, and, to attain any sort of accuracy combined with reasonable speed, the only alternative is to use the tape suspended in catenary.

Methods of Surface Taping. As the method of surface taping is seldom used outside of the United States or Canada the following description of it is based almost exclusively on the system adopted by the Coast and Geodetic Survey.

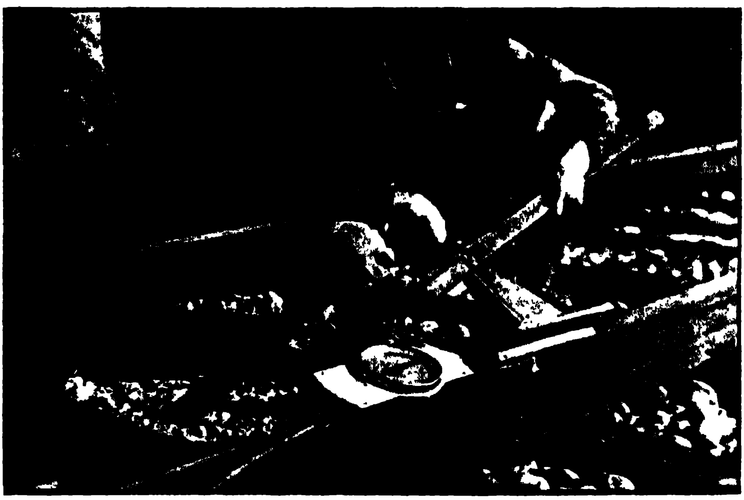


FIG. 104. FORWARD STRETCHER FOR INVAR TAPE WITH ATTACHED SPRING BALANCE. THE STRETCHER IS MADE OF THIN SHEET IRON AND HELD ON THE RAIL BY THE FOOT.

(Photo by U.S. Coast and Geodetic Survey.)

When the taping follows a railway, the tape is laid along the surface of one rail, except for short lengths at the end of the traverse leg where it is necessary to leave the rail in order to reach the main instrument station. This station is usually sited at, or near, the intersection of the tangents to curves, and the tape is supported on graded pegs between this point

and the point where it leaves the rail. The tape used is 50 metres long and the tension of 15 kilogrammes is applied to it at one end by means of a special lever or stretcher to which a spring balance is attached. This stretcher works on a shoe which fits easily over the top of the rail and is held firmly on the latter by the tapeman pressing his foot on top of the shoe. A similar stretcher, but without the spring balance, is used at the other end of the tape. Fig. 104 shows one of these stretchers with attached spring balance as used on a railway; when they have to be used on a road or pavement a small piece of wood is slipped between the flanges of the shoe to provide a smooth surface between the bottom of the latter and the surface of the road.

When measurements are being made along the top of the rail, the position of the forward-end mark is scratched on the rail with an ordinary glass cutter, and, in measuring over stakes, the mark is made by a knife on the top of the stake. The rear end of the tape is held against the mark at that end, so that, with this method of taping, end differences are not read at every set up of the tape.

Measurement of Temperature. Temperatures are measured by means of special thermometers which are attached to the tape at distances of 1 metre inside the zero and 50-metre graduations. Each thermometer is read for a full set up of the tape, but only one is read on end bays or on offset lines shorter than a complete tape length.

Measurement of Heights and Slopes. The correction for height above sea level is obtained from the results of the measurement of a line of first-order levels which, unless a completed line already exists in the neighbourhood, is surveyed at the same time or before the traverse is measured. The function of this line of levels is to determine the heights of the main traverse stations and to establish bench marks to which the minor levelling, required to determine slope correction, can be connected when desired. This minor levelling is done by the use of a wye level or sometimes by a special track level in the form of a clinometer that can be laid along the surface of a rail. During the course of the minor levelling, the slope or difference of height of each tape length, and at intermediate points where the slope changes, is determined.

Measurement of Odd Length Bays. Distances on the main lines shorter than a full tape length are measured by using the 5-metre graduations on the invar tape and measuring the distance from a suitable graduation to the mark by means of a steel tape graduated in feet and metres, the measurements being made in both units.

Measurements in Catenary. The methods of measurement of the lengths of traverse legs with the tape suspended in catenary are very similar to those used in base line measurement. The lengths of the tapes used, the manner of supporting them, and the methods of applying tension, vary in different countries. In the United States, the tape employed is 50 metres long, with one intermediate support at the 25-metre mark, and the tension is applied and controlled at one end by means of a spring balance carried on a special pole or stretcher, a similar stretcher, but without the spring balance, being used at the other end. Fig. 105 shows a spring balance with its special mounting on the stretcher pole.

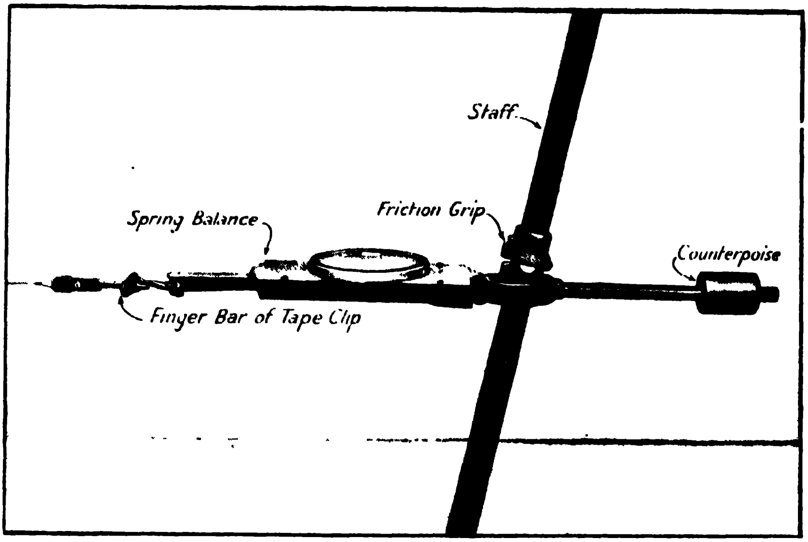
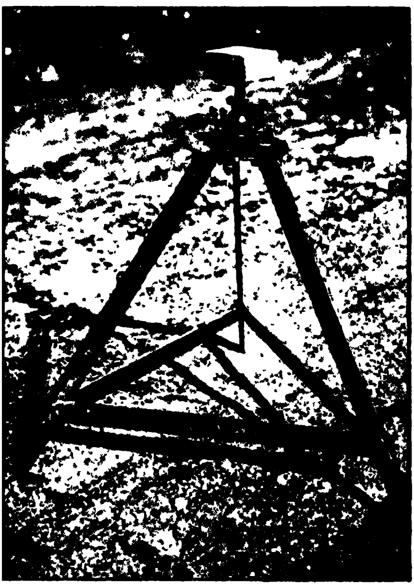


FIG. 105.

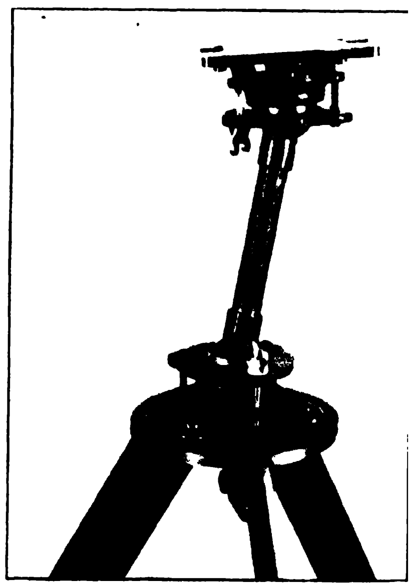
(*Photo by U.S. Coast and Geodetic Survey.*)



(*a*)

PORTABLE IRON TRIPOD FOR TAPE SUPPORT, SINGLE JOINT.

The wooden marking table, which carries a strip of copper on which the mark for the tape end is made, can be placed on correct slope by the ball-and-socket base and then clamped in position.



(*b*)

PORTABLE IRON TRIPOD FOR TABLE SUPPORT, DOUBLE JOINT.

This tripod permits the marking table to be adjusted over a point as well as placed on correct slope.

FIG. 106.

(*Photo by U.S. Coast and Geodetic Survey.*)

The ends of the tape are sometimes supported on stakes carrying a piece of metal on which the end marks can be scratched, but, in other cases, and more specially in work in cities, low movable iron tripods are used as tape supports. These tripods, shown in Figs. 106 (*a*) and 106 (*b*), are provided with a small wooden table mounted on a ball and socket joint, so that it can be placed and clamped on the same slope as the end of the tape. This table either carries a strip of copper on which the end mark can be stretched or else it carries an invar graduated scale so placed that the zero of the scale faces the direction in which the line is being measured. At the rear end of the tape, the zero mark is held against the zero of the scale, and, at the forward end, the small difference

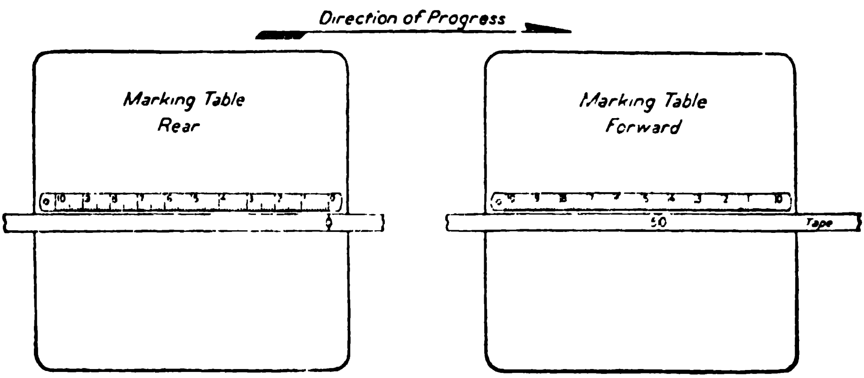


FIG. 107. METHOD OF USING GRADUATED INVAR STRIPS ON MARKING TABLE.

The marking tables of movable tripods for traverse measurement are frequently fitted with these strips. With the arrangement as shown, and the rear or zero mark of the tape held opposite the zero mark of the strip on the rear marking table, the reading of the forward mark on the tape would always be recorded as a set-up, or additive quantity.

(*By permission of the Director U.S.C. & G.S.*)

between the 50-metre mark on the tape and the zero mark on the scale is read and booked. Fig. 107 shows the method of reading when a scale is used.

The support at the 25-metre mark is a nail on a stake or a loop or hook held by a labourer. This support, whatever its nature, is carefully aligned, both horizontally and vertically, between the end marks.

Elsewhere than in America, invar tapes 300 ft. long and $\frac{1}{8}$ in. wide are often used in two or three equal spans, with supports at the 150-ft. or at the 100-ft. and 200-ft. marks. Tension is usually applied at one end either by weight or spring balance, the other end being "anchored" to some form of support or anchoring trestle or lever. The actual measurements are made to marks carried on stakes or on measuring tripods similar to the measuring tripods used in base line measurement.

When long tapes are used, intermediate supports are necessary and different types are used. One is simply a hook suspended from a long

pole held by a labourer. Another, shown in Fig. 108, consists of the hub of a bicycle wheel mounted with its ball bearings on a frame which can be slid up and down a long pole, and clamped at any height, this pole also being held by a labourer. In all cases, the intermediate support must be carefully lined in, both horizontally and vertically, between the end marks.

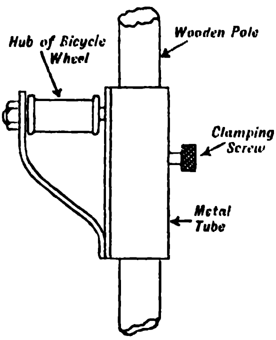
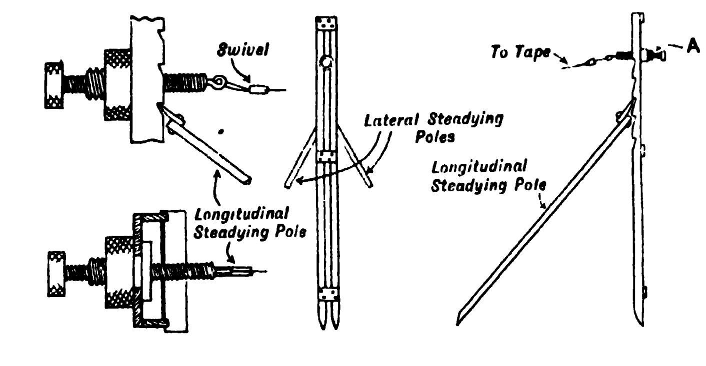


Fig. 108.

In applying tension by weight, a straining trestle, similar to the ones used in base line measurement, is often employed; in other cases, a pulley mounted on ball bearings and carried on a single pole held by a labourer is used. The other end of the tape is generally fastened to some form of anchoring pole or trestle, the trestle in some cases being a spare straining trestle similar to the one used for applying the weight. Fig. 109 shows a rather elaborate, but "home-made," anchoring trestle with a special fine motion "laying-on device" for laying the end mark of the tape exactly against the mark on the stake or measuring tripod. This trestle, intended primarily for accurate field standardisation work, is supported and controlled by a labourer aided by "steadying poles" similar to those often used for steadying levelling staffs, and it can be made out of two light angle irons, or out of stiff boards nailed together to form a shape like an



Simple Anchoring Trestle

Fig. 109.

angle iron. These two angle irons or boards are separated by a space about 1 in. wide, except at points where they are fastened together by plates or slats, this space forming a groove along which the laying-on device can be slid up and down and clamped where desired. The construction of the trestle and of the laying-on device will be more clearly understood from the diagram. A series of notches in the

frame serve to hold in position a strut to take the forward pull on the trestle.

Instead of trying to lay the end mark of the tape exactly against a mark on the measuring tripod, end differences are often read at both ends, generally by means of a metal scale, a magnifying glass being used to magnify the images of the marks when necessary.

On heavy, or fairly heavy, slopes a note should always be made in the field book to indicate whether the tension has been applied at the upper or the lower end of the tape. This information may be needed to work out the sag correction for a tape on a slope.

Measurement of Temperature. Temperatures are usually observed at each set up of the tape by means of two thermometers, held one near each end of the tape. The best type of thermometer is one cased in a metal tube with an open slot opposite the graduations in the stem. Each thermometer should be compared at reasonably frequent intervals against a standardised one, and during taping it should be supported near the tape in a horizontal position (see page 180).

Measurement of Heights and Slopes. As all measured lengths have to be reduced for slope and for height above sea level, it is necessary either to carry a line of levels along the line of the traverse or else to observe vertical angles, the latter being the quicker method and the one which is most generally used. The angles on the taping line can be observed either with a theodolite, or, more conveniently still, with a small aligning and levelling telescope that can be slipped over the vertical cylinder of a measuring head (page 174). In any event, vertical angles should also be observed at each main instrument station and these angles used to control the observations along the taping line as well as to give the data for working out the heights of the principal stations. The main vertical angles should be observed on both faces, but, provided the instrument is in good adjustment, the slopes along the taping lines may be observed on one face only.

In all cases, whenever bench marks established by spirit levelling exist near the line of the traverse, the trigonometrical heights resulting from the measurement of the vertical angles should be connected to the nearest bench mark.

Measurement of Odd Length Bays. Whenever practicable, the taping lines are laid out so as to avoid the use of odd length bays, but, when an odd length occurs, it is measured either directly from the graduations on the tape itself, when the latter is suitably graduated, or else by a special graduated tape provided for the purpose. If necessary, a special tape grip, similar to that shown in Fig. 118, Vol. I, may be used to apply the tension at an intermediate point of the tape, although it is better to lay out the latter beyond the end station of the line, exactly as if a fresh bay were to be measured. In any case, the use of tape grips with the main invar tapes should be avoided as much as possible.

For the determination of the sag correction for odd length bays see Vol. I, page 161.

Check Taping. Owing to the risk of error in taping it is essential to have some sort of check on the main taping. Usually, if care is taken in measuring end differences, particularly in the measurement of odd-

length bays, it is only necessary to carry out a check taping with sufficient accuracy to detect gross errors of a foot or more. Consequently, where the traverse is set out beforehand, and the setting out taping is done with sufficient accuracy, the setting out taping will serve as a check taping.

The United States Coast and Geodetic Survey makes a check measurement of each traverse section by means of a 300-ft. steel tape used on the flat under a tension of 5 kilogrammes, the object of this measurement being to detect gross errors only. The allowable discrepancies between the measurements with the invar and the steel tape, after all corrections for temperature, slope, standardisation, etc., have been made to both measures, are as follows :—

For sections exceeding 500 metres in length, 1 part in 5,000.

For sections under 500 metres in length, 1 part in 2,500.

If these limits are exceeded, a second measurement is made with the steel tape, and, if the second measurement agrees with the one made previously with that tape, the line must be re-measured with the invar tape.

In addition to these routine precautions against gross error, the chief of the party or other senior surveyor measures a section of traverse, 1 to 2 km. long between traverse stations, at the beginning of the work and at intervals of forty to fifty miles along it during the progress of the survey. This measurement is made with an invar tape to first-order standards of accuracy, and the discrepancy between the two measures is not expected to exceed 10 mm. $\sqrt{K}$, where K is the distance in kilometres.

Owing to the great risk of error in traverse work it is now the practice in some of the Colonies to duplicate all important measurements connected with first-order taping by separate observers, each observer recording his own observations in a book kept by himself.

Standardisations of Field Tapes. It is hardly necessary to point out that during all stages of the survey of a precise traverse it is most essential that the field tapes should be properly standardised, as errors of standardisation are cumulative in their effects. The field tapes must therefore be standardised at frequent intervals, and, for this purpose, the party should be provided with at least two invar tapes that have been properly standardised, and have had their co-efficients of expansion determined, in a properly equipped standardising laboratory. These standard tapes must be used for no purpose other than for standardising the field tapes used for the actual taping of the lines. The necessity for keeping at least two tapes arises from the fact that one tape may suffer damage or undergo a slight change in length, which, in the absence of a second tape, might remain undetected until the tape was re-standardised at the laboratory. Accordingly, all standardisations should be made against both standard tapes, and, if the results are inconsistent, the presumption is that the length of one standard tape has changed. In that event, earlier standardisations of the field tapes may indicate the standard tape with which the fault lies and it should be sent at once for re-standardisation. If conditions are such that the standard tapes cannot easily be re-standardised within a very short time, the party should be provided with a third standard invar tape which can at once be brought into use in the event of one of the others having to be sent for re-standardisation.

The standardisations in the testing laboratory, and those in the field, should be carried out as far as possible under conditions similar to those under which the tapes will be used in the field. This means that all the apparatus used in the standardisations, including weights, cords, swivels, hooks, etc., should be those which will actually be used in the field, and this apparatus should be sent with the standard tape when it is sent to the laboratory.

Tapes to be used on the flat should be standardised on the flat; those to be used in catenary should be standardised in catenary with supports at the points where supports will be used in the field.

When field tapes are to be used in catenary with one end anchored and the weight applied at the other, the tension should be applied in exactly the same way during the field standardisations. In this case, however, it is well to have straining tripods and weights at both ends of the standard tape when it is being used to determine the true length of the base against which the field tape will be compared, and to have it standardised at the laboratory with weights applied at both ends.

The field tapes should be compared against the standard tapes at very frequent intervals, say every two or three days. A field standardisation does not take long to complete, and invar tapes are easy to damage or deform. Consequently, frequent standardisations may easily avoid having to repeat work which would otherwise be of doubtful value.

Corrections to be Applied to Measured Lengths. The corrections to be applied to measured lengths are exactly the same as those applied to base measurement and have already been described in the earlier part of this chapter in pages 181 to 189. They are as follows :—

1. Correction for standardisation or absolute length.
2. Correction for temperature.
3. Correction for change of tension if the tensions vary.
4. Correction for sag, including sag on a slope.
5. Correction for index error of spring balance if a spring balance is used instead of weights.
6. Correction for change of gravity. Only to be applied to correct the lengths of the standard tapes if these have been standardised in a place where the value of gravity is different from its value at the place where the traverse is measured.
7. Correction for slope.
8. Correction during standardisations for inclination of end readings.
9. Correction for horizontal alignment.
10. Correction for height above sea level.

Of the above, (4) does not apply to surface taping unless the tape has been standardised in catenary, in which case the sag correction must be applied to the length in catenary to give the equivalent length on the flat.

In addition to the above corrections, there is another which is only applicable when the traverse is computed in terms of some special co-ordinate system, such as the Transverse Mercator system. This is what is called "scale correction," and, for traverses not running due north and south and computed on the Transverse Mercator system, it varies slightly in value for each leg. It is described for the Transverse Mercator system on pages 378 to 386.

SECOND-ORDER TRAVERSES

Second-order traverses are used to break down the work between the first-order traverses or between secondary or primary trigonometrical points which are not very far apart. Second-order, or secondary, traverses are therefore usually much shorter than first-order ones.

The accuracy specified by the United States Coast and Geodetic Survey for a second-order traverse is a position check not exceeding 1 : 10,000 of the length of the circuit when closure is made on a point of first or second-order triangulation or traverse. This is the limiting error allowed, but the instructions for survey have been drawn up with a view to ensuring that the error of each line after adjustment will not exceed the above limit and the average error of all the lines will be about 1 : 30,000. The latter figure is probably the average to be expected from the use of the methods now to be described.

The satisfactory closure of a traverse ending on itself as a closed figure is not a satisfactory test of accuracy, since compensating and systematic errors will not show up in the error of closure. Thus, a serious error in standardisation, constant throughout the whole traverse, merely alters the apparent size or area of the closed figure and does not appear in the closing error, which may still remain very small.

In second-order traverses, long lines for the main angular legs are not so important as they are on first-order work, although long legs are desirable when they can be obtained fairly easily. Accordingly, less use is made of deviations and loops. Fewer angular observations are taken at each station and these observations are generally made by day to some sort of non-luminous signal. A greater number of angular stations between azimuth stations is allowed, and distances may be measured with either steel or invar tapes.

Methods Used in the United States. The following is a brief summary of the instructions for the survey of second-order traverses in the United States :—

1. Distances between main scheme stations should not exceed 3 miles and an average distance between permanently marked main angular stations not exceeding one mile is desirable.

2. Second-order azimuths should not be further apart than 15 to 25 angular stations, the latter limit only being reached when observing conditions are favourable. The allowable discrepancy between the observed and computed azimuth at the end of a section is $2''{\cdot}0$ times the number of main angular stations in the section.

3. The error of a main deflection angle should seldom exceed three seconds of arc, whether the observations are made with a repeating or a direction theodolite. With a micrometer theodolite with 8 to 10 in. diameter circle, four to six positions of the circle, with a face-right and face-left observation at each setting, are considered sufficient ; with a smaller circle of 5 to 7 in., six to eight positions. The limit of rejection for the larger instrument is five seconds from the mean ; for the smaller instrument it is six seconds from the mean. In the case of the subsidiary angles, *i.e.* those on loops or deviations, two positions of the circle are used when the instrument is a small micrometer direction theodolite and the total length of the loop does not exceed a mile. For longer loops, the

accuracy of the angular measurements should approach that of the main angular stations.

4. The main linear measurements are made with invar tapes, 50 metres long, used under a tension of 15 kg. (about 33 lb.), which is applied by a spring balance attached between the forward end of the tape and the front stretcher. The stretchers are of the same type as those already described for use on first-order traverses. Measurements are made on the flat whenever possible, but, if conditions are such as to make surface taping inadvisable, the tapes may be used in catenary to stakes or light marking or measuring tripods. If the tape is used in catenary, it is supported in the middle, the support being lined in horizontally and vertically between the end marks.

5. Temperatures are observed at every set up of the tape on a special cased thermometer, the latter being attached by adhesive tape to the rear end of the tape about 1 metre forward of the rear mark.

6. Most of the second-order traverses of the United States Coast and Geodetic Survey are near the sea and at low elevation. Consequently, the correction for sea level is very small and no attempt is made to apply it. In other cases, it is estimated that it is not necessary to know the true level of the line closer than to about 300 metres, since an error in elevation of this amount would only affect the result by about 1 part in 20,000, and the mean elevation of a line can usually be estimated more closely than 300 metres.

7. Observations for slope are only taken when it is estimated that the slopes encountered are such that the introduction of slope correction would affect the measured length by more than 1 part in 20,000. Otherwise, slopes are observed by means of a wye-level.

8. A single check taping, to detect gross error only, is observed with a steel tape 300 ft. long, used on the flat under a tension of about 5 kg. only, and with such a light tension tape stretchers are not necessary. Observations for temperature and slope are not made during the course of this taping and consequently corrections on account of these factors are not introduced.

9. The invar tapes used for the main taping are compared with a standard invar tape at intervals not exceeding 20 to 25 miles of traverse.

Further particulars of the United States Coast and Geodetic Survey methods will be found in the *Manual of Second and Third Order Triangulation and Traverse*, U.S.C. and G.S. Special Publication No. 145.

Other Methods of Surveying Second-order or Secondary Traverses. Experience has shown that, by using the following methods, an average accuracy of somewhere about 1 : 30,000 may be expected in tropical conditions on traverses up to about 100 miles in length. Many of the traverses on which this average standard of accuracy has been obtained have been surveyed over rough country, involving lines cut through the bush.

1. Distances between main angular stations seldom exceed 1½ miles in length and are often less than ½ mile. As long lines as possible are, of course, desirable.

2. The distances between azimuth stations should not exceed 30 legs except in exceptional circumstances, and, where possible, should not be more than 10 miles apart. The allowable discrepancy between computed and observed bearings at the second azimuth station should not exceed

$9'' \sqrt{n}$, where n is the number of angular stations including one azimuth leg.

3. Angular observations are made with a 5-in. or 6-in. micrometer theodolite or with a small double reading instrument of the Tavistock, Wild, Zeiss, Watts-Zeiss or Casella types. On main angular lines at least four readings, each on a different zero and with a change of face between each zero, should be taken, the maximum discrepancy between a single reading and the mean of all readings not being allowed to exceed 6". On short deviations, two observations on two zeroes, with change of face and swing on each zero, will be sufficient, the allowable discrepancy between the two observations being 10".

The above rules hold for instruments carefully tested at regular short intervals for collimation error and the error adjusted when necessary. The observations may be made in daylight, the signals being either targets of the kind used for the three tripod system of observing, or, for the shorter lines, a plumb bob string suspended from a tripod and carrying a small piece of paper to enable the string to be picked up in the telescope. For longer lines, an ordinary ranging pole, very carefully plumbed and centered over the ground mark, may be used when targets for the three-tripod system of observing are not available.

4. The main linear measurements may be made with a 300-ft. steel tape, $\frac{1}{8}$ in. wide, used in catenary under a tension of 15 to 20 lb., with a lined-in support at the 150-ft. mark. Readings at one end are made to, or the end mark of the tape is held against, a mark on a nail in a stout wooden peg about 3 ft. high, or else to the mark on an ordinary measuring tripod. At the other end, the readings are taken to a mark at the centre of the trunnion axis of the theodolite in the manner described in Vol. I, pages 155–157, tension being applied at the theodolite end of the tape by means of a spring balance attached to a long straining pole, a similar pole being used at the other end.

The readings to the theodolite axis are taken to the nearest hundredth of a foot by means of a steel scale graduated to tenths and hundredths.

5. Temperatures are observed immediately after the reading of the tape has been taken. These readings are taken at every set up of the tape on a single cased thermometer suspended near the tape in a horizontal position.

6. Slopes and heights for slope correction, and for the reduction of the length of the line to its length at sea level, are obtained by means of vertical angles observed on the theodolite in the usual way.

7. Before work on actual measurement commences, the lines are set out and pegs put in on line at every 300-ft. length. This setting out chainage, which is measured with a 300-ft. tape stretched along the ground under a tension of 15 lb., is used as a check on the main chainage against gross error.

8. The field tapes are standardised at intervals of 20 to 25 miles against a properly standardised field standard tape.

ESTIMATION OF ALLOWABLE ERROR IN TRAVERSING

In chapters II and IV of Vol. I of this book we have used some of the results of the theory of least squares when discussing the probable errors of ordinary taping and ordinary traversing, and, although the description

of the main principles of the theory is reserved for the next chapter, it will be convenient if we use here some of the results already used in the previous volume to discuss one or two problems that are of importance in their application to precise traversing. These problems have their practical application from the point of view of deciding on the methods to be adopted, the care to be taken in different operations, and the limits of error to be allowed in order to obtain a given standard of accuracy. If, however, the reader is not familiar with the theory of least squares, he may, if he so desires or if he finds it more convenient to do so, postpone his reading of the following pages until he has read the chapter that follows.

The problems involved are best illustrated by a numerical example based on the discussion already contained in Vol. I, as the same principles apply to errors arising from measurements with an invar tape as to those which arise from measurements with a steel tape, and the errors which arise in precise traversing are similar in nature, though different in magnitude, to those which occur in ordinary traversing.

Suppose that we have a "straight" traverse, 480,000 ft. long, divided into ten equal sections, each section bounded by azimuth stations, with 16 legs of an average length of 3,000 ft. in each section, and it is desired to determine the probable error necessary in the angular observations to give a probable linear closing error in the total length of traverse of 1 part in 100,000. The linear measurements are to be made with a 300-ft. invar tape, of section 0·125 in. by 0·015 in., used in catenary under a tension of 20 lb. in two equal spans of 150 ft. each. Weight of tape = 14 oz. per 100 ft., co-efficient of expansion = 0·000 002 per 1° F., and E = Young's Modulus of Elasticity = 22,000,000 lbs. per square inch.

The first step is to estimate the probable error in displacement required at the end of a single section. Let r_s be this probable error. It is almost certain that there will be a constant linear error of unknown sign running throughout the whole of the ten sections which may be due to errors in the standardisation of the field standard tapes and possibly also to uncertainty as regards the height of the datum used in running the levels for the computation of sea level correction.* Let this constant error be estimated to be about $\pm$ 1 : 300,000. The probable error due to this cause in a length of 480,000 ft. is $\pm$ 1·6 ft., and the total probable error for the whole traverse is to be of the order 1 : 100,000 or $\pm$ 4·8 ft. Hence, assuming that the errors of the ten equal traverse sections are equal in value but of varying sign, and using the square root rule for the combination of errors of unknown and varying sign :—

$$(4{\cdot}8)^2 = [(1{\cdot}6)^2 + 10r_s^2]$$
$$\therefore\ 10r_s^2 = 20{\cdot}48$$
$$\therefore\ r_s = \pm\ 1{\cdot}431 \text{ ft.},$$

which, in a section of 48,000 ft., works out at 1 : 35,000 approximately.

We have next to investigate the probable error of the taping, and, to do this, we proceed according to the scheme outlined in Vol. I, pages 169–174. Here, in the absence of any other data or formula, we must estimate,

* The error in a length l due to an error dH in height above sea level is equal to $l\dfrac{dH}{R}$.

as best we can, the probable errors of the different factors that can introduce error into the taping, dividing the resulting errors into three classes :

1. Errors that are always of the same sign for all tapes and lines.
2. Errors that are of the same sign for one tape or line but that vary in sign with different tapes and lines.
3. Errors that are of variable signs for different bays of the same line.

Using the same notation as in Vol. I but noting that, since the only error that can arise in vertical alignment in a tape suspended in catenary in two equal spans is an error in the height of the single intermediate support, so that the error due to vertical deformation becomes $\frac{2(\delta h)^2}{l}$, where δh is the error of vertical alignment at the intermediate support and l is the total length of the tape, we form a table as follows :—

Errors due to	Errors always of same sign though of variable amounts	Errors of same sign for whole line but of different signs for different lines	Errors variable in sign for each bay of the line
Faulty alignment at end of tape	$u_1 = \frac{d^2}{2l}$ $d = 1$ in $l = 300$ ft $u_1 = 0.000011$	—	—
Error in horizontal alignment of intermediate support	$u_2 = \frac{2D^2}{l}$ $D = 1$ in, $l = 300$ ft $u_2 = 0.000046$	—	—
Error in vertical alignment at intermediate support	$u_3 = \frac{2(\delta h)^2}{l}$ $\delta h = 1$ in, $l = 300$ ft $u_3 = 0.000046$	—	—
Errors in measurement of slope	—	$v_1 = l \sin\theta \cdot \theta_1 \sin 1''$ $\theta_1 = \pm 10''$, $\theta = 1°$, $l = 300$ ft $v_1 = \pm 0.000254$	$\omega_1 = l \sin\theta \cdot \theta_2 \sin 1''$ $\theta_2 = \pm 10''$, $\theta = 1°$ $l = 300$ ft $\omega_1 = \pm 0.000254$
Error in measurement of temperature	—	$v_2 = l \times c \times \delta T_1$ $\delta T_1 = \pm 1°$ F, $l = 300$ ft $c = 0.000002$ per 1° F $v_2 = \pm 0.000600$	$\omega_2 = l \times c \times \delta T_2$ $\delta T_2 = \pm 0.5°$ F, $l = 300$ ft $c = 0.000002$ per 1° F $\omega_2 = \pm 0.000300$
Error in stretch due to error in pull	—	$v_3 = \frac{l \times \delta F_1}{A \times E}$ $\delta F_1 = \pm \frac{1}{8}$ lb, $l = 300$ ft $A = 0.125 \times 0.015$ square inches $E = 22{,}000{,}000$ lb per square inch $v_3 = \pm 0.000909$	$\omega_3 = \frac{l \times \delta F_2}{A \times E}$ $\delta F_2 = \pm \frac{1}{16}$ lb, $l = 300$ ft $A = 0.125 \times 0.015$ square inches $E = 22{,}000{,}000$ lb per square inch $\omega_3 = \pm 0.000455$
Error in sag correction due to errors in pull	—	$v_4 = \frac{w^2 l^3}{12n^2F^3}\,\delta F_1$ $w = \frac{14}{16 \times 100}$ lb per ft $l = 300$ ft, $n = 2$ $\delta F_1 = \pm \frac{1}{8}$ lb $F = 20$ lbs $v_4 = \pm 0.000672$	$\omega_4 = \frac{w^2 l^3}{12n^2F^3}\,\delta F_2$ $w = \frac{14}{16 \times 100}$ lb per ft $l = 300$ ft, $n = 2$ $\delta F_2 = \pm \frac{1}{16}$ lb $F = 20$ lbs $\omega_4 = \pm 0.000336$
Error in standardisation combined with error in height above sea level	—	$v_5 = \Delta_1 l$ $\Delta_1 = \pm 1/300{,}000$ $l = 300$ ft $v_5 = \pm 0.001000$	—
Error due to faulty end readings and settings	—	—	$\omega_5 = \pm 0.002000$

The sum of the terms in the second column is 0·0001 ft., which, in a length of 300 ft. works out at 1 : 3,000,000. This is a very small quantity and this term can therefore be neglected. Hence, we proceed to form the sum of the squares of the probable errors in the last two columns.

	v	v^2	ω	ω^2
1	0·000 254	$64{,}516 \times 10^{-12}$	0·000 254	$64{,}516 \times 10^{-12}$
2	0·000 600	$360{,}000 \times 10^{-12}$	0·000 300	$90{,}000 \times 10^{-12}$
3	0·000 909	$826{,}281 \times 10^{-12}$	0·000 455	$207{,}025 \times 10^{-12}$
4	0·000 672	$451{,}584 \times 10^{-12}$	0·000 336	$112{,}896 \times 10^{-12}$
5	0·001 000	$1{,}000{,}000 \times 10^{-12}$	0·002 000	$4{,}000{,}000 \times 10^{-12}$
		$\Sigma v^2 = 2{,}702{,}381 \times 10^{-12}$		$\Sigma\omega^2 = 4{,}474{,}437 \times 10^{-12}$

Thus, for this particular tape and under the conditions allowed for above, the probable error at the end of N bays or tape lengths will be given by :—

$$\text{P.E.} = \pm\,[0{\cdot}000\,0027\,N^2 + 0{\cdot}000\,0045\,N]^{\frac{1}{2}}$$

Hence, for a line 3,000 ft. long with 10 bays, $N = 10$ and :

$$\begin{aligned}\text{P.E.} = R_l &= \pm\,[0{\cdot}000\,2700 + 0{\cdot}000\,0450]^{\frac{1}{2}}\\ &= \pm\,[0{\cdot}000\,3150]^{\frac{1}{2}}\\ &= \pm\,0{\cdot}018 \text{ ft.}\\ &= \pm\,1 : 167{,}000 \text{ approximately.}\end{aligned}$$

Having determined as above the probable displacement at the end of a single line due to linear errors only, we see that the total probable displacement for the whole section of 16 legs will be in the same direction as the direction of the traverse and will be equal to $\pm\,0{\cdot}018 \times \sqrt{16} = \pm\,0{\cdot}072$ ft.

We now require the probable displacements due to the angular errors, which, since the traverse is a "straight" one, will be at right angles to the direction of the traverse. First of all, there will be a displacement due to the error in the azimuth at the initial station. Let the probable error of this azimuth be $r_z = \pm\,1''{\cdot}0$.* Then the probable displacement R_z at the end of the section due to this error will be $\pm\,1{\cdot}0 \times L \times \sin 1''$, where L is the length of the section. $L = 48{,}000$, so that $R_z = \pm 0{\cdot}233$ ft.

Assuming that the difference between the bearing at the end azimuth station as computed through the traverse section and that computed directly from the observed azimuth is distributed in the usual manner equally among the angles of the section, the probable displacement at the end of the section due to the angular errors only is given (Vol. I, page 238) by :—

$$R_a = r_a L \sin 1''\left[\frac{(n+1)\,(n+2)}{12n}\right]^{\frac{1}{2}}$$

* The probable error of the azimuth should include allowance for the error likely to occur through neglect of the Laplace correction if this is unknown, or has not been applied.

where n is the number of legs in the section, in this case 16, and $r_{''}$ is the probable error of an angle.*

Hence, $$R_{''} = r_{''} \times 48{,}000 \times \sin 1'' \times \left[\frac{17 \times 18}{12 \times 16}\right]^{\frac{1}{2}}$$

The total probable linear displacement due to the combined linear, azimuth and angular errors will therefore be :—

$$\pm \left[(0{\cdot}072)^2 + (0{\cdot}233)^2 + r_{''}^2 \times (48{,}000)^2 \times \sin^2 1'' \times \frac{17 \times 18}{12 \times 16}\right]^{\frac{1}{2}}$$

and on page 219 we found that we wanted this probable error not to exceed $\pm$ 1·431 ft. This gives the equation :—

$$r_{''}^2 \times (48{,}000)^2 \times \sin^2 1'' \times \frac{17 \times 18}{12 \times 16} = (1{\cdot}431)^2 - (0{\cdot}072)^2 - (0{\cdot}233)^2.$$
$$= 1{\cdot}9884$$
$$\therefore r_{''}^2 = 23{\cdot}034$$
$$\therefore r_{''} = \pm 4''{\cdot}81.$$

It follows, therefore, that, in order to secure the required degree of accuracy, the probable error of the angular observations should not be allowed to exceed $\pm 4''{\cdot}81$. Owing, however, to there being so many unknown factors which we have had to estimate, it would be advisable to reduce this figure to, say, $3''{\cdot}5$, or to $4''$ at the most.

In order to determine the number of observations on different zeroes necessary to secure a probable error of a given amount, it is necessary to know the probable error of a single observation. For any particular instrument this is best found from the actual misclosures of the bearings at the end of different sections. Thus (Vol. I, page 232) let :—

e_1 = closing error at the end of a section containing n_1 angular stations.
e_2 = closing error at the end of a section containing n_2 angular stations.
. . . .
e_p = closing error at the end of a section containing n_p angular stations.
N = total number of sections or of closing errors available.
$r_{''}$ = probable error of an observed angle.

Then. $$r_{''} = \pm 0{\cdot}6745 \left[\frac{\dfrac{e_1^2}{n_1} + \dfrac{e_2^2}{n_2} + \dots + \dfrac{e_p^2}{n_p}}{N}\right]^{\frac{1}{2}}$$

This formula † does not include any allowance for the probable errors of the observed azimuths and a better result is obtained if these are in-

* Formulæ for $R_{''}$ for traverses of different shapes other than "straight" are given in Vol. I, page 238.

† Compare this expression with the equivalent formula, known as Ferrero's, which is sometimes used in triangulation to determine the probable error of the observed angles from the closing errors of the different triangles. Ferrero's formula is :—

$$r_{''} = \pm 0{\cdot}6745\sqrt{\frac{\Sigma\Delta^2}{3N}}$$

where $\Sigma\Delta^2$ is the sum of the squares of the closing errors of the N triangles (See page 298.)

cluded, particularly if n is small. If the probable error of an observed azimuth is assumed to be q times the probable error of an observed angle, the formula becomes :—

$$r_a = \pm 0{\cdot}6745 \left[\frac{\dfrac{e_1^2}{n_1 + 2q^2} + \dfrac{e_2^2}{n_2 + 2q^2} + \dfrac{e_3^2}{n_3 + 2q^2} + \cdots \dfrac{e_n^2}{n_n + 2q^2}}{N} \right]^{\frac{1}{2}}$$

The value of r_a thus obtained will be the probable error of the mean of, say, m zeroes on the instrument. Hence (page 277) the probable error of a single zero will be $r_s = r_a\sqrt{m}$ and the number of zeroes necessary to obtain a probable error of r will be given by $n = \frac{r_s^2}{r^2}$.

With a new instrument, which has not been used before, or one for which no values for the closing errors at the end of the different traverse sections are available, the above method cannot be used, but the probable error of a single observation can be found by taking a large number of observations of a single angle. Let $\Sigma\epsilon^2$ be the sum of the squares of the residuals, *i.e.* the sum of the squares of the differences between each single observation and the mean of all the observations, and m the number of observations. Then

$$r_s = 0{\cdot}6745\sqrt{\frac{\Sigma\epsilon^2}{(m-1)}}$$

and, as before, $n = \frac{r_s^2}{r^2}$. This method, however, will generally give a lower value for r_s than the first one, and this lower value will not be so reliable as the other since the conditions of the test do not reproduce in the same way all the varying conditions that are met with in the field. Consequently, if the second method is used, it is well to adopt a value of n somewhat greater than that found from the observations.

It will be noted that we have worked out a value for r_a which is based on the assumption that the angles are adjusted by distributing equally between them the angular misclosure found at the terminal azimuth station. Angles, however, are not always adjusted in this way. For example, the United States Coast and Geodetic Survey obtain corrections to all bearings and lengths by means of a least square adjustment which takes into account the misclosure in position as well as the misclosure in bearing. However, as Yates has shown, the result of adjusting the angles by an equal and rather arbitrary distribution of the closing error in bearing is to reduce the probable linear displacement by about a half (Vol. I, page 239) and it is doubtful if a least square adjustment would reduce this further to any appreciable extent. Consequently, it may be assumed that the formula we have used above will still give a result which will be reliable enough for estimating purposes, even when the angles have been adjusted by least squares. If, however, it is desired to obtain the required degree of accuracy by the use of angles and bearings that have not been adjusted at all for misclosure, the formula to be used for the probable displacement at the end of a "straight" traverse, due to the angular errors only, will be :—

$$R_a = r_a L \sin 1'' \left[\frac{(n+1)(2n+1)}{6n} \right]^{\frac{1}{2}}$$

which, when n is fairly large, reduces to :—

$$R_a = r_a L \sin 1'' \sqrt{\frac{n}{3}}$$

Had the first of these last two formulæ been used in the example we would have found $r_a = \pm 2''{\cdot}54$.

There are several points that strike one about this investigation. The first is the large contribution which the angular errors make to the probable linear displacement when compared with that made by the linear errors. The standard of linear accuracy that has been assumed is by no means difficult to realise with well-standardised invar tapes when these are properly and carefully used. Hence, it follows that great care has to be taken to maintain a proper standard of accuracy in the angular measurements, and, considering how relatively easy it is to attain a high standard of linear accuracy, this standard should be maintained at as high a level as is conveniently possible.

A second point is that the ratio of the closing error of a traverse to the total length is not an entirely satisfactory method of defining the accuracy of a traverse, because the probable linear displacement at the end is not proportional to the length, and, in fact, provided standardisation and other cumulative errors are reduced to a minimum, the ratio should be much smaller for a long traverse than for a short one. It must, however, be said that, although sufficient reliable data from the field which would justify any definite law of closure being formulated do not yet exist, such data as are available tend to indicate that fractional errors of misclosure are now more directly dependent on the length of the traverse than the figures taken above would suggest. Hence, it is possible that errors of a cumulative nature are greater than those that we have allowed for.

It should also be noted that the errors here investigated are probable errors only and that the chances of the real error being greater than the probable error are equal to those of its being less. Consequently, if it is desired to make even reasonably certain (but not absolutely, which is impossible) that a given error should not be exceeded, the value to be adopted for the permissible probable error should be at most $\frac{1}{3}$ to $\frac{1}{4}$ of the given value.

EXAMPLES

1. The elevation of an instrument at A is 210·3 ft. Find the minimum height of signal required at B, 28·3 miles distant, where the elevation of the ground is 296·0 ft. The intervening ground may be assumed to have a uniform elevation of 150 ft., and the line of sight must nowhere be less than 6 ft. above the surface. Take $h = {\cdot}5738D^2$.

2. The elevation of an instrument at A is 142 ft. Find the minimum elevation required for a signal at B, 54 miles distant, if the line crosses an arm of the sea. The coefficient of refraction is to be taken as 0·08 and the mean radius of the earth as 3,960 miles.

3. The altitudes of two proposed stations A and B, 80 miles apart, are respectively 2,066 ft. and 3,487 ft. The altitudes of two points C and D on the profile between them are respectively 1,803 ft. and 2,216 ft., the distances being AC = 30 miles and

AD = 55 miles. Determine whether A and B are intervisible for $k = 0{\cdot}07$ and $R = 3{,}960$ miles.

4. Describe, in detail, the field operations necessary for measuring a long base-line with extreme accuracy by means of a steel tape or wire. Enumerate the corrections that must be made.

A line, 2 miles long, is measured with a tape of length 300 ft., which is standard under no pull at 60° F. The tape in section is $\frac{1}{8}$ inch wide and $\frac{1}{20}$ inch thick. If one half of the line is measured at a temperature of 70° F., and the other half at 80° F., and the tape is stretched with a pull of 50 lbs., find the correction on the total length. Coefficient of expansion = 0·0000065; weight of 1 cubic inch of steel = 0·28 lbs.; E = 29,000,000 lbs. per square inch. (Univ. of Lond., 1908.)

5. A base line is simultaneously measured with a steel wire and a brass wire. The length given by the steel component is 31,342·622 ft., and that by the brass component is 31,339·144 ft., both referred to the absolute lengths of the wires at 32° F. The coefficients of expansion of the steel and the brass components are respectively 0·0000063 and ·0000100 per 1° F.

Find the length of the base corrected for temperature.

6. The horizontal distance between two points on a roadway which is on a slope is required very accurately.

The error of the standard tape at a known temperature and the coefficient of expansion of the standard tape and of the tape used to measure the line and the weight of the measuring tape are known.

Having measured the line at a known temperature, explain carefully how you obtain the correct horizontal distance between the ends of the line.

A line is measured on a uniform slope with a 100 ft. steel tape pulled with a force of 10 lb., and is found to be 1,725 ft. long. The temperature at the time of measurement is 34° F. The tape is correct at 62° F., when the pull on the tape is 10 lb. The difference of level of the two ends of the line is 30 ft. Determine the horizontal distance between the two ends of the line. Coefficient of expansion of the tape is 0·000006 per degree F. (Univ. of Lond., 1915.)

7. The corrected measured length of the Semliki base in Uganda is 16,534·05438 m. Its mean height above sea level is 645·4 m., and the radius of curvature of the earth computed for the latitude and azimuth of the base is 6,358,982 m.

Calculate the length of the base reduced to sea level.

8. Compute the value of the correction due to change of gravity in the case of the Lossiemouth base line, the length of which is 23,526 ft. The 100 ft. tapes were standardised at Southampton, in latitude 50° 54′, at an elevation of 76 ft., the tension being applied by 20 lb. weights. The same weights were used in the field measurement in latitude 57° 42′ and at an elevation of 23 ft. The tapes stretch ·00208 ft. per lb. tension per tape length. Take the radius of the earth as 21×10^6 ft.

9. The circle of a theodolite is graduated to 5′ spaces, and 5 turns of the micrometer screw are required to carry the hairs from one graduation to the next. The forward and back readings to be applied to the approximate reading of 97° 5′ are 3′ 39″·4 and 3′ 41″·2 respectively. What is the correct reading?

10. The horizontal axis of a theodolite has an inclination of 5″·4. For a single observation in which the left hand pivot is the higher, compute the correction for dislevelment applicable to an angle the left hand station of which has an angle of elevation of 9° 35′ and the right hand station a depression of 4° 21′.

11. Compute the value of the correction to angle AOB for phase of cylindrical signals. The observed directions are OA = 39° 17′ 34″·2, OB = 86° 52′ 07″·4, and sun = 43° 25′. The diameter of the signal at A subtends 5″ at the instrument, and that at B subtends 3″, and the pointings are made upon the bright line.

12. Directions are observed from a satellite station, 204 ft. from station C, with the following results; A, zero; B, 71° 54′ 32″·25; C, 296° 12′. The approximate lengths of AC and BC are respectively 54,072 ft. and 71,283 ft. Compute the angle subtended at station C.

CHAPTER IV

SURVEY ADJUSTMENT

THE following two chapters deal with the computing work which follows the derivation of the final field results of extended triangulation. The method of obtaining the final data from the field book figures by the application of corrections has already been indicated in the preceding chapter. Those results are subjected to a process of correction to eliminate apparent inconsistencies, and finally the triangle sides and the geodetic co-ordinates of the stations are calculated.

THEORY OF ERRORS

Even with the most refined methods of angular or linear measurement, errors, however small, are unavoidable. These are evidenced by discrepancies between the results of repeated measurements of the same quantity and by non-fulfilment of geometrical relationships which should obtain between different quantities. Such inconsistencies must be eliminated by subjecting the field results to a process of adjustment designed to yield the most probable value of each measured quantity based upon the observed values and the conditions binding them together

Classification of Errors. From whatever cause arising, errors can be classed as: (*a*) Mistakes; (*b*) Constant errors; (*c*) Systematic errors; (*d*) Accidental errors.

Mistakes arise from carelessness or inattention on the part of the observer. While their occurrence is always possible, mistakes of serious magnitude should never influence the final result, since the system of checking required in every surveying operation should expose them.

Constant errors are those which have the same effect, and the same sign, for all observations. Systematic errors are those in which the algebraic sign and the magnitude bear a fixed relation to one of the quantities that are being measured. Both constant and systematic errors tend, therefore, to be cumulative in their effects. In some cases, though not in all, their effects are understood and can be eliminated, some by the adoption of a particular routine in measurement, and others by the application of computed corrections. Such errors do not become apparent because of discrepancies between the results of repeated measurements of the same quantity. Owing to their cumulative nature, it is very important to eliminate their effects by the methods sufficiently emphasised throughout the text.

Systematic errors need not always have the same sign throughout the whole of the work but may have one sign for certain parts of it and the opposite sign for other parts, so that, taken over a very large number of measurements, they may tend, as regards their ultimate effect, to behave as accidental errors. An example of this occurs in precise levelling where systematic error may occur throughout the whole of one line, or throughout part of it, but, when taken over a whole series of

lines, or over one long line, the different systematic errors tend to vary both in sign and in magnitude, and so to behave as accidental errors. Much the same sort of thing happens in taping and in traversing.

Accidental errors include all unavoidable and unknown errors which are entirely beyond the control of the observer, their chief characteristic being that they vary in sign, positive and negative errors tending to occur with equal frequency. Every observation is subject to numerous accidental errors, and, as these happen by mere chance, their algebraic sum, representing the accidental error of the observation, can be treated by the laws of probability alone.

Examples of Different Classes of Error. In the case of base measurement, mistakes might conceivably occur in reading the tape, counting tape lengths, or booking. Taping with a tape that is too long or too short, or with one for which there is an error in the standardisation, is a good example of a constant error. Systematic errors in taping can arise from differences in temperature during measurement and these can be computed and allowed for when the temperatures and coefficient of expansion are known. Accidental errors arise from, among other causes, imperfection of eyesight influencing the operations of standardising and using the tape, imperfections in thermometers and spring balances, and the effects of small and momentary changes of temperature and tension.

In angle measurement, blunders, however unlikely, might occur by sighting the wrong signal, by repeating an erroneous micrometer or vernier reading, or by using the wrong tangent screw. Constant and systematic errors are due to imperfect adjustment of the instrument and defects in the non-adjustable parts. Constant errors may arise in the measurement of vertical angles through index error in the vertical circle. The errors of graduation of a circle tend to be systematic and to obey a definite law of periodicity but their effects can be largely eliminated in the measurement of horizontal angles, though not of vertical ones, by using different zeroes. Accidental errors are due to such causes as limited refinement of reading, imperfect estimation of readings and signal bisections, the effects of irregular atmospheric refraction, imperfection of the observer's sight and touch in the levelling and manipulating of the instrument, etc.

Definitions. A *Direct Measurement* is one made directly on the quantity being determined, *e.g.* the measurement of a base, the single measurement of an angle, etc.

An *Indirect Measurement* is one in which the observed value is deduced from the measurement of some related quantities, *e.g.* the sides of a triangulation, the measurement of an angle by repetition (a multiple of the angle being measured).

A *Conditioned Quantity* as opposed to an independent quantity, is one whose value must bear a rigid relationship to some other quantity or quantities.

The *Most Probable Value* of a quantity is that more likely than any other to be its true value, and is the best result which can be attained.

A *True Error* is the difference between the true and observed values of a quantity. Since the true value of a quantity cannot be ascertained, the true error is never known. In the case, however, of the summation of the angles of triangles or polygons or of angles to close the horizon, the true error of their sum is known.

A *Residual Error* or *Residual* is the difference between an observed value and the most probable value of the quantity.

The *Weight* of an observation is a measure of its relative trustworthiness

In the course of a series of observations, the personal, instrumental, or atmospheric conditions may vary so that a uniform degree of reliability in the results would not be expected. The results are then said to be of unequal weight. Numerical values assigned as the weights of a series of observations are simply ratios indicating the relative precision of the observations. Weights are assigned by estimation, or in terms of the number of separate observations of equal reliability from which the result being weighted is derived, or by calculation from the probable error.

The Laws of Accidental Error. After all known errors have been corrected out, the results of observations contain accidental errors only, and these are treated in accordance with the laws of probability. It must be understood that these laws are quite inapplicable to constant and systematic errors, which are not considered further.

Experience shows that for a prolonged series of observations of the same quantity or quantities, under apparently constant conditions, accidental errors obey the following laws:

(1) Small errors are more frequent than large ones.
(2) Positive and negative errors are equally frequent.
(3) Very large errors do not occur.

The graphical representation of these laws is shown in Fig. 110, in which abscissæ represent magnitudes of errors and ordinates their respective frequencies or probabilities of occurrence. The third law as stated does not lend itself to graphical interpretation with the others, but, if it is expressed in the form that very large errors occur with very small frequency, it is represented mathematically by making the x axis an asymptote to the curve.

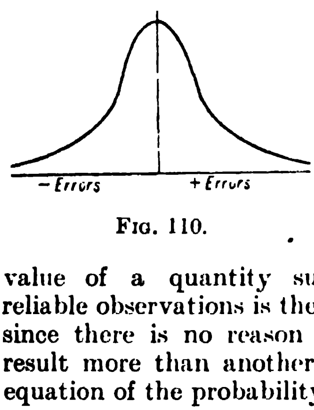


FIG. 110.

Equation of the Probability Curve. It may be taken that the most probable value of a quantity subjected to a series of direct and equally reliable observations is the arithmetic mean of the several measurements, since there is no reason why one determination should influence the result more than another. This principle being accepted, the general equation of the probability curve is found to be

$$y = ke^{-c^2x^2},$$

where y is the probability of occurrence of an error x, k is a constant, c depends upon the precision of the observations, and e is the base of the Napierian logarithms.

General Principle of Least Squares. When a quantity is being deduced from a series of observations, then to every value which may be adopted for the result there will correspond a series of errors or differences between the adopted value and the several observed results. The most probable series of errors is necessarily that derived from the adoption of the most probable value of the quantity. It is found from the probability equation that the most probable values of a series of errors arising from observations of equal weight are those for which the sum of the squares is a minimum.

In this manner is derived the fundamental law of least squares, which states that, for observations of equal weight, the most probable value of an observed quantity is that which makes the sum of the squares of the residual errors a minimum. It may likewise be deduced that, for observations of unequal weight, the most probable value of the observed quantity is that which makes the sum of the weighted squares of the residuals a minimum.

Most Probable Values of Directly Observed Independent Quantities. It has been stated that the most probable value of a quantity subjected to a series of direct observations of equal weight is given by the arithmetic mean of the observed results. This is in accordance with the law of least squares. Similarly, when the observations are of unequal weight the most probable value is the weighted arithmetic mean, obtained by multiplying each result by its weight and dividing the sum of the products by the sum of the weights.

Example.—Find the most probable value of an angle of which measurements under different conditions are 56° 12′ 3″·2, weight 2, and 56° 12′ 8″·0, weight 1.

	°	′	″			°	′	″
	56	12	3·2	× 2 =		112	24	6·4
	56	12	8·0	× 1 =		56	12	8·0
					3)	168	36	14·4
Most probable value =						56	12	4·8

Most Probable Values of Indirectly Observed Independent Quantities. In the general case of indirectly observed quantities the most probable values are determined by the solution of "normal equations" compiled from "observation equations" expressing the observed results. These normal equations are formed for each quantity of which the most probable value is required, and the latter are obtained by simultaneous solution.

To form the normal equation for any quantity, each observation equation is multiplied by the algebraic coefficient of the quantity in that observation equation, and the products are added. When the observations are of unequal weight, the normal equation for any quantity is formed by multiplying each observation equation by the weight of that observation as well as by the algebraic coefficient of the quantity in that observation equation, and the products are added.

Example. Find the most probable values of the angles a, b, and c at one station from the following observations:—

	°	′	″	
$a =$	40	13	28·7,	weight 1.
$b =$	34	46	15·4,	1.
$a + b =$	74	59	43·0,	2.
$a + b + c =$	132	31	07·2,	1.
$b + c =$	92	17	42·2,	3.

For the sake of convenience arrange the equations in columns thus:—

				°	′	″	
a			=	40	13	28·7,	weight 1.
	b		=	34	46	15·4,	1.
$a +$	b		=	74	59	43·0,	2.
$a +$	$b +$	c	=	132	31	07·2,	1.
	$b +$	c	=	92	17	42·2	3.

To form the normal equation in a, multiply each equation by the algebraic coefficient of a and by the weight and add the results together, giving :—

$$\begin{array}{rlll} a & = 40 & 13 & 28 \cdot 7, \\ 2a + 2b & = 149 & 59 & 26 \cdot 0, \\ a + b + c & = 132 & 31 & 07 \cdot 2, \\ \hline 4a + 3b + c & = 322 & 44 & 01 \cdot 9 \end{array}$$

By the same process,

$$\begin{array}{rllll} & {}^{\circ} & & & \\ 3a + 7b + 4c = & 594 & 09 & 55 \cdot 2 & = \text{normal equation in } b. \\ a + 4b + 4c = & 409 & 24 & 13 \cdot 8 & = \text{normal equation in } c. \end{array}$$

On solving these equations, the most probable values are found to be :

$$\begin{array}{rlll} & {}^{\circ} & {}' & {}'' \\ a = & 40 & 13 & 27 \cdot 25, \\ b = & 34 & 46 & 15 \cdot 63, \\ c = & 57 & 31 & 26 \cdot 00. \end{array}$$

It should be noted that, when the normal equations are written down in order and arranged in columns, the coefficients in the first equation are reproduced in the first column, those of the second equation in the second column, and so on. For this reason, the three normal equations in the example would usually be written in the purely conventional form :—

$$\begin{array}{rlll} 4a + 3b + c = & 322^{\circ} & 44' & 01'' \cdot 9, \\ 7b + 4c = & 594 & 09 & 55 \cdot 2, \\ 4c = & 409 & 24 & 13 \cdot 8. \end{array}$$

All normal equations, owing to this peculiar symmetry, can be written in this abbreviated manner, and it will be found that, by doing so, there results a saving in copying and writing and that the actual solution is considerably simplified and shortened.

The labour of solution is greatly diminished by assuming a set of approximate values for the unknown quantities and then determining by normal equations the corrections necessary to give the most probable values. The corrections so found must then be added algebraically to the respective assumed values to yield the most probable values of the unknowns. This method is almost invariably employed in ordinary computing work.

Example. For the previous case let it be assumed that the approximate values of a, b and c are respectively $40^{\circ}\ 31'\ 28''\cdot 7$, $34^{\circ}\ 46'\ 15''\cdot 4$ and $57^{\circ}\ 31'\ 26''\cdot 8$, taken from observation equations 1, 2 and 5, and let r_1, r_2 and r_3 represent the unknown residual errors made in the assumption, so that the most probable values of a, b and c are $40^{\circ}\ 31'\ 28''\cdot 7 + r_1$, $34^{\circ}\ 46'\ 15''\cdot 4 + r_2$ and $57^{\circ}\ 31'\ 26''\cdot 8 + r_3$ respectively.

The observation equations are to be reduced in terms of r_1, r_2 and r_3 to express discrepancies between the various observed results and those given by the assumed values, the latter being subtracted from the former. The reduced observation equations then become :—

$$\begin{array}{rll} r_1 = & 0, & \text{weight } 1, \\ r_2 = & 0, & 1, \\ r_1 + r_2 = & -1 \cdot 1, & 2, \\ + r_2 + r_3 = & -3 \cdot 7, & 1, \\ r_2 + r_3 = & 0, & 3, \end{array}$$

It will be seen later that it is more convenient to transfer the absolute terms on the right-hand side of the equations to the left-hand sides, the signs, of course,

being changed at the same time and the whole of each equation put equal to zero. When this has been done, and the equations arranged in columns, we have :—

$$\begin{array}{llll}
r_1 & = 0, & \text{weight } 1, & s = 1, \\
\quad r_2 & = 0, & 1, & 1, \\
r_1 + r_2 \qquad + 1{\cdot}1 & = 0, & 2, & 3{\cdot}1, \\
r_1 + r_2 + r_3 + 3{\cdot}7 & = 0, & 1, & 6{\cdot}7, \\
\quad r_2 + r_3 & = 0, & 3, & 2.
\end{array}$$

Here we have added an extra column at the right, the use of which will be explained later. Proceeding as before, the normal equations are found to be :

$$\begin{array}{ll}
4r_1 + 3r_2 + r_3 + 5{\cdot}9 = 0, & s = +13{\cdot}9 \\
3r_1 + 7r_2 + 4r_3 + 5{\cdot}9 = 0, & \quad +19{\cdot}9 \\
r_1 + 4r_2 + 4r_3 + 3{\cdot}7 = 0, & \quad +12{\cdot}7.
\end{array}$$

These last three equations have the same coefficients for the unknowns as before and exhibit the same characteristic form of symmetry already noticed in the previous example. Hence, they can be written in the conventional form :—

$$\begin{array}{rl}
4r_1 + 3r_2 + r_3 + 5{\cdot}9 = 0, & s = +13{\cdot}9 \\
7r_2 + 4r_3 + 5{\cdot}9 = 0 & \quad +19{\cdot}9 \\
4r_3 + 3{\cdot}7 = 0 & \quad +12{\cdot}7,
\end{array}$$

where it is understood that the first term in the second equation, and the first two terms in the third equation, have been omitted but their coefficients are identical with the numbers appearing above the coefficients of r_2 and r_3 in the extreme left-hand terms of the second and third equations.

The solution of these three normal equations gives $r_1 = -1''{\cdot}45$, $r_2 = +0''{\cdot}23$, $r_3 = -0''{\cdot}80$. By applying these corrections to the assumed values, we have, as before, the most probable values: $a = 40^\circ\ 13'\ 27''{\cdot}25$, $b = 34^\circ\ 46'\ 15''{\cdot}63$, and $c = 57^\circ\ 31'\ 26''{\cdot}00$.

There is an extremely useful check on the formation of the normal equations which should always be employed. To the right of each observation equation in the above example we have written the sum of the absolute term and the coefficients of the r's which occur in that equation. Multiply each of these quantities by the corresponding coefficient of r_1 and by the weight and add the results. The figure obtained is $1 + 0 + 6{\cdot}2 + 6{\cdot}7 + 0 = 13{\cdot}9$, which is equal to the sum of the absolute term and the coefficients of the r's in the first normal equation. This is the check on the formation of that equation. Similarly, multiply each s by the coefficient of r_2 in the second observation equation and by the weight and add the results. This gives $+19{\cdot}9$, which is the sum of the coefficients of the r's and of the absolute in the second normal equation. This checks the formation of that equation. Similarly, for the third normal equation, $6{\cdot}7 + 6 \quad + 12{\cdot}7 = 1 + 4 + 4 + 3{\cdot}7$.

It will be seen later that the summations which appear to the right of the normal equations not only check the formation of these equations but also provide most useful checks at almost every stage of the solution.

General Formulæ for Observational Normal Equations. Let the n observation equations be :—

$$\begin{array}{lr}
a_1r_1 + b_1r_2 + c_1r_3 + d_1r_4 + \ldots + k_1r_m + l_1 = 0, & \text{wt} = w_1, \\
a_2r_1 + b_2r_2 + c_2r_3 + d_2r_4 + \ldots + k_2r_m + l_2 = 0, & w_2, \\
a_3r_1 + b_3r_2 + c_3r_3 + d_3r_4 + \ldots + k_3r_m + l_3 = 0, & w_3, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\
a_nr_1 + b_nr_2 + c_nr_3 + d_nr_4 + \ldots + k_nr_m + l_n = 0, & w
\end{array}$$

where the r's are the corrections to be found, the l's are the absolute terms transferred to the left-hand side of the equations, the w's are the weights of the different observations and the a's, b's, c's . . . k's are numerical coefficients. Then the normal equations which have to be solved to give $r_1, r_2, r_3, \ldots r_m$ are :—

$$\begin{array}{l}
[wa^2]r_1 + [wab]r_2 + [wac]r_3 + \ldots + [wak]r_m + [wal] = 0, \\
[wba]r_1 + [wb^2]r_2 + [wbc]r_3 + \ldots + [wbk]r_m + [wbl] = 0, \\
[wca]r_1 + [wcb]r_2 + [wc^2]r_3 + \ldots + [wck]r_m + [wcl] = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
[wka]r_1 + [wkb]r_2 + [wkc]r_3 + \ldots + [wk^2]r_m + [wkl] = 0,
\end{array}$$

where [ ] denotes summation,* so that :—

$$\begin{array}{l}
[wa^2] = w_1a_1^2 + w_2a_2^2 + w_3a_3^2 + \ldots + w_na_n^2, \\
[wab] = w_1a_1b_1 + w_2a_2b_2 + w_3a_3b_3 + \ldots + w_na_nb_n, \\
[wac] = w_1a_1c_1 + w_2a_2c_2 + w_3a_3c_3 + \ldots + w_na_nc_n, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
[wb^2] = w_1b_1^2 + w_2b_2^2 + w_3b_3^2 + \ldots + w_nb_n^2, \\
[wba] = w_1b_1a_1 + w_2b_2a_2 + w_3b_3a_3 + \ldots + w_nb_na_n,
\end{array}$$

and so on.

Since $[wba] = [wab]$; $[wca] = [wac]$; $[wcb] = [wbc]$, etc., the terms lying to the left of those containing $[wb^2]$; $[wc^2]$; . . . $[wk^2]$ in the second, third, etc., equations are usually not written down, so that the normal equations are expressed in the abbreviated, but purely conventional, form :—

$$\begin{array}{rrrrr}
[wa^2]r_1 + [wab]r_2 + [wac]r_3 + \ldots & + [wak]r_m & + [wal] & = 0, \\
[wb^2]r_2 + [wbc]r_3 + \ldots & + [wbk]r_m & + [wbl] & = 0, \\
[wc^2]r_3 + \ldots & + [wck]r_m & + [wcl] & = 0, \\
\cdot & & & \\
& [wk^2]r_m & + [wkl] & = 0.
\end{array}$$

When written in this abbreviated form, the second, third, fourth, etc., equations are, of course, not complete, as it must be understood that there are omitted terms in $r_1, r_2, r_3, \ldots r_m$, whose coefficients are identical with those that appear immediately above the term at the extreme left of each equation.

We also note that if

$$\begin{array}{l}
s_1 = a_1 + b_1 + c_1 + d_1 + \ldots + k_1 + l_1, \\
s_2 = a_2 + b_2 + c_2 + d_2 + \ldots + k_2 + l_2, \\
\\
s_n = a_n + b_n + c_n + d_n + \ldots + k_n + l_n.
\end{array}$$

Then :—

$$\begin{array}{l}
w_1a_1s_1 = w_1a_1^2 + w_1a_1b_1 + w_1a_1c_1 + \ldots + w_1a_1k_1 + w_1a_1l_1, \\
w_2a_2s_2 = w_2a_2^2 + w_2a_2b_2 + w_2a_2c_2 + \ldots + w_2a_2k_2 + w_2a_2l_2, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
w_na_ns_n = w_na_n^2 + w_na_nb_n + w_na_nc_n + \ldots + w_na_nk_n + w_na_nl_n.
\end{array}$$

* The symbol Σ (Greek capital sigma) is often used in mathematical work to denote summations of this kind, but, in books on least squares applied to surveying, the symbol [ ], as used here, is more generally employed.

Hence,

$$[was] = [wa^2] + [wab] + [wac] + \ldots + [wak] + [wal].$$

Similarly,

$$[wbs] = [wba] + [wb^2] + [wbc] + \ldots + [wbk] + [wbl],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$[wks] = [wka] + [wkb] + [wkc] + \ldots + [wk^2] + [wkl],$$

These relations give the summation tests for the formation of the normal equations.

Unless care is taken to carry out the arithmetical work methodically and to apply the summation test for each equation as it is formed, it is very easy to make a mistake when forming the normal equations. A good plan is to proceed as follows :—

First form a table giving the weights, coefficients, absolute term and the sum of the coefficients and absolute term for each observation equation. Thus :—

w	x_1	x_2	x_3	x_4	...	x_k	l	s
w_1	a_1	b_1	c_1	d_1	...	k_1	l_1	s_1
w_2	a_2	b_2	c_2	d_2	...	k_2	l_2	s_2
w_3	a_3	b_3	c_3	d_3	...	k_3	l_3	s_3
...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...
w_n	a_n	b_n	c_n	d_n	...	k_n	l_n	s_n

Next form a table giving the products of these weights and coefficients, column by column, up to the products of the l's and s's by the w's and k's, and take the sum in each column as in the table on page 264.

In this table, the sums of the individual columns give the coefficients in the normal equations, and each normal equation as it is formed is checked by seeing if the sums of the different coefficients add up to the sum in the columns containing the s's.

Thus, with the example already given :—

Observation Equations.

w	x_1	x_2	x_3	l	s
1	1	0	0	0	1
1	0	1	0	0	1
2	1	1	0	+ 1·1	+ 3·1
1	1	1	1	+ 3·7	+ 6·7
3	0	1	1	0	2

TABLE FOR FORMATION OF OBSERVATIONAL NORMAL EQUATIONS

wa^2	wab	wac	. . .	wak	wal	was	wb^2	wbc	. . .
$w_1a_1^2$	$w_1a_1b_1$	$w_1a_1c_1$	. . .	$w_1a_1k_1$	$w_1a_1l_1$	$w_1a_1s_1$	$w_1b_1^2$	$w_1b_1c_1$	. . .
$w_2a_2^2$	$w_2a_2b_2$	$w_2a_2c_2$	. . .	$w_2a_2k_2$	$w_2a_2l_2$	$w_2a_2s_2$	$w_2b_2^2$	$w_2b_2c_2$	. . .
$w_3a_3^2$	$w_3a_3b_3$	$w_3a_3c_3$	. . .	$w_3a_3k_3$	$w_3a_3l_3$	$w_3a_3s_3$	$w_3b_3^2$	$w_3b_3c_3$	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
$w_na_n^2$	$w_na_nb_n$	$w_na_nc_n$	. . .	$w_na_nk_n$	$w_na_nl_n$	$w_na_ns_n$	$w_nb_n^2$	$w_nb_nc_n$	. . .
$[wa^2]$	$[wab]$	$[wac]$	. . .	$[wak]$	$[wal]$	$[was]$	$[wb^2]$	$[wbc]$	. . .

Formation of Normal Equations.

$[wa^2]$	$[wab]$	$[wac]$	$[wal]$	$[was]$	$[wb^2]$	$[wbc]$	$[wbl]$	$[wbs]$	$[wc^2]$	$[wcl]$	$[wcs]$
1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	1	0	0	0
2	2	0	+ 2·2	+ 6·2	2	0	+ 2·2	+ 6·2	0	0	0
1	1	1	+ 3·7	+ 6·7	1	1	+ 3·7	+ 6·7	1	+ 3·7	+ 6·7
0	0	0	0	0	3	3	0	6	3	0	6
4	3	1	+ 5·9	+ 13·9	7	4	+ 5·9	+ 19·9	4	+ 3·7	+ 12·7

In addition to the ordinary checks given on page 263, there are two additional ones which can be formed by adding three columns (not shown above) headed $[wl^2]$, $[wls]$, and $[ws^2]$ to the table for the formation of the normal equations. Then $[wls]$ = sum of the absolute terms in the normal equations plus $[wl^2]$, and $[ws^2] = [was] + [wbs] + [wcs] + [wds] + \ldots + [wls]$. These additional terms are set out in the last two rows of the following tabulation of the coefficients, absolute terms and sum or check terms in the normal equations, the first three rows being the normal equations themselves The whole of the checks are therefore :—

$$
\begin{array}{l}
4 + 3 + 1 + 5{\cdot}9 = + 13{\cdot}9 \\
3 + 7 + 4 + 5{\cdot}9 = + 19{\cdot}9 \\
1 + 4 + 4 + 3{\cdot}7 = + 12{\cdot}7 \\
5{\cdot}9 + 5{\cdot}9 + 3{\cdot}7 + 16{\cdot}11 = + 31{\cdot}61 \\
13{\cdot}9 + 19{\cdot}9 + 12{\cdot}7 + 31{\cdot}61 = + 78{\cdot}11.
\end{array}
$$

Normal Equations.

r_1	r_2	r_3	l	s
4	3	1	+ 5·9	+ 13·9
	7	4	+ 5·9	+ 19·9
		4	+ 3·7	+ 12·7
			+ 16·11	+ 31·61
				+ 78·11

Solution of Normal Equations. The labour of solution of normal equations, which may be very heavy if the number of equations is large, will be greatly reduced, and the chances of error lessened, if a sound and systematic method is followed throughout. Very fortunately, simple checks are available which can be used to check the work in all its stages and to locate an error when one is made. The method can best be followed by studying a simple example, and, although the one now to be given consists of only three equations and three unknowns, a close examination

of it should enable the reader to solve any number of equations. The actual working is first set out and then follows an explanation of it line by line. Another example, with four equations and four unknowns, is given on page 276.

Take the three normal equations which we have just formed. The full solution is in two parts, and the first part, often called the "forward solution," is set out as follows :—

Line	r_1	r_2	r_3	l	s
1	4	3	1	5·9	13·9
2	− 1	− 0·75	− 0·25	− 1·475	− 3·475
3		7	4	5·9	19·9
4		− 2·25	− 0·75	− 4·425	− 10·425
5		+ 4·75	+ 3·25	+ 1·475	+ 9·475
6		− 1	− 0·6842	− 0·3105	− 1·9947
7			4	3·7	12·7
8			− 0·25	− 1·475	− 3·475
9			− 2·2237	− 1·0092	− 6·4829
10			+ 1·5263	+ 1·2158	+ 2·7421
11			− 1	− 0·7966	− 1·7966

Here :

Line 1 = The first normal equation.
Line 2 = Line 1 divided by 4, the coefficient of r_1, with signs reversed.
Line 3 = The second normal equation.
Line 4 = Line 2 multiplied by 3, the coefficient of r_2 in line 1.
Line 5 = Algebraic sum of lines 3 and 4.
Line 6 = Line 5 divided by + 4·75, the coefficient of r_2 in that line, with signs reversed.
Line 7 = The third normal equation.
Line 8 = Line 2 multiplied by 1, the coefficient of r_3 in line 1.
Line 9 = Line 6 multiplied by + 3·25, the coefficient of r_3 in line 5.
Line 10 = Algebraic sum of lines 7, 8 and 9.
Line 11 = Line 10 divided by + 1·5263, the coefficient of r_3 in that line, with signs reversed.

This completes the first, and by far the larger, part of the solution and leads to the value for r_3, which is the figure, −0·7966, in the last line of the l column.

The checks so far are :—

(1) The sum of the figures in the "r" and "l" columns of lines 5 and 10 add up to the figure in the "s" column of the same lines.

(2) The sum of the figures in the "r" and "l" columns of lines 2, 6 and 11 add up to the figure in the "s" column in the same lines.

If these checks are not fulfilled, there is an error somewhere, and, if this error is not to be found in the additions, it is most likely in one of the lines 4, 8 or 9 which can be examined line by line as follows :—

Line 4 is the product of the entries in line 2 multiplied by 3, the co-

efficient of r_2 in line 1. But, in line 4 the product 3×-1 has not been written down under r_1. Hence, taking this term into account, the sum of the entries in the " r " and " l " columns of line 4 is $-3 - 2{\cdot}25 - 0{\cdot}75 - 4{\cdot}425 = -10{\cdot}425$, which is the entry in the " s " column of that line. Again, take line 9, which comes from lines 5 and 6. The omitted term in this line is $+3{\cdot}25 \times -1 = -3{\cdot}25$. Then we have $-3{\cdot}25 - 2{\cdot}2237 - 1{\cdot}0092 = -6{\cdot}4829$, which agrees with the figure in the " s " column.

The whole process, perhaps, will be more clearly understood if each equation and the different steps are written out in full as follows :—

$$
\begin{array}{lrrrrl}
\text{Line (1A)} & +4r_1 & +3{\cdot}0000r_2 & +1{\cdot}0000r_3 & +5{\cdot}9000 & = 0. \\
\quad\text{,, (2A)} & -r_1 & -0{\cdot}7500r_2 & -0{\cdot}2500r_3 & -1{\cdot}4750 & = 0. \\
 & & & & & \\
\quad\text{,, (3A)} & +3r_1 & +7{\cdot}0000r_2 & +4{\cdot}0000r_3 & +5{\cdot}9000 & = 0. \\
\quad\text{,, (4A)} & 3r_1 & -2{\cdot}2500r_2 & -0{\cdot}7500r_3 & -4{\cdot}4250 & = 0. \\
\quad\text{,, (5A)} & 0 & +4{\cdot}7500r_2 & +3{\cdot}2500r_3 & +1{\cdot}4750 & = 0. \\
\quad\text{,, (6A)} & 0 & -1{\cdot}0000r_2 & -0{\cdot}6842r_3 & -0{\cdot}3105 & = 0. \\
 & & & & & \\
\quad\text{,, (7A)} & +r_1 & +4{\cdot}0000r_2 & +4{\cdot}0000r_3 & +3{\cdot}7000 & = 0. \\
\quad\text{,, (8A)} & -r_1 & -0{\cdot}7500r_2 & -0{\cdot}2500r_3 & -1{\cdot}4750 & = 0. \\
\quad\text{,, (9A)} & 0 & -3{\cdot}2500r_2 & -2{\cdot}2237r_3 & -1{\cdot}0092 & = 0. \\
\quad\text{,, (10A)} & 0 & 0 & +1{\cdot}5263r_3 & +1{\cdot}2158 & = 0. \\
\quad\text{,, (11A)} & 0 & 0 & -1{\cdot}0000r_3 & -0{\cdot}7966 & = 0.
\end{array}
$$

It will be seen that the 3rd and 4th lines lead to the elimination of r_1, and the 6th line to an equation in r_2 and r_3 with minus one as the coefficient of r_2. The 7th, 8th and 9th lines lead to the elimination of r_2, and the 11th to an equation in r_3 with minus one as the coefficient of r_3. The equations represented by the 2nd, 6th and 11th lines are called the " derived normal equations."

So far, we have obtained the value for r_3 but we also want the values for r_2 and r_1. The arrangement of this part of the work is as follows :—

r_3	Check	r_2	Check	r_1	Check
$-0{\cdot}7966$	$-1{\cdot}7966$	$-0{\cdot}3105$	$-1{\cdot}9947$	$-1{\cdot}4750$	$-3{\cdot}4750$
		$+0{\cdot}5450$	$+1{\cdot}2292$	$+0{\cdot}1992$	$+0{\cdot}4492$
		$+0{\cdot}2345$	$-0{\cdot}7655$	$-0{\cdot}1759$	$+0{\cdot}5741$
				$-1{\cdot}4517$	$-2{\cdot}4517$

giving $r_1 = -1''{\cdot}45$; $r_2 = +0''{\cdot}23$; $r_3 = -0''{\cdot}80$ and the corrected values $a = 40^\circ\ 31'\ 27''{\cdot}25$; $b = 34^\circ\ 46'\ 15''{\cdot}63$; $57^\circ\ 31'\ 26''{\cdot}00$.

In this it will be noticed that -1 added to each of the summations in the r_1, r_2 and r_3 columns is equal to the summation in each of the respective check columns.

In this part of the solution, which is often called the " back solution," we merely substitute the value of r_3, found in equation (11A) above, in the derived normal equation (6A) to get r_2, and then the values of r_3 and r_2 in the derived normal equation (2A) to get r_1. Thus, in the third and fourth columns of the back solution, the quantities $-0{\cdot}3105$ and $-1{\cdot}9947$ in the first line are the " l " and " s " terms in line 6 of the first part of the solution. The quantities $+0{\cdot}5450$ and $+1{\cdot}2292$ are the values of

r_3 and its check multiplied by $-0{\cdot}6842$, the coefficient of r_3 in line 6. Similarly, in the last two columns of the back solution, $-1{\cdot}4750$ and $-3{\cdot}4750$ come from the "l" and "s" terms in line 2 of the first part of the solution. $+0{\cdot}1992$ and $+0{\cdot}4492$ are equal to $-0{\cdot}7966$ and $-1{\cdot}7966$, the values of r_3 and its check, each multiplied by $-0{\cdot}25$, the coefficient of r_3 in line 2, and $-0{\cdot}1759$ and $+0{\cdot}5741$ are $+0{\cdot}2345$ and $-0{\cdot}7655$, each multiplied by $-0{\cdot}75$, the coefficient of r_2 in line 2.

Normal equations are most conveniently solved by using a computing machine, but logarithms may also be employed. If logarithms are used it will be noticed that many of them occur more than once. Hence, the logarithmic work should be preserved as it proceeds. The best plan is to work on a definite organised scheme, and such schemes for logarithmic solutions are given in most text-books on least squares. A convenient scheme for use with a machine, as devised by Daubrasse, is described by E. H. Thompson in the *Empire Survey Review*, Vol. III, No. 20, April, 1936.

The final check, both on the formation of the reduced observation equations and on the formation and solution of the normal equations, is to apply the corrections to the assumed approximate values of the unknowns and then substitute the corrected values in the original observation equations. The latter will not be entirely satisfied but will leave small residuals. The check then is to take the squares of these residuals and multiply each square by the weight of the corresponding observation equation. The sum of the quantities so obtained should then be equal to the sum of the quantities formed by multiplying the square of each absolute term in the observation equations by the weight of that equation plus the sum of the products of each correction and the absolute term in the corresponding normal equation. Symbolically, this may be expressed as :—

$$[wv^2] = r_1[wal] + r_2[wbl] + \ldots + r_m[wkl] + [wl^2].$$

where $[wv^2]$ is the sum of the weighted squares of the residuals found by substitution of the corrected quantities in the original observation equations, $r_1, r_2, \ldots r_m$ are the corrections and $[wal]$, $[wbl]$, $\ldots$ $[wkl]$ are the absolute terms in the normal equations.

In the example :—

Sum of weighted squares of the absolute terms of the observation equations $= 1 \times 0 + 1 \times 0 + 2 \times (1{\cdot}1)^2 + 1 \times (3{\cdot}7)^2 + 3 \times 0 \quad = +16{\cdot}110$

Sum of the products of each correction and the absolute terms in the corresponding normal equation $= -1{\cdot}45 \times 5{\cdot}9 + 0{\cdot}23 \times 5{\cdot}9 - 0{\cdot}80 \times 3{\cdot}7 \quad = -10{\cdot}158$

$+5{\cdot}952$

Sum of weighted squares of the residuals $= 1 \times (1{\cdot}45)^2 + 1 \times (0{\cdot}23)^2 + 2 \times (0{\cdot}12)^2 + 1 \times (1{\cdot}68)^2 + 3 \times (0{\cdot}57)^2 \quad = +5{\cdot}941$

The small difference of 0·011 is due to dropped decimal places and hence the work may be assumed to be correct.

Most Probable Values of Conditioned Quantities. When the quantities of a set must fulfil rigorous geometrical relationships, the most probable value of any one quantity is influenced by the results of observation of the others. The problem is to ascribe a most probable set of values to the

unknown quantities and not those which, if no conditions had to be met, would appear from the observation equations to be the most probable for each.

With conditioned quantities, therefore, there are, in addition to the observation equations, one or more equations of condition, which are always fewer in number than the unknowns, and which exhibit the conditions to be fulfilled. Thus, if *a*, *b*, *c*, and *d* are angles closing the horizon at a station, the equation of condition to be satisfied is $(a+b+c+d) = 360°$. If this is written as $d = 360° - (a + b + c)$, it is only necessary to find by the previous methods the most probable values of the independents *a*, *b*, and *c*, and then evaluate the dependent *d*. Generally, if there are *n* unknowns connected by *m* independent conditional equations, the number of independent quantities is $(n - m)$, the remaining unknowns being expressible in terms of these independent quantities and obtainable from them.

This method of dealing with conditioned quantities is suitable for many of the simple cases which may be encountered by the civil engineer, and is illustrated by the following examples.

Example 1. The observations closing the horizon at a station are :—

$a =$	72	30′	42″·2,	weight 1,
$b =$	61	07	20 ·4,	2,
$c =$	75	45	12 ·8,	3,
$d =$	42	23	38 ·0,	2,
$e =$	108	13	08 ·9,	1,
$a + b =$	133	38	04 ·7,	1,
$c + d =$	118	08	48 ·9,	3,
$e + a =$	180	43	53 ·0,	1,

Conditional equation : $a + b + c + d + e = 360°$.

Find the most probable values.

The number of independent quantities is 4. Let *e* be the dependent quantity, and let the most probable values of *a*, *b*, *c* and *d* be :

$$a = 72° 30′ 42″·2 + r_1 ; \quad b = 61° 07′ 20″·4 + r_2 ;$$
$$c = 75° 45′ 12″·8 + r_3 ; \quad d = 42° 23′ 38″·0 + r_4 .$$

Then the correct value of *e* is $360° - (a + b + c + d + r_1 + r_2 + r_3 + r_4) = 108° 13′ 06″·6 - (r_1 + r_2 + r_3 + r_4)$, and the reduced observation equations, written in tabular form, become :

w	r_1	r_2	r_3	r_4	l	s
1	1	0	0	0	0	1
2	0	1	0	0	0	1
3	0	0	1	0	0	1
2	0	0	0	1	0	1
1	1	1	1	1	2·3	− 6·3
1	1	1	0	0	2·1	− 0·1
3	0	0	1	1	1·9	3·9
1	0	1	1	1	4·2	7·2

giving the normal equations

r_1	r_2	r_3	r_4	l	s
3	2	1	1	0·2	7·2
	5	2	2	4·4	15·4
		8	5	12·2	28·2
			7	12·2	27·2

Solution of these equations gives :—

$$r_1 = +0''.89,\ r_2 = -0''.45,\ r_3 = -0''.79,\ r_4 = -1.18.$$

Hence, the corrected values are :—

$$\begin{array}{llll} a = 72^\circ\ 30'\ 43''.09\ ; & b = 61^\circ\ 07'\ 19''.95\ ; \\ c = 75\ \ 45\ \ 12\ .01\ ; & d = 42\ \ 23\ \ 36\ .82\ ; \\ e = 108\ \ 13\ \ 08\ .13. & \end{array}$$

In this example the final check is :—

$$\begin{array}{lr} [wl^2] & = +38.17 \\ r_1[wal] + r_2[wbl] + r_3[wcl] + r_4[wdl] & = -25.836 \\ \hline \text{Sum} & = +12.334 \\ [wv^2] & = +12.385 \end{array}$$

The small difference of 0·051 is due to dropped decimal places, as may be verified by working to three or four, instead of two, decimal places.

Example 2. The angle observations of two triangles ABC and BCD, having a common side, gave the following results of uniform weight,

$$\begin{array}{llrrr} & & ^\circ & ' & '' \\ BAC = a & = & 48 & 01 & 17.0, \\ ABC = b_1 & = & 56 & 34 & 02.4, \\ CBD = b_2 & = & 67 & 12 & 00.8, \\ ABD = b & = & 123 & 46 & 05.6, \\ ACB = c_1 & = & 75 & 24 & 42.1, \\ BCD = c_2 & = & 42 & 50 & 07.0, \\ ACD = c & = & 118 & 14 & 46.2, \\ BDC = d & = & 69 & 57 & 50.1. \end{array}$$

The conditional equations are :—

$$\begin{array}{l} b_1 + b_2 = b, \\ c_1 + c_2 = c, \\ a + b_1 + c_1 = 180^\circ, \\ d + b_2 + c_2 = 180^\circ. \end{array}$$

The number of independent quantities is 4. Let these be b_1, b_2, c_1, and c_2.

Using reduced observation equations with assumed values for the independents as given in the 2nd, 3rd, 5th, and 6th observation equations, and letting r_1, r_2, r_3, and r_4 represent the corrections for b_1, b_2, c_1, and c_2 respectively, we have, on eliminating the dependents, the reduced observation equations,

$$\begin{array}{lllll} r_1 & & + r_3 & & + 1.5 = 0, \\ r_1 & & & & = 0, \\ & r_2 & & & = 0, \\ r_1 & + r_2 & & & - 2.4 = 0, \\ & & r_3 & & = 0, \\ & & & r_4 & = 0, \\ & & r_3 & + r_4 & + 2.9 = 0, \\ & r_2 & & + r_4 & - 2.1 = 0, \end{array}$$

yielding the normal equations :—

$$\begin{array}{lllll} 3r_1 & + r_2 & + r_3 & & - 0.9 = 0, \\ & 3r_2 & & + r_4 & - 4.5 = 0, \\ & & 3r_3 & + r_4 & + 4.4 = 0, \\ & & & 3r_4 & + 0.8 = 0, \end{array}$$

solution of which gives $r_1 = +0''.29$, $r_2 = +1''.49$, $r_3 = -1''.47$, $r_4 = -0''.27$.

On application of these, the most probable values are :—

$$\begin{array}{lrrr} & ^\circ & ' & '' \\ a = & 48 & 01 & 16.68, \\ b_1 = & 56 & 34 & 02.69, \\ b_2 = & 67 & 12 & 02.29, \\ c_1 = & 75 & 24 & 40.63, \\ c_2 = & 42 & 50 & 06.73, \\ d = & 69 & 57 & 50.98. \end{array}$$

Method of Correlatives. When there are several conditions to be fulfilled, the most probable values are usually obtained by the use of undetermined multipliers, called "correlatives."

Taking the common case in which the observations are direct and equal in number to the unknowns, let there be n observed quantities, of which the most probable values are x_1, x_2, etc., and the observed values are o_1, o_2, etc.

The n observation equations are:

$$x_1 = o_1 + r_1, \text{ weight } w_1,$$
$$x_2 = o_2 + r_2, \text{ weight } w_2, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Let the m conditional equations be

$$a_1x_1 + a_2x_2 + \ldots a_nx_n = q_1,$$
$$b_1x_1 + b_2x_2 + \ldots b_nx_n = q_2, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Substituting (1) in (2) we have

$$a_1r_1 + a_2r_2 + \ldots a_nr_n = q_1 - [ao] = -u_1,$$
$$b_1r_1 + b_2r_2 + \ldots b_nr_n = q_2 - [bo] = -u_2, \text{ etc.} \quad . \quad . \quad . \quad . \qquad (3)$$

For the most probable values of the residuals,

$$w_1r_1^2 + w_2r_2^2 + \ldots w_nr_n^2 = \text{a minimum},$$
$$\text{or} \quad w_1r_1dr_1 + w_2r_2dr_2 + \ldots w_nr_ndr_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

But, differentiating (3),

$$a_1dr_1 + a_2dr_2 + \ldots a_ndr_n = 0,$$
$$b_1dr_1 + b_2dr_2 + \ldots b_ndr_n = 0, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Forming the m equations (5) into a single equivalent equation, we have

$$k_1(a_1dr_1 + a_2dr_2 + \ldots a_ndr_n) + k_2(b_1dr_1 + b_2dr_2 + \ldots b_ndr_n) + \ldots$$
$$= 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where $k_1, k_2, \ldots k_m$ are independent constants or correlatives.

Equating (6) and (4), since the values of dr_1, dr_2, etc. are simultaneous, we obtain, on rearranging the terms,

$$(k_1a_1 + k_2b_1 + \ldots k_mm_1 - w_1r_1)\,dr_1 + (k_1a_2 + k_2b_2 + \ldots k_mm_2$$
$$- w_2r_2)\,dr_2 + \ldots = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

in which the coefficients of the differentials must each be zero so that

$$k_1a_1 + k_2b_1 + \ldots k_mm_1 = w_1r_1,$$
$$k_1a_2 + k_2b_2 + \ldots k_mm_2 = w_2r_2, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Substituting these values of r_1, r_2, etc. in (3) we obtain

$$k_1\left[\frac{a^2}{w}\right] + k_2\left[\frac{ab}{w}\right] + \ldots k_m\left[\frac{am}{w}\right] = -u_1,$$

$$k_1\left[\frac{ba}{w}\right] + k_2\left[\frac{b^2}{w}\right] + \ldots k_m\left[\frac{bm}{w}\right] = -u_2, \text{ etc} \quad . \quad . \quad . \quad . \qquad (9)$$

in which

$$\left[\frac{a^2}{w}\right] = \frac{a_1^2}{w_1} + \frac{a_2^2}{w_2} + \ldots \frac{a_n^2}{w_n},$$

$$\left[\frac{ab}{w}\right] = \frac{a_1b_1}{w_1} + \frac{a_2b_2}{w_2} + \ldots \frac{a_nb_n}{w_n}, \text{ etc.}$$

Equations (9) are solved simultaneously for $k_1, k_2, \ldots k_m$; and $r_1, r_2, \ldots r_n$ are then obtained from (8), whence $x_1, x_2, \ldots x_n$ from (1).

As in the case of the normal equations derived from observation equations, we observe that, as $\left[\frac{ba}{w}\right] = \left[\frac{ab}{w}\right]$, $\left[\frac{ca}{w}\right] = \left[\frac{ac}{w}\right]$, $\left[\frac{cb}{w}\right] = \left[\frac{bc}{w}\right]$, etc., equations (9) can be written in the conventional and abbreviated form :—

$$
\begin{aligned}
k_1\left[\frac{a^2}{w}\right] + k_2\left[\frac{ab}{w}\right] + k_3\left[\frac{ac}{w}\right] + \ldots + k_m\left[\frac{am}{w}\right] + u_1 &= 0 \\
k_2\left[\frac{b^2}{w}\right] + k_3\left[\frac{bc}{w}\right] + \ldots + k_m\left[\frac{bm}{w}\right] + u_2 &= 0 \\
k_3\left[\frac{c^2}{w}\right] + \ldots + k_m\left[\frac{cm}{w}\right] + u_3 &= 0 \\
\ldots\ldots\ldots\ldots\ldots \\
k_m\left[\frac{m^2}{w}\right] + u_m &= 0
\end{aligned}
$$

It will also be observed that, in the normal equations formed from observation equations, each absolute term in the observation equations is multiplied by the weight of observation and by the coefficient of one of the unknowns, and the results are added together to give the absolute term in the normal equations. In the above equations, which are called the "correlative normal equations," the absolute terms are not the result of similar summations, but are simply the absolute terms in the conditional equations themselves. Another difference between observational and correlative normal equations is that, when forming the observational normal equations, the coefficients that are to be multiplied together occur in the same equation; in forming the correlative normal equations, the coefficients that are to be multiplied together occur in the different conditional equations. Hence, in the table used for forming the observational normal equations, the coefficients in each observation equation are written in horizontal lines, but, as will be seen later, those of conditional equations are most conveniently tabulated in vertical columns.

Note also that in forming observational normal equations the squares and products of coefficients are multiplied by the weights themselves; in forming conditional normal equations these squares and products are multiplied by the reciprocals of the weights.

Corresponding to the summation checks in the formation of ordinary observational normal equations, there are similar checks in the case of the formation of the correlative normal equations, but in the latter case the absolute terms are not included in the check.

Let $\quad s_1 = a_1 + b_1 + c_1 + d_1 + \ldots + m_1$

$\quad\quad = \text{sum of coefficients of } r_1 \text{ in the conditional equations.}$

$\quad s_2 = a_2 + b_2 + c_2 + d_2 + \ldots + m_2$

$\quad s_3 = a_3 + b_3 + c_3 + d_3 + \ldots + m_3$, etc.

Then

$$\frac{a_1 s_1}{w_1} = \frac{a_1^2}{w_1} + \frac{a_1 b_1}{w_1} + \frac{a_1 c_1}{w_1} + \frac{a_1 d_1}{w_1} + \ldots + \frac{a_1 m_1}{w_1}$$

$$\frac{a_2 s_2}{w_2} = \frac{a_2^2}{w_2} + \frac{a_2 b_2}{w_2} + \frac{a_2 c_2}{w_2} + \frac{a_2 d_2}{w_2} + \ldots + \frac{a_2 m_2}{w_2}, \text{ etc.}$$

and we have

$$\left[\frac{as}{w}\right] = \left[\frac{a^2}{w}\right] + \left[\frac{ab}{w}\right] + \left[\frac{ac}{w}\right] + \left[\frac{ad}{w}\right] + \ldots + \left[\frac{am}{w}\right]$$

$= \text{sum of coefficients in the first correlative normal equation.}$

Similarly,

$$\left[\frac{bs}{w}\right] = \text{sum of coefficients in second correlative normal equation.}$$

In addition, we have

$$\frac{s_1 s_1}{w_1} = \frac{s_1 a_1}{w_1} + \frac{s_1 b_1}{w_1} + \frac{s_1 c_1}{w_1} + \frac{s_1 d_1}{w_1} + \ldots + \frac{s_1 m_1}{w_1}$$

$$\frac{s_2 s_2}{w_2} = \frac{s_2 a_2}{w_2} + \frac{s_2 b_2}{w_2} + \frac{s_2 c_2}{w_2} + \frac{s_2 d_2}{w_2} + \ldots + \frac{s_2 m_2}{w_2}, \text{ etc.}$$

$$\left[\frac{ss}{w}\right] = \left[\frac{as}{w}\right] + \left[\frac{bs}{w}\right] + \left[\frac{cs}{w}\right] + \left[\frac{ds}{w}\right] + \ldots + \left[\frac{ms}{w}\right].$$

The method of forming the correlative normal equations and of checking them as they are formed will be seen from the following example in which all the weights are unity.

Example. Find the most probable values of the angles of *Ex.* 2, page 270, by the method of correlatives.

Let $r_1, r_2, \ldots r_8$ be the corrections to the angles $a, b_1, \ldots d$ in the order given. Here the weights are all unity and the conditional equations (3) are

$$r_4 - r_2 - r_3 + 2''{\cdot}4 = 0,$$
$$r_7 - r_5 - r_6 - 2''{\cdot}9 = 0,$$
$$r_1 + r_2 + r_5 + 1''{\cdot}5 = 0,$$
$$r_8 + r_3 + r_6 - 2''{\cdot}1 = 0,$$

the coefficients in which may be tabulated as follows :—

r	$\frac{1}{w}$	a	b	c	d	s
r_1	1	0	0	+1	0	+1
r_2	1	−1	0	+1	0	0
r_3	1	−1	0	0	+1	0
r_4	1	+1	0	0	0	+1
r_5	1	0	−1	+1	0	0
r_6	1	0	−1	0	+1	0
r_7	1	0	+1	0	0	+1
r_8	1	0	0	0	+1	+1

The coefficients in the correlative normal equations are then found by tabulating :—

r	$\frac{a^2}{w}$	$\frac{ab}{w}$	$\frac{ac}{w}$	$\frac{ad}{w}$	$\frac{as}{w}$	$\frac{b^2}{w}$	$\frac{bc}{w}$	$\frac{bd}{w}$	$\frac{bs}{w}$	$\frac{c^2}{w}$	$\frac{cd}{w}$	$\frac{cs}{w}$	$\frac{d^2}{w}$	$\frac{ds}{w}$	$\frac{s^2}{w}$	r
r_1	0	0	0	0	0	0	0	0	0	+ 1	0	+ 1	0	0	+ 1	r_1
r_2	+ 1	0	− 1	0	0	0	0	0	0	+ 1	0	0	0	0	0	r_2
r_3	+ 1	0	0	− 1	0	0	0	0	0	0	0	0	+ 1	0	0	r_3
r_4	+ 1	0	0	0	+ 1	0	0	0	0	0	0	0	0	0	+ 1	r_4
r_5	0	0	0	0	0	+ 1	− 1	0	0	+ 1	0	0	0	0	0	r_5
r_6	0	0	0	0	0	+ 1	0	− 1	0	0	0	0	+ 1	0	0	r_6
r_7	0	0	0	0	0	+ 1	0	0	+ 1	0	0	0	0	0	+ 1	r_7
r_8	0	0	0	0	0	0	0	0	0	0	0	0	+ 1	+ 1	+ 1	r_8
[ ]	+ 3	0	− 1	− 1	+ 1	+ 3	− 1	− 1	+ 1	+ 3	0	+ 1	+ 3	+ 1	+ 4	[ ]

Here all the checks are fulfilled since we have :—

$$\left[\frac{a^2}{w}\right] + \left[\frac{ab}{w}\right] + \left[\frac{ac}{w}\right] + \left[\frac{ad}{w}\right] = +3 + 0 - 1 - 1 = +1 = \left[\frac{as}{w}\right]$$

$$\left[\frac{ba}{w}\right] + \left[\frac{b^2}{w}\right] + \left[\frac{bc}{w}\right] + \left[\frac{bd}{w}\right] = 0 + 3 - 1 - 1 = +1 = \left[\frac{bs}{w}\right]$$

$$\left[\frac{ca}{w}\right] + \left[\frac{cb}{w}\right] + \left[\frac{c^2}{w}\right] + \left[\frac{cd}{w}\right] = -1 - 1 + 3 + 0 = +1 = \left[\frac{cs}{w}\right]$$

$$\left[\frac{da}{w}\right] + \left[\frac{db}{w}\right] + \left[\frac{dc}{w}\right] + \left[\frac{d^2}{w}\right] = -1 - 1 + 0 + 3 = +1 = \left[\frac{ds}{w}\right]$$

$$\left[\frac{sa}{w}\right] + \left[\frac{sb}{w}\right] + \left[\frac{sc}{w}\right] + \left[\frac{sd}{w}\right] = +1 + 1 + 1 + 1 = +4 = \left[\frac{s^2}{w}\right]$$

The correlative normal equations, when written in conventional form, are therefore :—

$$\begin{array}{rrrrrl}
3k_1 & + 0 & - k_3 & - k_4 & + 2''{\cdot}4 & = 0 \\
 & 3k_2 & - k_3 & - k_4 & - 2{\cdot}9 & = 0 \\
 & & 3k_3 & + 0 & + 1{\cdot}5 & = 0 \\
 & & & 3k_4 & - 2{\cdot}1 & = 0
\end{array}$$

solution of which gives :—

$$\begin{aligned}
k_1 &= -\,0{\cdot}613 \\
k_2 &= +\,1{\cdot}153 \\
k_3 &= -\,0{\cdot}320 \\
k_4 &= +\,0{\cdot}880
\end{aligned}$$

In obtaining $r_1, r_2 \ldots r_8$, the coefficients in equations (8) are those

given in the table on page 273 which was used in forming the correlative normal equations. Then

$$\begin{aligned}
r_1 &= +k_3 &&= -0''\!\cdot\!32\\
r_2 &= -k_1 + k_3 &&= +0\cdot29\\
r_3 &= -k_1 + k_4 &&= +1\cdot49\\
r_4 &= +k_1 &&= -0\cdot61\\
r_5 &= -k_2 + k_3 &&= -1\cdot47\\
r_6 &= -k_2 + k_4 &&= -0\cdot27\\
r_7 &= +k_2 &&= +1\cdot15\\
r_8 &= +k_4 &&= +0\cdot88
\end{aligned}$$

On applying these corrections to the observed values, we have the most probable values.

$a = 48° \; 01' \; 17''\!\cdot\!0 - 0''\!\cdot\!32 = 48° \; 01' \; 16''\!\cdot\!68$, etc., as before.

When these values are substituted in the conditional equations, the latter should be satisfied to within one or two units in the last decimal place. This is the final check.*

Solution of Correlative Normal Equations. The solution of the correlative normal equations is carried out in exactly the same manner as in the case of normal equations derived from observation equations. The coefficients of the correlatives and the absolute term in each normal equation are added together to give a sum term and this sum term is used as a check in exactly the same manner as before. The solution of the four correlative normal equations found in the last example is given on page 276.

Precision of the Most Probable Value. When the most probable value of a quantity has been obtained, it is desirable to have some index of the precision of the result. Thereby the relative accuracy of different series of observations may be ascertained, and, for purposes of adjustment, weights based on the value of the index may be ascribed to each. The number of individual measures of the quantity and their consistency afford data for estimating the probable precision of the result. Three criteria are employed, *viz.* probable error, mean square error, and mean error.

The *Probable Error* (p.e.) is of such magnitude that the probability of the true error being larger is equal to the probability of the true error being smaller than the probable error. In other words, in a large series of observations the probability is that there are as many errors numerically greater than the probable error as there are smaller.

The *Mean Square Error* (m.s.e.) equals the square root of the arithmetic mean of the squares of the individual true errors.

The *Mean Error* (m.e.) is the arithmetic mean of the individual true errors without regard to sign.

As the determination of the values of these errors is based on the theory of probabilities, it must again be emphasised that the employment of such standards presupposes the entire elimination of all but accidental errors.

When the number of observations is large, these errors have the relationship

$$\text{p.e.} = 0\cdot6745 \text{ m.s.e.} = \cdot8453 \text{ m.e.},$$

so that it is comparatively unimportant which is adopted.

* Another example of an adjustment by conditional equations, but with weights other than unity, is given on pages 423–425.

SOLUTION OF NORMAL EQUATIONS FOUND ON PAGE 274

Part I

Line	k_1	k_2	k_3	k_4	u	s	Explanation.
1	+ 3	0	− 1·0000	− 1·0000	+ 2·4000	+ 3·4000	First normal equation.
2	− 1	0	+ 0·3333	+ 0·3333	− 0·8000	− 1·1333	Line 1 ÷ + 3, the coefficient of k_1 in line 1, with signs reversed.
3		+ 3	− 1·0000	− 1·0000	− 2·9000	− 1·9000	Second normal equation.
4		0	0·0000	0·0000	0·0000	0·0000	Line 2 × 0, the coefficient of k_2 in line 1.
5		+ 3	− 1·0000	− 1·0000	− 2·9000	− 1·9000	Sum of lines 3 and 4.
6		− 1	+ 0·3333	+ 0·3333	+ 0·9667	+ 0·6333	Line 5 ÷ + 3, the coefficient of k_2 in line 5 with signs reversed.
7			+ 3·0000	0·0000	+ 1·5000	+ 2·5000	Third normal equation.
8			− 0·3333	− 0·3333	+ 0·8000	+ 1·1333	Line 2 × − 1, the coefficient of k_3 in line 1.
9			− 0·3333	− 0·3333	− 0·9667	− 0·6333	Line 6 × − 1, the coefficient of k_3 in line 5.
10			+ 2·3333	− 0·6667	+ 1·3333	+ 3·0000	Sum of lines 7, 8 and 9.
11			− 1	+ 0·2857	− 0·5714	− 1·2857	Line 10 ÷ + 2·3333, the coefficient of k_3 in line 10, with signs reversed.
12				+ 3·0000	− 2·1000	− 1·1000	Fourth normal equation.
13				− 0·3333	+ 0·8000	+ 1·1333	Line 2 × − 1, the coefficient of k_4 in line 1.
14				− 0·3333	− 0·9667	− 0·6333	Line 6 × − 1, the coefficient of k_4 in line 5.
15				− 0·1904	+ 0·3809	+ 0·8571	Line 11 × − 0·6667, the coefficient of k_4 in line 10.
16				+ 2·1430	− 1·8858	+ 0·2571	Sums of lines 12, 13, 14 and 15.
17				− 1	− 0·8800	− 0·1200	Line 16 ÷ + 2·1430, the coefficient of k_4 in line 16, with signs reversed.

Part II (Back Solution)

k_4	Check	k_3	Check	k_2	Check.	k_1	Check.	
+ 0·8800	− 0·1200	− 0·5714	− 1·2857	+ 0·9667	+ 0·6333	− 0·8000	− 1·1333	k_4 = + 0·8800 is given by equation − k_4 + 0·8800 = 0 in line 17.
		− 0·2514	− 0·0343	+ 0·2933	− 0·0400	+ 0·2933	− 0·0400	k_3 = − 0·3200 is obtained by inserting value of k_4 in equation − k_3 + 0·2857 k_4 − 0·5714 = 0 from line 11.
		− 0·3200	− 1·3200	− 0·1067	− 0·4400	− 0·1067	− 0·4400	k_2 = + 1·1533 is obtained by inserting values of k_3 and k_4 in equation − k_2 + 0·3333 k_3 + 0·3333 k_4 + 0·9667 = 0 from line 6.
				+ 1·1533	+ 0·1533	0·0000	0·0000	k_1 = − 0·6134 is obtained by inserting values of k_2, k_3 and k_4 in equation − k_1 + 0 + 0·3333 k_3 − 0·3333 k_4 − 0·8000 [illegible]
						− 0·6134	− 1·6133	

Probable Error. Probable error is the criterion generally employed in British and American practice, and the formulæ for its evaluation in the simplest cases are quoted below. As the use of the word "probable" is confusing, the reader is cautioned at the outset against attaching any significance to the term other than that contained in the definition. The probable error is not the most probable error, as this is shown by the probability curve to be always zero. Many authorities prefer to use the mean square error as a measure of precision.

In giving the result of a measurement from which constant errors are eliminated, the value of the probable error is written with a positive and negative sign after the most probable value. Thus, the standardised length of a tape obtained from repeated comparisons might be stated as $100{\cdot}00042 \pm 0{\cdot}00005$ ft., indicating that the true length is as likely to lie within the limits 100·00037 and 100·00047 ft. as to have any value outside them.

Formulæ for Probable Error.

Let n = the number of measures contributing to the most probable value,
$[r^2]$ = the sum of the squares of the residual errors,
w = the weight of a measure,
$[w]$ = the sum of the weights of the measures.
$[wr^2]$ = the sum of the weighted squares in the residuals.

For Direct Observations of Equal Weight on a Single Quantity.

$$\text{p.e. of a single measure} = 0{\cdot}6745\sqrt{\{[r^2]/(n-1)\}}.$$
$$\text{p.e. of the arithmetic mean} = 0{\cdot}6745\sqrt{\{[r^2]/n(n-1)\}}.$$

For Direct Observations of Unequal Weight on a Single Quantity.

$$\text{p.e. of a single measure of unit weight} = 0{\cdot}6745\sqrt{\{[wr^2]/(n-1)\}}$$
$$\text{p.e. of the weighted mean} = 0{\cdot}6745\sqrt{\{[wr^2]/(n-1)[w]\}}.$$

These formulæ are strictly applicable only when n is indefinitely great, but are commonly used for cases when n is small. In such cases the result does not truly represent the probable error, but nevertheless serves to indicate the relative precision of similar sets of observations.

Example 1. Eight measures of an angle are as tabulated. Compute the p.e. of the arithmetic mean, the observations being of uniform weight.

Measure.			r	r^2
°	′	″		
67	34	14·2	− 1·64	2·69
		18·9	+ 3·06	9·36
		13·2	− 2·64	6·97
		17·8	+ 1·96	3·84
		16·9	+ 1·06	1·12
		15·5	− 0·34	0·12
		14·0	− 1·84	3·39
		16·2	+ 0·36	0·13
		8) 126·7	− 0·02	$[r^2]$ = 27·62
		15·84		

p.e. of arithmetic mean $= 0{\cdot}674\sqrt{\{27{\cdot}62/(8 \times 7)\}}$
$= \pm\ 0''{\cdot}47.$

p.e. of a single measure $= 0{\cdot}674\sqrt{\{27{\cdot}62/7\}}.$
$= \pm\ 1''{\cdot}34.$

Example 2. If the last four measures above are given a weight three times that of the first four, compute the p.e. of the weighted mean.

Measure			w	wm	r*	r^2	wr^2
°	′	″		″			
67	34	14·2	1	14·2	1·54	2·37	2·37
		18·9	1	18·9	3·16	9·99	9·99
		13·2	1	13·2	2·54	6·45	6·45
		17·8	1	17·8	2·06	4·24	4·24
		16·9	3	50·7	1·16	1·35	4·05
		15·5	3	46·5	0·24	0·06	0·18
		14·0	3	42·0	1·74	3·03	9·09
		16·2	3	48·6	0·46	0·21	0·63
		$[w] =$	16	) 251·9		$[wr^2] =$	37·00
				15·74			

p.e. of weighted mean $= 0{\cdot}674\sqrt{\{37{\cdot}00/(7 \times 16)\}}$
$= \pm\ 0{\cdot}39''.$

For Computed Quantities.

Let R = the result computed from the most probable values, m, n, etc., of one or more independent quantities $= f(m, n, \text{etc.})$,

e_1, e_2, etc. = the probable errors of m, n, etc.

$$\text{then p.e. of } R = \sqrt{\left(e_1\frac{dR}{dm}\right)^2 + \left(e_2\frac{dR}{dn}\right)^2 + \ldots}$$

In the common case where $f(m, n, \text{etc.})$ is linear, it follows that,

if R = the sum or difference of a constant and an observed quantity m, having a p.e. of e_1,

p.e. of $R = e_1$.

If R = the sum or difference of a number of observed quantities, m, n, etc.,

p.e. of $R = \sqrt{e_1^2 + e_2^2 + \ldots}$

This result has been used in the previous chapter in connection with the propagation of error in base line measurement and precise traversing and also in Chapters II and IV of Vol. I in connection with the propagation of error in linear measurements and in traversing.

Example 3. If the most probable value of an angle is 48° 12′ 32″·24 ± 0″·28, that of its explement is 311° 47′ 27″·76 ± 0″·28.

* Since the most probable value to be obtained from a number of independent measures of a single directly observed quantity is the weighted mean of all the observed values, a "residual error" is the difference between an individual measure and the weighted mean of all the measures.

Example 4. The most probable value of an angle AOC is 84° 40′ 21″·20 ± 0″·90, and that of its part AOB is 36° 14′ 08″·52 ± 1″·30. Find the probable error of the value of BOC as obtained by subtraction.

$$\text{p.e. of BOC} = \sqrt{(\cdot 90)^2 + (1\cdot 30)^2} = \pm\ 1''\cdot 58.$$

Example 5. Find the probable error of the area of a rectangle the sides of which are in feet, 210·55 ± 0·01 and 372·40 ± 0·02.

$$R = mn.$$

$$\frac{dR}{dm} = n, \text{ and } \frac{dR}{dn} = m.$$

$$\text{p.e. of } R = \sqrt{\left(e_1\frac{dR}{dm}\right)^2 + \left(e_2\frac{dR}{dn}\right)^2} = \sqrt{(e_1\,n)^2 + (e_2\,m)^2}.$$

$$= \sqrt{(\cdot 01 \times 372\cdot 40)^2 + (\cdot 02 \times 210\cdot 55)^2} = \pm\ 5\cdot 62 \text{ sq. ft.}$$

Application of Probable Error to Weighting. If the results of two or more series of observations of a quantity are available, and the probable error of each is known, the most probable value may be obtained from the individual results by giving to each a weight based on its probable error. It may be shown that weight is inversely proportional to the square of the probable error, and, in the absence of other data, the individual results are so weighted.

Example. Telegraphic longitude differences between two stations gave the following results:

h.	m.	s.		s.
0	4	10·48	±	0·60,
		9·50	±	0·30,
		8·56	±	0·75,
		11·06	±	0·50,
		9·12	±	0·42,
		10·06	±	0·72

Calculate the most probable value and its probable error.

The weights are proportional to $\frac{1}{\cdot 60^2}$, $\frac{1}{\cdot 30^2}$, $\frac{1}{\cdot 75^2}$, $\frac{1}{\cdot 50^2}$, $\frac{1}{\cdot 42^2}$, $\frac{1}{\cdot 72^2}$,

or to 28, 111, 18, 40, 57, 19,

whence

m	w	wm	r	r^2	wr^2
10·48	28	293·4	0·75	0·563	15·76
9·50	111	1,054·5	0·23	0·053	5·88
8·56	18	154·1	1·17	1·369	24·64
11·06	40	142·4	1·33	1·769	70·76
9·12	57	519·8	0·61	0·372	21·20
10·06	19	191·1	0·33	0·109	2·07
$[w]$ 273		) 2,655·3			$[wv^2]$ = 140·31
		9·73			

$$\text{p.e. of weighted mean} = \cdot 674\sqrt{\frac{140\cdot 31}{273 \times 5}} = \pm\ 0\cdot 22 \text{ s.,}$$

and difference of longitude = 0h. 4 m. 9·73 s. ± 0·22 s.

Probable Error of Indirectly Observed Independent Quantities. When an indirectly observed quantity is derived from observation equations let q be the number of observations, n the number of independent unknowns and let $[wv^2] = w_1v_1^2 + w_2v_2^2 + w_3v_3^2 + \ldots$ where v_1, v_2, v_3, etc., are

the residuals found by substituting the corrections in each original (unweighted) observation equation and w_1, w_2, w_3, etc., are, as before, the weights of the different observations. Then the average probable error of an unadjusted observation of weight unity is given by :—

$$\text{p.e.} = \pm\, 0{\cdot}6745\sqrt{\frac{[wv^2]}{q-n}}$$

and the average probable error of an adjusted observation of unit weight is given by :—

$$\text{p.e.} = \pm\, 0{\cdot}6745\sqrt{\frac{n[wv^2]}{(q-n)[w]}}$$

Probable Error of Conditioned Quantities. When the quantities to be determined are related by, and the corrections are obtained from, equations of condition, the residuals are the corrections themselves. Let n_c be the number of condition equations and q the number of observed quantities. The probable error of an unadjusted observation of unit weight is then given approximately by :—

$$\text{p.e.} = \pm\, 0{\cdot}6745\sqrt{\frac{[wv^2]}{n}},$$

and the probable error of an adjusted observation of unit weight is given approximately by :—

$$\text{p.e.} = \pm\, 0{\cdot}6745\sqrt{\frac{(q-n_c)[wv^2]}{n_c[w]}},$$

in which, as before, $[wv^2] = w_1v_1^2 + w_2v_2^2 + \ldots$ and $[w] = w_1 + w_2 + w_3 + \ldots$

In all the above cases, the probable error, r_w, of an observation of weight w is obtained from

$$r_w = r_1/\sqrt{w},$$

where r_1 is the probable error of an observation of unit weight.

Rejection of Doubtful Observations. In a series of observations it not infrequently happens that one or more measures differ considerably more than the others from the mean. This may be caused by a mistake or by some external influence which does not affect the other observations, and, if so, the discrepant measures should not be used in computing the final result. Since, however, each observation is subject to an indefinitely large number of very small accidental errors, it is in accordance with the laws of probability that these should occasionally combine to form a large accidental error, and this circumstance would not justify the rejection of the measure affected. It is therefore a matter of considerable difficulty to decide whether an observation at variance with the others should be discarded or retained.

In the field an observer should not cancel an observation merely because it differs rather widely from the others, unless the discrepancy is so great that a mistake is obvious. When he is of opinion that an observation will not prove as trustworthy as usual, he should record the full circumstances affecting its accuracy. Such notes form a valuable guide to the computer, who may either use his own judgment as to

whether any observations should be rejected or be guided by a rule based on mathematical principles Many such criteria have been proposed That recommended by Wright and Hayford * is as follows

> Reject each observation for which the residual exceeds five times the probable error of a single observation as derived from all the measures Examine each observation for which the residual exceeds 3½ times the probable error of a single observation, and reject it if any of the conditions under which the observation was made were such as to produce any lack of confidence

Peirce's criterion has been largely used for the same purpose The rule and the table of constants required in its application will be found in The Royal Geographical Society's *Hints to Travellers*, Chauvenet's *Astronomy*, Wilson's *Topographic Surveying*, etc

ADJUSTMENT OF TRIANGULATION

Theoretically, all the angles of a triangulation system should be treated together by least squares to yield their simultaneous most probable values The labour of such a solution is so very great that even in primary work it is usual to divide the system into sections, which are separately adjusted More generally, it is sufficient to adjust the angles of each triangle or simple system of triangles under the heads, (1) station adjustment, (2) figure adjustment

Station Adjustment is directed to finding the most probable values of the angles at a station without reference to the results of observations at other stations The only geometrical conditions which may have to be fulfilled occur when the horizon is closed, so that the angles must sum to 360°, and when angles are measured in combination, so that certain observed results should equal the sums of others

Figure Adjustment involves the adjustment of the angles of each triangle or simple system of triangles so that the figures may be geometrically consistent The conditions to be satisfied are (1) that the angles of each triangle or polygon should sum correctly, and (2) that the length of any side as computed from any other should in a system of interlaced triangles, be independent of the route chosen

Station Adjustment. Any case may be solved by application of the method of least squares and the numerical examples given on pages 259 and 269 are examples of station adjustments For much ordinary work however it is unnecessary to have recourse to a least square adjustment When the horizon is closed by measuring each angle independently, the error of closure is distributed to the angles by applying equal corrections, if the weights of the observations are equal, and in amounts inversely proportional to the weights if these are unequal In the case where two or more angles and their sum form the station observations, a discrepancy between the observed value of the total angle and the sum of the observed values of the parts is distributed to all the measurements in amounts inversely proportional to their respective weights, and with the opposite sign for the correction of the total angle to that of the parts

* *Adjustment of Observations* 2nd edition p 90

Example 1. Adjust the following angles closing the horizon at a station.

	°	′	″	
$a =$	124	05	58·6, weight	2,
$b =$	88	43	16·3,	1,
$c =$	70	52	31·2,	3,
$d =$	76	18	16·7,	1.
	360	00	02·8	

The excess of 2″·8 falls to be distributed in the ratio $\frac{1}{2} : 1 : \frac{1}{3} : 1$, or $3 : 6 : 2 : 6$, so that the negative corrections are :

$$\frac{3}{17} \times 2''{\cdot}8 = 0''{\cdot}5,\ \frac{6}{17} \times 2''{\cdot}8 = 1''{\cdot}0,\ \frac{2}{17} \times 2''{\cdot}8 = 0''{\cdot}3, \text{ and } \frac{6}{17} \times 2''{\cdot}8 = 1''{\cdot}0,$$

giving the adjusted values,

$a =$	124	05	58·1,
$b =$	88	43	15·3,
$c =$	70	52	30·9,
$d =$	76	18	15·7,
	360	00	00·0.

Example 2. Adjust the angles a and b observations of which give

	°	′	″	
$a =$	54	28	17·4, weight	1,
$b =$	63	51	41·3,	2,
$a + b =$	118	19	55·1,	2.

The discrepancy of 3″·6 must be distributed in the ratio $1 : \frac{1}{2} : \frac{1}{2}$, or $2 : 1 : 1$, giving corrections of $-1''{\cdot}8$, $-0''{\cdot}9$, and $+0''{\cdot}9$, and yielding the adjusted values,

	°	′	″
$a =$	54	28	15·6,
$b =$	63	51	40·4,
$a + b =$	118	19	56·0.

Figure Adjustment—Spherical Excess. In geodetic triangulation the triangles or polygons may, for all practical purposes, be taken as spherical, and the sum of the angles exceeds that of the corresponding plane figure by an amount termed the spherical excess. On a true sphere spherical excess is given by :—

$$\epsilon = \frac{A}{r^2 \sin 1''},$$

where ϵ = spherical excess in seconds of arc,
A = area of figure,
r = radius of sphere.

For the terrestrial spheroid the excess for a given area varies with the latitude, decreasing from the equator to the poles, and for large figures the *spheroidal* excess is given more precisely by

$$= \frac{A(1 - e^2 \sin^2 \phi)^2}{a^2(1 - e^2) \sin 1''}$$

$$= \frac{A}{RN \sin 1''},$$

where ϕ = mean of the latitudes of the bounding stations.
a = the earth's equatorial semi-axis,
e = the earth's eccentricity (page 317),

R = the radius of curvature of a meridian section, at latitude ϕ.
N = the length of the normal, or the radius of curvature of the arc cut out on the surface by a normal section, perpendicular to the meridian, at latitude ϕ (pages 317-318).

If b and c are the two sides of a triangle, and A is the included angle, $\Delta = \frac{1}{2}bc \, . \sin A$, and

$$\epsilon = \frac{1}{2} \frac{bc \sin A.}{RN \sin 1''}$$

Most geodetic tables give values for the logarithms of $\frac{1}{2RN}$ or for $\frac{1}{2RN \sin 1''}$, for different latitudes, and hence the value for $\frac{1}{2RN \sin 1''}$ can easily be obtained.* The value of the excess for triangles of moderate size is very small so that for RN we can use the square of the mean radius of the earth (3,963 miles). For many purposes, spherical excess may be taken with sufficient accuracy as 1″ for every 76 square miles of area. For primary work, of course, and for large triangles, the spheroidal formula should be used.

Adjustment of Single Triangle. The only requirement to be satisfied is the summation of the angles to $(180° + \epsilon)$. To obtain the angles required in computing the sides (page 312), the observed value of each angle is first reduced by $\frac{1}{3}\epsilon$. The difference between the sum of the resulting angles and 180° is then distributed as equal corrections if the observations are of uniform weight, or in amounts inversely proportional to the weights if these are unequal. The values required for the computation of azimuths are not reduced for spherical excess, and are similarly adjusted to sum to $(180° + \epsilon)$.

Adjustment of Quadrilateral. If observations have been taken along one diagonal only, the figure consists of two triangles having a common side, and, after reducing, if necessary, for spherical excess, the conditions to be met and methods of solution have been shown in *Ex.* 2, page 270 and the example on page 273.

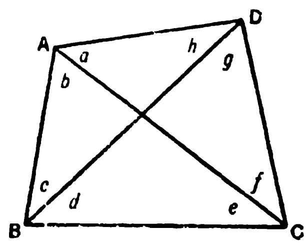


FIG. 111.

In the *geodetic* quadrilateral, observations are made along both diagonals and the angles lettered in Fig. 111 are measured. The observed values are first reduced for spherical excess as required in computing the triangle sides (page 312). In the most refined work the excess is computed for each triangle formed by the intersection of the diagonals, and one-third is deducted from each of the two appropriate angles: otherwise, all eight angles are reduced by one-eighth the excess for the whole figure.

The angle requirements are that: (1) the sum of the observed angles

* A table giving the logarithm of the factor $\frac{1}{2RN \sin 1''}$, is given as Table II in Close and Winterbotham's *Text book of Topographical Surveying*.

should be 360° ; (2) the sum of the angles in any of the triangles formed should be 180° ; (3) the opposite angles at the intersection of the diagonals should be equal. These conditions are not independent, and are fulfilled if three independent angle equations are satisfied. Several sets of three equations are available.

That used below is

$$a + b + c + d + e + f + g + h = 360°,$$
$$a + h - d - e = 0,$$
$$b + c - f - g = 0.$$

The requirement that the evaluation of one side from another should yield the same result through whatever triangles the calculation is carried gives rise to one side equation of condition. If, for example, CD is calculated from AB through triangles ABD and ADC, the result is given by

$$CD = \frac{\sin a \sin c}{\sin f \sin h} AB,$$

and if through triangles ABC and BCD,

$$CD = \frac{\sin b \sin d}{\sin e \sin g} AB.$$

For consistency, therefore,

$$\frac{\sin a \sin c \sin e \sin g}{\sin b \sin d \sin f \sin h} = 1,$$

or, more conveniently,

$$(\log \sin a + \log \sin c + \log \sin e + \log \sin g) - (\log \sin b + \log \sin d + \log \sin f + \log \sin h) = 0,$$

which is the side equation.

It can easily be seen from consideration of Fig. 112 that the angles may close satisfactorily but the sides may not. Here :—

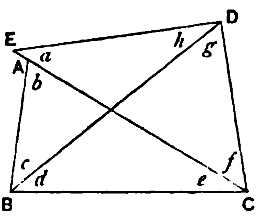


Fig. 112.

$$(a + h + g + f) = 180°,$$
$$(b + c + d + e) = 180°,$$

so that—

$$a + b + c + d + e + f + g + h = 360°,$$

and we also have :—

$$a + h - d - e = 0$$
$$b + c - f - g = 0$$

Thus, all the angle equations given above are satisfied, but the points E and A do not coincide. Other angles may be chosen, but they will still satisfy the angle conditions. Even the angles a, b, c and h in the figure BAEDA, which is not a triangle, add up to 180°, because, in the triangle BDC, $d + e + f + g = 180°$. Also, for the angles opposite the intersection of the diagonals, $a + h = d + e$ and $b + c = f + g$. Hence, $a + b + c + h = 180°$.

Adjustment of Geodetic Quadrilateral. This is best performed by means of correlatives as illustrated below.

Example. Adjust the following observed angles lettered as in Fig. 111. The observations are of equal weight, and are already corrected for spherical excess.

	°	′	″		°	′	″
$a =$	30	27	07·2	$e =$	44	01	23·2
$b =$	37	10	32·6	$f =$	30	56	45·3
$c =$	48	26	09·0	$g =$	54	39	48·8
$d =$	50	21	54·6	$h =$	63	56	14·5

Denoting the corrections for $a, b, \ldots h$ by $r_1, r_2, \ldots r_8$, the angle conditional equations are:

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 = + 4{\cdot}8,$$
$$r_1 + r_2 - r_4 - r_5 = - 3{\cdot}9,$$
$$r_2 + r_3 - r_6 - r_7 = - 7{\cdot}5.$$

The logarithmic form of the side equation in terms of the corrections becomes:

$$d_1r_1 - d_2r_2 + \ldots - d_8r_8 + \log\sin o_1 - \log\sin o_2 + \ldots - \log\sin o_8 = 0,$$

giving

$$d_1r_1 - d_2r_2 + \ldots - d_8r_8 = u_4,$$

where o_1, o_2, etc., are the observed values of the angles as given above, d_1, d_2, etc. are the tabular differences for 1″ for the log sines of o_1, o_2, etc., and u_4 is, as before, the amount by which the observed values fail to satisfy the side equation.*

In this case, taking the unit as the 6th place of logarithms,

Angle	+ log sin.	d.	Angle.	− log sin.	d
a	9·7048506	3·58	b	9·7812249	2·78
c	9·8740253	1·87	d	9·8865616	1·74
e	9·8419526	2·18	f	9·7111563	3·51
g	9·9115677	1·49	h	9·9534283	1·03
	9·3323962			9·3323711	
	711				

$u_4 = -25{\cdot}1$, and the side equation is

$$3{\cdot}58r_1 - 2{\cdot}78r_2 + 1{\cdot}87r_3 - 1{\cdot}74r_4 + 2{\cdot}18r_5 - 3{\cdot}51r_6 + 1{\cdot}49r_7 - 1{\cdot}03r_8 = -25{\cdot}1$$

Write down the coefficients of the conditional equations in vertical columns and form the sums of the horizontal lines, thus:

r	a	b	c	d	s
r_1	+ 1	+ 1		+ 3·58	+ 5·58
r_2	+ 1		+ 1	− 2·78	− 0·78
r_3	+ 1		+ 1	+ 1·87	+ 3·87
r_4	+ 1	− 1		− 1·74	− 1·74
r_5	+ 1	− 1		+ 2·18	+ 2·18
r_6	+ 1		− 1	− 3·51	− 3·51
r_7	+ 1		− 1	+ 1·49	+ 1·49
r_8	+ 1	+ 1		− 1·03	− 0·97

The coefficients of the correlative normal equations are then formed in the usual tabular form as on page 286.

* This equation follows by taking the side equation found on page 254 and writing $\log\sin a = \log\sin o_1 + r_1d_1$, $\log\sin b = \log\sin o_2 + r_2d_2$, etc. The d's are easily written down from the values of the common differences given in most tables of logarithmic sines. Thus, in Shortrede's *Logarithmic Tables* the common differences are given at the bottom of the minute columns in units of the 7th place.

TABLE FOR FORMATION OF NORMAL EQUATIONS FOR EXAMPLE ON PAGE 285.

r	a²	ab	ac	ad	as	b²	bc	bd	bs	c²	cd	cs	d²	ds	s²
r_1	+ 1	+ 1		+ 3·58	+ 5·58	+ 1		− 3·58	+ 5·58				+ 12·816	+ 19·976	+ 31·136
r_2	+ 1		+ 1	− 2·78	− 0·78					+ 1	− 2·78	− 0·78	+ 7·728	+ 2·168	+ 0·608
r_3	+ 1		+ 1	− 1·87	+ 3·87					+ 1	+ 1·87	+ 3·87	+ 3·497	+ 7·237	+ 14·977
r_4	+ 1	− 1		− 1·74	− 1·71	+ 1		+ 1·74	+ 1·74				− 3·028	− 3·028	+ 3·028
r_5	+ 1	− 1		+ 2·18	+ 2·18	+ 1		− 2·18	− 2·18				+ 4·752	+ 4·752	+ 4·752
r_6	+ 1		− 1	− 3·51	− 3·51					+ 1	+ 3·51	+ 3·51	+ 12·320	+ 12.320	+ 12·320
r_7	+ 1		− 1	− 1·49	− 1·49					+ 1	− 1·49	− 1·49	+ 2·220	+ 2·220	+ 2·220
r_8	+ 1	+ 1		− 1·03	+ 0·97	+ 1		− 1·03	+ 0·97				+ 1·061	− 0·999	+ 0·941
[ ]	+ 8	0	0	+ 0·06	+ 8·06	+ 4	0	+ 2·11	+ 6·11	+ 4	+ 1·11	+ 5·11	+ 47·42	+ 50·702	+ 69·981

Hence the normal equations are:—

$$\begin{aligned} 8k_1 \qquad\qquad & + 0{\cdot}06k_4 - 4{\cdot}8 = 0 \\ 4k_2 \qquad & + 2{\cdot}11k_4 + 3{\cdot}9 = 0 \\ 4k_3 & + 1{\cdot}11k_4 + 7{\cdot}5 = 0 \\ & 47{\cdot}42k_4 + 25{\cdot}1 = 0 \end{aligned}$$

whence,

$$\begin{aligned} k_1 &= + 0{\cdot}6034, \\ k_2 &= - 0{\cdot}7342, \\ k_3 &= - 1{\cdot}7483, \\ k_4 &= - 0{\cdot}4565. \end{aligned}$$

so that

$$\begin{aligned} r_1 &= k_1 + k_2 + 3{\cdot}58k_4 = - 1{\cdot}76, \\ r_2 &= k_1 + k_3 - 2{\cdot}78k_4 = + 0{\cdot}12, \\ r_3 &= k_1 + k_3 + 1{\cdot}87k_4 = - 2{\cdot}00, \\ r_4 &= k_1 - k_2 - 1{\cdot}74k_4 = + 2{\cdot}13, \\ r_5 &= k_1 - k_2 + 2{\cdot}18k_4 = + 0{\cdot}34, \\ r_6 &= k_1 - k_3 - 3{\cdot}51k_4 = + 3{\cdot}95, \\ r_7 &= k_1 - k_2 + 1{\cdot}49k_4 = + 1{\cdot}67, \\ r_8 &= k_1 + k_2 - 1{\cdot}03k_4 = + 0{\cdot}34. \end{aligned}$$

Application of these to the observed values gives the adjusted values:—

	°	′	″		°	′	″
$a =$	30	27	05·44	$e =$	44	01	23·54
$b =$	37	10	32·72	$f =$	30	56	49·25
$c =$	48	26	07·00	$g =$	54	39	50·47
$d =$	50	21	56·73	$h =$	63	56	14·84

Adjustment of more Complicated Figures. When more complicated figures are involved, it is very important that the number of equations chosen to fulfil the geometrical conditions should be no more and no less than those that are essential for the purpose. If too few conditions are taken, the essential conditions will naturally not be fulfilled and this will not be found out until the final solution of the triangles. If too many conditions are inserted, and the number of conditional equations is too large, the result will be an unnecessary waste of labour. This will show itself during the solution of the normal equations by the coefficients above and to the right of one diagonal term becoming equal to zero.

In choosing the conditions, and judging the number necessary, proceed as follows. Draw the figure triangle by triangle, using a single new triangle to fix a new point, and continue until all points are fixed. Each triangle so drawn gives one angle equation, or equation to satisfy the closure of the triangle, or, if all triangles are not fully observed, the number of angular equations will be equal to the number of triangles that are fully observed. Next draw all other rays sighted in one or in both directions. These lines may be drawn in dotted, but lines sighted in a single direction should be differentiated from those sighted in both directions. The extra equations introduced by these lines will be (1) a side equation for each line and (2) an angle equation for every line observed in both directions. To these add a condition equation for every internal station where the sum of all the observed angles must equal 360°, or, if the figure is only an addition to an old figure, for each set of angles which must have a fixed sum.

If one or more stations is unoccupied, that is, if the point is fixed by intersection from other points and no observations at all have been taken from it, the condition equations will contain no corrections for the

unobserved angles. Approximate values for each of these angles will be obtained from the two observed angles in each triangle which has the point as one apex, and the correction to the unobserved angle will be minus the sum of the corrections to the other two angles in the triangle. Hence, there will be no condition equation for closure of all triangles which have the unobserved point as one apex, but at a centre point there will still be an equation expressing the condition that the sum of the angles at the unoccupied point must equal 360°.

Thus, in Fig. 113 the points C, D, E and F are fixed in turn from the line AB by the triangles ABC, BCD, DCE and ECF, and, provided the triangles are fully observed, *i.e.*, that all three angles in each are measured, each triangle introduces an angle equation. The dotted lines AD, BF and FA are extra lines. Each introduces a side equation, and, for each line observed in both directions, an extra angle equation—say for the closures of the triangles ABD, ABF and ACF. In addition, the angles at C must add up to 360°, and this introduces an additional angle equation. Hence, there are 3 side equations and 4 + 3 + 1 = 8 angle equations, or 11 conditional equations in all.

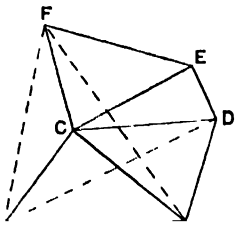


Fig. 113.

If the side FA had only been sighted in one direction, the angle CFA not having been observed, there would be one less angle equation.

The number of condition equations required may also be checked from the following rule :—

The number of conditions is equal to the number of observed quantities for which corrections are required, minus twice the number of new points to be fixed, plus the number of unoccupied stations.

Thus, in the above example, the number of observed angles for which corrections are needed is 19, and there are 4 new points to be fixed. Hence, the number of conditional equations is 19 − 8 = 11. If the angle CFA was not observed, the number of conditional equations is 18−8=10.

As another example, take the case of a secondary point E fixed from an existing and finally adjusted quadrilateral ABCD, Fig. 114. The observed angles are numbered 1 to 12. The number of corrections is 12 and the number of new points is 1. The number of conditional equations is therefore 12 − 2 = 10, and these arise as follows :—

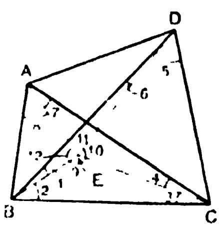


Fig. 114.

	Angle	Side
Triangle BEC to fix E	1	—
Lines AE and ED	2	2
Sums of angles 1 and 2 ; 3 and 4 ; 5 and 6 ; 7 and 8 ; 9, 10, 11 and 12	5	—
Total	8	2

In this last example, if the point E in Fig. 114 is not occupied, the number of observed quantities for which corrections are required is now 8, there is 1 new point to be fixed, and this also counts as an unoccupied station. Hence, the number of condition equations is $8 - 2 + 1 = 7$. Here the side equations remain as before but there are no closing errors for the triangle BEC or for the triangles involving the sides AE and ED. The angular condition equations are for the sums of the angles 1 and 2; 3 and 4; 5 and 6; 7 and 8; 9, 10, 11 and 12. In the case of this last sum we have

$$180^\circ - (2 + 3) + 180^\circ - (4 + ACD + 5) + 180^\circ - (6 + ADB + DAC + 7) + 180^\circ - (8 + ABD + 1) = 360^\circ,$$

and, since the angles ACD, ADB, DAC and ABD are fixed, no corrections to them are involved or appear in the equations. Consequently, the derived condition equation is of the form

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 + q = 0,$$

and the corrections to the angles marked 9, 10, 11 and 12 are $-(r_2 + r_3)$, $-(r_4 + r_5)$, $-(r_6 + r_7)$ and $-(r_8 + r_1)$. Similarly, when forming the side equations, if the angle 9 enters into the equation, the sine to be used in working out the absolute term is $\sin(180^\circ - 2 - 3) = \sin(2 + 3)$ and the corresponding correction term to be used in forming the side equation is $-(r_2 + r_3) \cdot d$, where d is the tabular difference for 1″ for log sin $(2 + 3)$.

If the method of directions (page 291) is used, the above rules require slight modification. In this case, the condition for the closure of the horizon or for the fixed sums of certain angles is not required and *the number of condition equations is now the number of new observed directions minus three times the number of new points.*

If this rule is applied to a figure which has no side that is to be held fixed in direction or length, the forward and reverse directions along one side and the two stations at the end of that side are not to be considered to be new observed directions or new points, as one side must serve as a base for computing the figure.

Applying this rule to Fig. 113, if all sides are observed in both directions, including the side AB as base, there are 22 new observed directions and 4 new points to be fixed, the directions AB and BA and the points A and B not being reckoned as new directions or new points. Hence, the number of condition equations is $22 - 3 \times 4 = 10$. This is the same as the number found by the rule for adjustment by angles after subtracting 1 for the equation for closing the horizon at C. In the case of Fig. 114, there are 8 new directions (AE, EA, BE, EB, CE, EC, DE and ED) and 1 new point, so the number of condition equations is $8 - 3 \times 1 = 5$, or 5 less than are needed for the method of angles.

Formation of Side Equations. The following simple rule is useful for forming the side equations. Choose as a pole a suitable point on the figure fixing a new line and write down the sides on the figure which radiate from that pole in order of azimuth. Write the second line under the first, the third under the second, etc., until the first line comes under the last. Replace these sides by the sines of the angles opposite them and put the resulting expression equal to unity.

When using this rule, it is not necessary that the point chosen as a pole should be an actual station. Any point where two lines cross will serve as a pole; for instance, the point of intersection of the diagonals of a quadrilateral.

As an example, take the case of the side equation for the line AD in Fig. 113. If D is chosen as pole to fix this line, the lines are :—

$$\frac{DB}{DA} \cdot \frac{DA}{DC} \cdot \frac{DC}{DB},$$

and the side equation for the quadrilateral ABDC is :-

$$\frac{\sin DAB}{\sin ABD} \cdot \frac{\sin ACD}{\sin DAC} \cdot \frac{\sin DBC}{\sin DCB} = 1.$$

Similarly, for the lines FB and FA with poles at B and C :—

$$\frac{BA}{BC} \cdot \frac{BC}{BF} \cdot \frac{BF}{BA} = 1,$$

and

$$\frac{CA}{CB} \cdot \frac{CB}{CD} \cdot \frac{CD}{CE} \cdot \frac{CE}{CF} \cdot \frac{CF}{CA} = 1,$$

giving

$$\frac{\sin ACB}{\sin CAB} \cdot \frac{\sin CFB}{\sin BCF} \cdot \frac{\sin FAB}{\sin AFB} = 1,$$

and

$$\frac{\sin CBA}{\sin CAB} \cdot \frac{\sin CDB}{\sin CBD} \cdot \frac{\sin CED}{\sin CDE} \cdot \frac{\sin CFE}{\sin CEF} \cdot \frac{\sin CAF}{\sin CFA} = 1.$$

Choice of Conditional Equations. Angle and side equations may be chosen in several alternative ways but nearly always there are some that are better than others. The triangles to be chosen for the angle equations are preferably those which have one or more sides on the exterior of the figure, and triangles with small angles are to be avoided wherever possible. In addition, it is well to avoid a triangle which has a side in common with one that has small angles. On the other hand, when choosing the pole for forming a side equation, choose that in which the more acute angles enter into the equation. This is because the differences in the log sines are greatest for small angles, and, in the condition equations, these differences occur as coefficients of the unknowns. Each small angle, however, should not be used for forming more than one equation.

It is often advisable to divide a side equation throughout, including the absolute term, by 10 so as to reduce the order of magnitude of the coefficients. This is because it is desirable to have the diagonal terms of the normal equations of about the same order of magnitude, so that as many as possible may lie between 1 and 10. If a side equation is divided throughout by 10, the effect is to divide the diagonal term in the original corresponding normal equation by 100 since this term comes from the squares of the coefficients that produce it, although all the other terms in that normal equation will be divided by 10 only. This fact may be used to reduce the order of magnitude of the coefficients in a normal equation if, after this equation has been formed, it is found that the magnitude of the

coefficient of the diagonal term is too great. When dividing the ordinary terms by 10, however, it must not be forgotten that the coefficients lying above the diagonal term must be divided by 10, as well as those which lie to the right of it, the diagonal term itself, of course, being divided by 100. In addition, if the corrections are found from the coefficients of the conditional equations as originally written, the corresponding correlative must also be divided by 10 when working out these corrections. All this, of course, should not be done if the coefficients of the corresponding conditional equation are divided by 10 before the formation of the normal equations is begun.

It is also advisable to make the absolute terms in either conditional or observational equations of the same order of magnitude as the coefficients in the equations themselves. This can be done by altering the unit in which the absolute term is expressed, *e.g.* using a hundredth of a foot instead of a foot as the unit of length. In the adjustment of triangulation, however, this point does not arise as the absolute terms are already of the right order of magnitude.

Adjustment by Method of Directions. In the rules for adjusting angles already given the corrections have been applied to the measured angles, but even an observed angle is not a directly observed quantity. In measuring it, two separate pointings of the telescope, combined with two separate sets of readings of the circle, are necessary. The value obtained for the angle is, in fact, derived as the difference between the measurements of two directions. Hence, it would appear to be more logical if the errors in the observations were considered to be in the observed directions rather than in the derived angles, and for the corrections to be applied to the directions rather than to the angles. The method of correcting the directions instead of the angles is fairly extensively used in first-class work since, in general, it tends to eliminate more completely the accidental errors, such as phase of signals, eccentricity of signal and instrument, horizontal refraction, etc., that are inherent in the different pointings. The method, which was that used by Clarke for the adjustment of the Principal Triangulation of Great Britain, is also particularly suitable for the adjustment of figures in which certain angles or bearings and sides have to be held fixed as the result of a previous adjustment, as in this case there are, in general, fewer condition equations than there are if the method of adjusting the angles is used. Hence, it is often used for the adjustment of secondary triangulation to primary triangulation, or of tertiary to secondary.

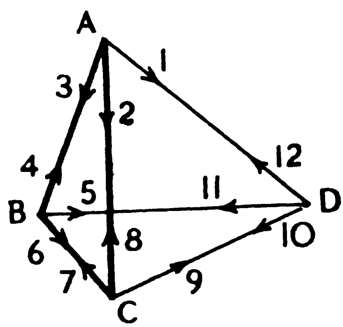


FIG. 115.

In this method angles are expressed as differences between directions or bearings, and the corrections are applied directly, not to the angles, but to the directions. Thus, in Fig. 115, in which the triangle ABC has already been fixed from a previous adjustment, as indicated by the thick lines, each line along which a sight has been taken is marked with an arrow and a number. Denoting

directions by numbers in this way, angle CAD is the difference between the directions 2 and 1 and may be written [2 − 1]; angle ACD is [9 − 8]; angle DBC is [6 − 5] and angle DCB is [9 − 7], and so on, the angles being written down as if the directions were bearings on the whole-circle system. The fixed sides and angles of the triangle ABC are :—

$$\begin{array}{llllll} \text{BAC} = [3 - 2] = 37^\circ & 10' & 32''{\cdot}7\,; & \log \text{BC} & = 4{\cdot}349\,2226\,; \\ \text{ABC} = [6 - 4] = 98 & 48 & 03\,{\cdot}7\,; & \log \text{AC} & = 4{\cdot}562\,8535\,; \\ \text{ACB} = [8 - 7] = 44 & 01 & 23\,{\cdot}6\,; & \log \text{AB} & = 4{\cdot}409\,9509\,; \end{array}$$
$$\text{Bearing AB} = 221^\circ\ 13'\ 47''{\cdot}6,$$

and the observed angles are :—

$$\begin{array}{llllllll} \text{ABD} = [5 - 4] = 48^\circ & 26' & 09''{\cdot}0\,; & \text{CDB} = [11 - 10] = & 54^\circ & 39' & 48''{\cdot}8\,; \\ \text{DBC} = [6 - 5] = 50 & 21 & 53\,{\cdot}7\,; & \text{ADB} = [12 - 11] = & 63 & 56 & 14\,{\cdot}5\,; \\ \text{ACD} = [9 - 8] = 30 & 56 & 45\,{\cdot}3\,; & \text{DAC} = [2 - 1] = & 30 & 27 & 07\,{\cdot}2. \end{array}$$

Here the angles [5 − 4] and [6 − 5] do not add up to [6 − 4] as

$$\begin{array}{llll} [5 - 4] = 48^\circ & 26' & 09''{\cdot}0 \\ [6 - 5] = 50 & 21 & 53\,{\cdot}7 \\ [6 - 4] = 98 & 48 & 02\,{\cdot}7. \end{array}$$

But the value of [6 − 4] fixed from the triangle ABC is 98° 48′ 03″·7. Hence, the difference is 1″·0. Accordingly, we split this difference between the two angles so that, adding 0″·5 to each, these become

$$\begin{array}{llll} [5 - 4] = 48^\circ & 26' & 09''{\cdot}5, \\ [6 - 5] = 50 & 21 & 54\,{\cdot}2, \end{array}$$

and these are the values to be used in the adjustment.* The revised observed angles are therefore :—

$$\begin{array}{lllllll} [5 - 4] = 48^\circ & 26' & 09''{\cdot}5\,; & [11 - 10] = 54^\circ & 39' & 48''{\cdot}8\,; \\ [6 - 5] = 50 & 21 & 54\,{\cdot}2\,; & [12 - 11] = 63 & 56 & 14\,{\cdot}5\,; \\ [9 - 8] = 30 & 56 & 45\,{\cdot}3\,; & [\ 2 - 1] = 30 & 27 & 07\,{\cdot}2. \end{array}$$

In this figure there are six new directions for which corrections are required, namely 5, 9, 10, 11, 12 and 1, and there is one new point, D, to be fixed. Hence, the number of condition equations is 6 − 3 × 1 = 3.† We shall take these as the conditions for closure of the triangles DBC and ACD and for equal values of the length of BD as computed from the sides AB and BC.

Denote the corrections to the various directions by the number of each direction enclosed in round brackets, so that, for instance, (1) is the

* This form of preliminary station adjustment is necessary because we have to assume that the directions have definite values, and the observed angles give different values for the direction of BD according as to whether this direction is computed from the fixed directions BC or BA. In addition, if (4), (5) and (6) represent corrections to be applied to the directions 4, 5 and 6, we could not use the condition that ABD + DBC = ABC because this condition would be written (5) − (4) + (6) − (5), giving (6) − (4), so that there would be no term for the correction to the direction 5 in the condition equation, which would then merely become an identity.

† If the method of adjustment of angles were used, there are six observed angles and one new point, so that the number of conditions would be 6 − 2 × 1 = 4. Thus, in this case the method of directions saves one condition equation.

correction to direction 1, (9) the correction to direction 9, etc. Then we have for the closure of the triangle DBC :—

$$[6 - 5] + (6) - (5) + [9 - 7] + (9) - (7) + [11 - 10] + (11) - (10) = 180°.$$

The directions 6 and 7 are fixed because they are parts of the fixed triangle ABC, and so there will be no corrections for them. Hence we must drop the corrections (6) and (7) from the equation, as their numerical value is zero. Accordingly, we have for the first condition equation,

$$179° \; 59' \; 51''·9 - (5) + (9) + (11) - (10) = 180°,$$

or,

$$-(5) + (9) + (11) - (10) - 8·1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the triangle ACD we get similarly :—

$$(1) + (12) - (10) + (9) - 4·2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for the side equation we have :—

$$\log AB + \log \sin [3 - 1] + \log \sin [11 - 10] - \log BC - \log \sin [9 - 7] - \log \sin [12 - 11] + (1)d_1 + \{(11) - (10)\}d_2 - (9)d_3 - \{(12) - (11)\}d_4 = 0.$$

where d_1, d_2, d_3 and d_4 are the differences for 1″ in the log sines of the angles [3 − 1], [11 − 10], [9 − 7] and [12 − 11]. This gives

$$8·67(1) + 14·93\{(11) - (10)\} - 5·65(9) - 10·28\{(12) - (11)\} - 19 = 0,$$

where the numerical coefficients are in units of the 7th place of decimals. After adding together algebraically the coefficients of the correction (11) which occurs twice in the equation, this simplifies to

$$8''·67(1) - 5·65(9) - 14·93(10) + 25·21(11) - 10·28(12) - 19 = 0,$$

or, dividing throughout by 10, and, as a matter of convenience, changing all the signs,

$$-0·87(1) + 0·56(9) + 1·49(10) - 2·52(11) + 1·03(12) + 1·9 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Using equations (1), (2) and (3) to compile a correlative table for the formation of the normal conditional equations, we write

r	a	b	c	sum
(1)	—	+ 1	− 0·87	+ 0·13
(5)	− 1	—		− 1
(9)	+ 1	+ 1	+ 0·56	+ 2·56
(10)	− 1	− 1	+ 1·49	− 0·51
(11)	+ 1		− 2·52	− 1·52
(12)	—	+ 1	+ 1·03	+ 2·03

The normal equations then are :—

$$+4k_1 + 2k_2 - 3·45k_3 - 8·1 = 0$$
$$+4k_2 - 0·77k_3 - 4·2 = 0$$
$$+10·7019k_3 + 1·9 = 0.$$

Whence, the correlatives are $k_1 = +3{\cdot}217$, $k_2 = -0{\cdot}329$ and $k_3 = +1{\cdot}191$, and the corrections are :—

$$(1)\ +1''{\cdot}36\,;\quad (9)\ +3''{\cdot}55\,;\quad (11)\ +0''{\cdot}22\,;$$
$$(5)\ -3{\cdot}22\,;\quad (10)\ -1{\cdot}10\,;\quad (12)\ +0{\cdot}90.$$

The correction to any angle is now the difference of the corrections to the directions containing the angle. Thus, the correction to the angle [2 − 1] is 0 − (1) = − 1″·36 and to [12 − 11] it is (12) − (11) = + 0″·90 − 0″·22 = + 0″·68, etc. In this way, we get the corrected angles :—

[5 − 4] =	48°	26′	06″·3 ;	[11 − 10] =	54	39′	50″·1 ;
[6 − 5] =	50	21	57 ·4 ;	[12 − 11] =	63	56	15 ·2 ;
[9 − 8] =	30	56	48 ·9 ;	[2 − 1] =	30	27	05 ·8 ;

and it will be found that these values satisfy all the conditions.

In practice, it is well to work with definite bearings and to set out the data and results methodically as in the table below. Starting with a line of known bearing, or with an assumed bearing if the real bearing of none of the lines is known, we compute the unadjusted bearings of the different lines from the observed angles and set these out in column 4.

1	2	3	4			5	6	7		
Stations		Lines	Unadjusted Bearing			Adjustment	Station Adjustment	Adjusted Bearing		
From	To									
			°	′	″	″		°	′	″
A	D	1	153	36	07·7	+ 1·4	—	153	36	09·1
	C	2	184	03	14·9	—	—	184	03	14·9
	B	3	221	13	47·6	—	—	221	13	47·6
B	A	4	41	13	47·6	—	—	41	13	47·6
	D	5	89	39	57·1	3·2	—	89	39	53·9
	C	6	140	01	51·3	—		140	01	51·3
C	B	7	320	01	51·3	—	—	320	01	51·3
	A	8	4	03	14·9	—	—	4	03	14·9
	D	9	35	00	00·2	+ 3·6	—	35	00	03·8
D	C	10	215	00	00·2	1·1	+ 4·7	215	00	03·8
	B	11	269	39	49·0	+ 0·2	+ 4·7	269	39	53·9
	A	12	333	36	03·5	+ 0·9	+ 4·7	333	36	09·1

In the case of the fixed lines, the forward and reverse bearings will differ by 180° exactly, assuming that we are working with plane bearings. When we come to a station, such as D in the example, which is not at the end of a fixed line, we must assume a bearing for one line from that station and then work out the other unadjusted bearings from that and from the unadjusted angles. In the table, we have taken the reverse bearing of the line CD as the approximate bearing of the line DC, and the unadjusted bearings of the lines DB and DA are worked out from the unadjusted angles. As a result of the adjustment of the figure, however, the forward bearing of CD will be altered and a correction will have to be

applied to the bearing of the line DC. Thus, the unadjusted bearing of CD is 35° 00′ 00″·2 and the adjusted value is 35° 00′ 03″·8. The figural adjustment to line DC is −1″·1, giving a new bearing of 214° 59′ 59″·1, which differs in the seconds by 4″·7 from the bearing CD. Hence, to bring the bearing of DC to differ by 180° exactly from the bearing of CD, we must add +4″·7 to the −1″·1 of the figural adjustment. This figure is inserted in the column headed "Station Adjustment" and it will have the same value for all lines from station D. After it has been added to the correction for figural adjustment of the lines DB and DA, we get the final adjusted bearings of these lines, and it will now be seen that these differ by 180° exactly from the forward bearings of the same lines. This shows that the angle equations in the adjustment are satisfied.

Adjustment by Method of Differential Displacements. By simple differentiation of the expression

$$\tan \alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

and using the formulae $x_2 - x_1 = l \cos \alpha$, $y_2 - y_1 = l \sin \alpha$, where l is the length and α the bearing of the line joining the point 1 to the point 2, we obtain

$$d\alpha = \frac{(dy_2 - dy_1) \cos \alpha}{l} - \frac{(dx_2 - dx_1) \sin \alpha}{l}.$$

which, when $d\alpha$ is expressed in seconds of arc, may be written

$$d\alpha = p.dx_1 - q.dy_1 - p.dx_2 + q.dy_2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$p = \frac{\sin \alpha}{l \sin 1''}, \text{ and } q = \frac{\cos \alpha}{l \sin 1''}$$

This expression may be used to adjust triangulation figures. For, if preliminary approximate co-ordinates of the new points to be fixed are computed from the unadjusted angles and lengths, we can compute an approximate bearing α for each line from the approximate co-ordinates. Starting with a line of known bearing and the observed angles, we can also calculate an observed bearing α_o for each line. When the bearings at the stations to be fixed are computed from the observed angles, however, the true bearing of the line from which the other bearings are calculated is not known until the figure is adjusted and consequently we must assume that there is a station adjustment $d\alpha_s$ which is common to all the observed bearings from that station and which must be applied to all these bearings. This adjustment is similar to the station adjustment used in the method of adjustment by bearings described in the last section. Consequently, we get an observation equation of the form

$$\alpha_o + d\alpha = \alpha_c + d\alpha_s$$

or

$$\alpha_o - \alpha_c - d\alpha_s + p.dx_1 - q.dy_1 - p.dx_2 + q.dy_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the initial point is a fixed point, $d\alpha_s$, dx_1, and dy_1 for all lines starting from that point must each be put equal to zero.

In this way, an observation equation is formed for each line observed

and normal equations formed with the $d\alpha_s$'s, dx's and dy's as the unknowns, there being a new $d\alpha_s$ for each new station to be fixed which is common to all lines from that station. Solution of the normal equations gives the unknowns, the dx's and dy's are then applied to the preliminary assumed co-ordinates, and the corrected bearings found from

$$\alpha = \alpha_c + p.dx_1 - q.dy_1 - p.dx_2 + q.dy_2 - d\alpha_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This method of solution is often used for the adjustment of lower order triangulation to that of a higher order.

In a similar manner, if the co-ordinates used are geographical co-ordinates in terms of latitude ϕ, longitude L, and azimuth A, we can, by using the first terms in Puissant's formulae (page 332), viz. $(\phi_2 - \phi_1) = B.l. \cos A$ and $(L_2 - L_1) = A.l. \sin A \sec \phi_2$, in which A and B have the meanings assigned to them on page 333, obtain

$$dA = p.d\phi_1 - q.dL_1 - p.d\phi_2 + q.dL_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$p = \frac{\sin A}{B\, l \sin 1''} \text{ and } q = \frac{\cos \phi_2 \cos A}{A\, l \sin 1''},$$

and the observation equation then becomes

$$A_c - A_o - dA_s + p \, . \, d\phi_1 - q \, . \, dL_1 - p \, . \, d\phi_2 + q \, . \, dL_2 = 0 \quad . \quad . \quad . \quad (5)$$

Approximate Adjustment of Geodetic Quadrilateral. For observations of equal weight, an approximate adjustment, sufficient for minor work, may be performed as follows.

(1) Satisfy the first angle equation (page 284) by distributing the error of summation of the observed values as equal corrections to all the angles.

(2) Satisfy the second angle equation by distributing one-fourth of the discrepancy to each angle therein.

(3) Satisfy the third angle equation in the same way.

(4) Adjust the values so obtained to suit the side equation by the following method.

Let a, b, etc. = values obtained after the angle conditions are met,
r_1, r_2, etc. = corrections to a, b, etc., required to satisfy the side equation,
d_1, d_2, etc. = tabular differences for 1″ for log sin a, log sin b, etc.

As in the rigorous adjustment, the side equation may be expressed as

$$d_1 r_1 - d_2 r_2 + \ . \ . \ . \ - d_8 r_8 = u_4.$$

This equation is now independent of the angle equations, and the most probable set of values for r_1, r_2, etc. is that in which r_1, r_2, etc. are numerically proportional to d_1, d_2, etc., or

$$\frac{r_1}{d_1} = -\frac{r_2}{d_2} = \frac{r_3}{d_3} = \text{etc.}$$

Dividing the side equation, term by term, by these ratios, we have

$$d_1^2 + d_2^2 + \ . \ . \ . + d_8^2 = \frac{u_4 d_1}{r_1} = -\frac{u_4 d_2}{r_2} = \text{etc.},$$

$$\text{or } r_1 = \frac{u_4 d_1}{[d^2]},\ r_2 = -\frac{u_4 d_2}{[d^2]}, \text{ etc.}$$

APPROXIMATE ADJUSTMENT OF GEODETIC QUADRILATERAL

Angle	Observed Values			1st Corrn.	2nd Corrn.	Adjusted Values			Log Sines	d	d^2	Side Corrn.	Adjusted Values			Check Log Sines
	°	′	″	″	″	°	′	″				″	°	′	″	
a	30	27	07·2	0·6	− 0·98	30	27	06·82	9·7048492	3·58	12·816	− 1·59	30	27	05·23	+ 9·7048135
b	37	10	32·6	0·6	− 1·88	37	10	31·32	− 9·7812214	2·78	7·728	+ 1·23	37	10	32·55	− 9·7812248
c	48	26	09·0	0·6	− 1·88	48	26	07·72	+ 9·8740229	1·87	3·497	− 0·83	48	26	06·89	+ 9·8740214
d	50	21	54·6	0·6	+ 0·98	50	21	56·18	− 9·8865643	1·74	3·028	+ 0·77	50	21	56·95	− 9·8865656
e	44	01	23·2	0·6	0·98	44	01	24·78	+ 9·8419561	2·18	4·752	− 0·97	44	01	23·81	+ 9·8419540
f	30	56	45·3	0·6	+ 1·88	30	56	47·78	− 9·7111650	3·51	12·320	+ 1·55	30	56	49·33	− 9·7111704
g	54	39	48·8	0·6	+ 1·88	54	39	51·28	9·9115714	1·49	2·220	− 0·66	54	39	50·62	− 9·9115704
h	63	56	14·5	0·6	− 0·98	63	56	14·12	− 9·9534279	1·03	1·061	+ 0·46	63	56	14·58	− 9·9534284
	[illegible]	[illegible]	[illegible]			360	0	00·00	$u_1 = -21·0$	$[d^2] = 47·42$			359	59	59·96	1

The final adjusted values in the table on page 297 do not satisfy the angle equations: a second adjustment gives results very nearly the same as those previously obtained by the rigorous method.

Since these corrections have been determined without reference to the requirements of the angle equations, their application is likely to disturb the former adjustments. If the discrepancies thus caused are greater than is desirable, the results obtained may be treated as observed values, and the complete adjustment is repeated by the same process.

Example. Apply the approximate method to the adjustment of the last case.

(1) The observed angles sum to $4''\cdot 8$ short of $360°$,

$$\therefore \text{ correction to each} = +\frac{4''\cdot 8}{8} = +0''\cdot 6.$$

(2) $a + h$ exceeds $d + e$ by $3''\cdot 9$,

$$\therefore \text{ correction to each} = \frac{3''\cdot 9}{4} = 0''\cdot 98, \ - \text{ for } a \text{ and } h, \ + \text{ for } d \text{ and } e.$$

(3) $b + c$ exceeds $f + g$ by $7''\cdot 5$,

$$\therefore \text{ correction to each} = \frac{7''\cdot 5}{4} = 1''\cdot 88, \ - \text{ for } b \text{ and } c, \ + \text{ for } f \text{ and } g.$$

The values obtained on application of these corrections are as tabulated on page 297.

(4) The algebraic sum of the log sines of the angles as adjusted is $21\cdot 0$ in units of the 6th decimal place of logarithms,

$$\therefore u_4 = -21\cdot 0,$$

$$\text{and } r_1 = -\frac{21\cdot 0 \times 3\cdot 58}{47\cdot 42} = -1''\cdot 59, \text{ etc., as tabulated and applied on page 297.}$$

This method of approximate adjustment can be extended to other simple figures besides the quadrilateral, and, for several of those that occur fairly commonly in practical work, McCaw has worked out empirical rules for the corrections to be applied to the angles to satisfy the angle conditional equations. Other simple methods and rules, also due to him, which are based on the method of successive approximations, are given in the War Office publication *The Approximate Rigorous Adjustment of Simple Figures.*

Ferrero's Formula for the Probable Error of an Observed Angle in Triangulation. Let there be N triangles and let $\Delta_1, \Delta_2, \Delta_3, \ldots \Delta_n$ be the individual closing errors. Then, if the observations of the angles are assumed to be of equal weight, the probable error of closure of a single triangle will be given by :—

$$\text{p.e.} = \pm\sqrt{r_0^2 + r_0^2 + r_0^2} = \pm\sqrt{3r_0^2},$$

in which r_0 is the probable error of a single angle. But, treating the various closing errors as residuals, the probable error of closure of a triangle will be :—

$$\text{p.e.} = \pm 0\cdot 6745\sqrt{\frac{[\Delta^2]}{N}}$$

where $$[\Delta^2] = \Delta_1^2 + \Delta_2^2 + \Delta_3^2 + \ldots + \Delta_n^2.$$

Hence,

$$\sqrt{3\, r_0^2} = \pm\, 0{\cdot}6745\sqrt{\frac{[\Delta^2]}{N}},$$

and therefore :—

$$r_0 = \pm\, 0{\cdot}6745\sqrt{\frac{[\Delta^2]}{3N}}.$$

This formula is generally known as Ferrero's formula. It is only to be used for calculating approximate probable errors from unadjusted results, as it only takes into account the angular equations and ignores the side equations.

Probable Error of an Adjusted Angle in Triangulation. When the angles of a figure have been adjusted by least squares the probable error of an adjusted angle may be found by using the formula for the probable error of conditioned quantities given on page 280. Let S be the total number of new stations in the figure and n the number of angles to which corrections have been applied. Then (page 288) the number of conditions, n_c, is given by $n_c = n - 2S$. Assuming that all the angles are of equal weight and putting $q = n$ in the formula of page 280,

$$r_a = \pm\, 0{\cdot}6745\sqrt{\frac{2S[v^2]}{n(n-2S)}},$$

in which $[v^2]$ is the sum of the squares of the corrections to the angles.

Propagation of Linear Error in Triangulation. If the angles of a triangle ABC are adjusted for closure it can be shown that the probable error in the length of the side a is given by :—

$$r_a^2 = \left(r_b \cdot \frac{a}{b}\right)^2 + \tfrac{2}{3}a^2 r_a^2 \sin^2 1''\,(\cot^2 A + \cot^2 B + \cot A \,.\, \cot B)$$

in which r_b is the probable error in the length of the side b and r_a is the probable error of an *adjusted* angle. It is easy to extend this formula to the case of a chain of single triangles when it will be found that the probable error in the computed length of the side g is given by :—

$$r_g^2 = \left(r_b \cdot \frac{g}{b}\right)^2 + \tfrac{2}{3}g^2 r_a^2 \sin^2 1''\Big[(\cot^2 A + \cot^2 B + \cot A \,.\, \cot B)\Big]$$

where r_a is the probable error of an *adjusted* angle and the square brackets [ ] in the second term denote summation over the distance angles used in computing the length of the side g from the base b. This formula means looking out the co-tangents of the angles, as these quantities have not been found in any previous computation. If the effect of the probable error of the base line is neglected, or is assumed to be zero, it is more convenient to use the formula :—

$$r_e^2 = \tfrac{2}{3} r_a^2 [d_A^2 + d_B^2 + d_A d_B]$$

in which r_e is the probable error in the *logarithm* of the length g. d_A and d_B

are the differences for one second in the logarithms of the sines of the distance angles that were used in the formation of the side or length equations, and, as before, the square bracket denotes summation. The formula is derived from the other by remembering that $\cot A = \frac{d(\log \sin A)}{dA}$.

Adjustment of a Chain of Triangulation between Fixed Bases and Azimuths. *When a check base and a check azimuth are observed at the end of a chain of triangulation, it is necessary to adjust the latter so that* the length of the second base and the check azimuth, as computed through the triangulation from the initial base and azimuth, agree with their measured values. If the chain is adjusted as a whole, this can easily be done by adding two additional conditional equations, one for closure between bases and the other for closure between azimuths. To adjust a long complicated chain in one single solution would usually mean a very great amount of labour and it is therefore a common practice to divide the chain into a series of single figures, some or all of which may be relatively complicated, and to adjust each figure independently for the ordinary geometrical conditions of closure of the sides and angles. After this has been done, a chain of well-shaped single triangles to connect the two bases and the two azimuth stations is chosen, and, with the adjusted values for the angles as obtained from the figural solutions, the second azimuth and the length of the second base are computed from the initial azimuth and base. These computed values will not agree with the measured values. The chain of single triangles is now adjusted as a whole, the conditional equations including one for the closure of each triangle, one for the closure in azimuth and a length equation for the closure of the bases. As the triangles have all been adjusted for closure during the adjustment of the different figures, the absolute terms in the ordinary angle conditional equations will be zero and the only equations in which absolute terms apart from zero will occur will be the azimuth and length equations.

When the terminal side is also fixed in position as well as in length and azimuth it will be necessary to add two extra condition equations, one for closure in latitude and the other for closure in longitude. These equations are rather complicated when stated in a general form and will not be given here. If required, they will be found in Wright and Hayford's *Adjustment of Observations*, pages 255–259 (second edition, 1906) or in the Gold Coast Survey Professional Paper No. 1 *Notes on the Application of the Method of Least Squares to the Adjustment of Triangulation and Level and Traverse Networks*, pages 51–54. As an alternative, if the triangulation is computed in terms of rectangular co-ordinates, the method of differential displacements described on page 295 may be used, the dx_s's, dx's and dy's of all the fixed stations being put equal to zero.

The adjustment between bases will naturally disturb the figural adjustments, and each figure will therefore have to be re-adjusted to hold fixed the new values for the angles in those triangles that occur in the chain of single triangles used in the adjustment just described.

ADJUSTMENT OF PRECISE TRAVERSES

Traverses may be computed either on a system, or modified system, of rectangular co-ordinates or else directly in terms of latitude and longitude on the spheroid, the latter being the method used by the United States Coast and Geodetic Survey. Whichever method is used, the principles applicable to the adjustment will be the same for each, although slight variations in the formulæ will naturally be necessary.

The most common problem in the adjustment of a traverse is that in which the traverse begins and ends on trigonometrical points whose positions are considered to be without error and which therefore have to be held fixed in the adjustment. There are then three main conditions to be fulfilled. First, there is the condition for the closure in X, or in latitude if geographical co-ordinates are used, and secondly there is the similar condition for the closure in Y, or in longitude. Finally, there is the condition for the closure of the bearings or azimuths at the azimuth stations. The system of adjustment may then involve corrections applied directly to the co-ordinates or else to the observed lengths and angles. In all systems of adjusting traverses one of the main difficulties is in estimating and assigning the relative weights of the linear and angular measures.

When networks of traverses are involved, the problem becomes more complicated, because one or both of the terminal points of an individual traverse is no longer fixed and the positions of the junction points of different traverses have to be found. The conditions then include closure in X and Y, or in latitude and longitude, for the different closed circuits which comprise the network, and for the closure of the angles or bearings on directly observed values. Here again, the main difficulty is one of assigning suitable weights to the angular and linear measures or to each single traverse as a whole.

Adjustment by Means of Corrections Applied Directly to the Co-ordinates. The most common of the methods of applying corrections directly to the co-ordinates is the conventional Bowditch rule which is described in Vol. I, page 272. This rule, although it can be derived directly from least squares, is based on various assumptions that are not altogether sound even when applied to ordinary traverses, and much less so when applied to precise traverses. These assumptions imply that bearings, and not angles, are the directly observed quantities, and they involve a system of weighting which is hardly consistent with the actual conditions of measurement.

For further details of the Bowditch rule reference may be made to Vol. I. Proofs of the formulæ involved, and of various modifications of them, are given in different text-books, such as Crandall's *Text-book of Geodesy and Least Squares* or Jameson's *Advanced Surveying*. The rule is also very fully discussed in a series of articles which have appeared in recent issues of the *Empire Survey Review*.*

Adjustment by Means of Corrections Applied to the Measured Angles and Lengths. In Vol. I, page 236, it was proved that the displacement in the X co-ordinate of a traverse, due to disturbances $\delta l_1, \delta l_2, \delta l_3 \ . \ . \ . \ \delta l_n$

* *Empire Survey Review*, Vol. IV, Nos. 29 and 30; Vol. V, Nos. 31, 32 and 38.

in the lengths, and $\delta\theta_1, \delta\theta_2, \delta\theta_3 \ldots \delta\theta_n$ in the angles, is given by :—

$$\delta X = \left[\frac{\delta l_1}{l_1}x_1 + \frac{\delta l_2}{l_2}x_2 + \frac{\delta l_3}{l_3}x_3 + \ldots + \frac{\delta l_n}{l_n}x_n\right] - \left[\delta\theta_1(y_1 + y_2 + y_3 + \ldots + y_n) + \delta\theta_2(y_2 + y_3 + \ldots + y_n) + \delta\theta_3(y_3 + \ldots + y_n) + \ldots + \delta\theta_n y_n\right].$$

where $l_1, l_2, l_3 \ldots l_n$ are the lengths of the legs ; $x_1, x_2, x_3 \ldots x_n$ are the latitudes and $y_1, y_2, y_3 \ldots y_n$ their departures.

The similar expression in Y is :—

$$\delta Y = \left[\frac{\delta l_1}{l_1}y_1 + \frac{\delta l_2}{l_2}y_2 + \frac{\delta l_3}{l_3}y_3 + \ldots + \frac{\delta l_n}{l_n}y_n\right] + \left[\delta\theta_1(x_1 + x_2 + x_3 + \ldots + x_n) + \delta\theta_2(x_2 + x_3 + \ldots + x_n) + \delta\theta_3(x_3 + \ldots + x_n) + \ldots + \delta\theta_n x_n\right].$$

If (X_1, Y_1) ; (X_2, Y_2) ; (X_3, Y_3) ; . . . (X_{n+1}, Y_{n+1}) are the co-ordinates of the different traverse stations, (X_1, Y_1) and (X_{n+1}, Y_{n+1}) being the co-ordinates of the terminal points of the traverse, these expressions may be written :—

$$\delta X = \left[\frac{\delta l_1}{l_1}x_1 + \frac{\delta l_2}{l_2}x_2 + \frac{\delta l_3}{l_3}x_3 + \ldots + \frac{\delta l_n}{l_n}x_n\right] - \left[\delta\theta_1(Y_{n+1} - Y_1) + \delta\theta_2(Y_{n+1} - Y_2) + \delta\theta_3(Y_{n+1} - Y_3) + \ldots + \delta\theta_n(Y_{n+1} - Y_n)\right],$$

$$\delta Y = \left[\frac{\delta l_1}{l_1}y_1 + \frac{\delta l_2}{l_2}y_2 + \frac{\delta l_3}{l_3}y_3 + \ldots + \frac{\delta l_n}{l_n}y_n\right] + \left[\delta\theta_1(X_{n+1} - X_1) + \delta\theta_2(X_{n+1} - X_2) + \delta\theta_3(X_{n+1} - X_3) + \ldots + \delta\theta_n(X_{n+1} - X_n)\right].$$

If, instead of considering them as errors, as we did when deriving these expressions, we consider $\delta l_1, \delta l_2, \delta l_3 \ldots \delta l_n$; $\delta\theta_1, \delta\theta_2, \delta\theta_3 \ldots \delta\theta_n$ as corrections to be applied to the observed lengths and angles, we can regard the above formulæ as conditional equations in which δX and δY are the closing errors (fixed minus computed values) in X and Y at the terminal point of the traverse, which is supposed to be fixed. The linear and angular observations, however, will hardly be of the same weight as regards the probable errors in linear displacement produced at the end of a line of given length. Let the probable error of an angular observation be the same for all angles and independent of the length of the legs, an

assumption which is justifiable when legs are long, and let the probable error of the measurement of l_1, expressed as a fraction of l_1, be $\sqrt{p_1}$ times the probable error of an observed angle. Then, taking the weights of the angles as unity, the weight of the measurement l_1 will be $\frac{1}{p_1}$, that of l_2 will be $\frac{1}{p_2}$, and so on. When the $\delta\theta$'s are expressed in seconds of arc, the conditional equations become :—

$$\left[p_1\frac{\delta l_1}{l_1}x_1 + p_2\frac{\delta l_2}{l_2}x_2 + p_3\frac{\delta l_3}{l_3}x_3 + \cdots + p_n\frac{\delta l_n}{l_n}x_n\right] - \left[\delta\theta_1 Q_1 + \delta\theta_2 Q_2 + \delta\theta_3 Q_3 + \cdots + \delta\theta_n Q_n\right] - \delta X = 0$$

$$\left[p_1\frac{\delta l_1}{l_1}y_1 + p_2\frac{\delta l_2}{l_2}y_2 + p_3\frac{\delta l_3}{l_3}y_3 + \cdots + p_n\frac{\delta l_n}{l_n}y_n\right] + \left[\delta\theta_1 P_1 + \delta\theta_2 P_2 + \delta\theta_3 P_3 + \cdots + \delta\theta_n P_n\right] - \delta Y = 0$$

where $P_1 = (X_{n+1} - X_1)\sin 1''$; $P_2 = (X_{n+1} - X_2)\sin 1''$; $P_3 = (X_{n+1} - X_3)\sin 1''$; . . $P_n = (X_{n+1} - X_n)\sin 1''$; $Q_1 = (Y_{n+1} - Y_1)\sin 1''$; $Q_2 = (Y_{n+1} - Y_2)\sin 1''$; $Q_3 = (Y_{n+1} - Y_3)\sin 1''$; . . . . ; $Q_n = (Y_{n+1} - Y_n)\sin 1''$.

Let the traverse be split up into, say, g sections, each section beginning and ending at an azimuth station, so that for each section there is an equation for the closure of the bearings. Then we have g angular conditional equations of the form :—

$$\begin{array}{l}
\delta\theta_1 + \delta\theta_2 + \delta\theta_3 + \cdots + \delta\theta_a - \delta\alpha_1 = 0 \\
\delta\theta_{a+1} + \delta\theta_{a+2} + \cdots + \delta\theta_b - \delta\alpha_2 = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\delta\theta_{r+1} + \delta\theta_{r+2} + \cdots + \delta\theta_n - \delta\alpha_g = 0
\end{array}$$

where $\delta\alpha_1$, $\delta\alpha_2$ . . . $\delta\alpha_g$ are the closing errors (fixed minus computed) of the bearings at the ends of the different sections. These equations, together with the two equations for misclosure in position, will give $(g + 2)$ conditional equations and $(g + 2)$ normal equations, the solution of which will yield $(g + 2)$ correlatives, and, from the latter, the individual corrections δl_1, δl_2, δl_3 . . . δl_n; $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ . . . $\delta\theta_n$ can easily be obtained. From the corrected angles, the corrected bearings can be computed and corrections to the latitudes and departures can then be obtained from :—

$$\delta x = \delta l\cos\beta - l\delta\beta\,.\sin 1''\sin\beta = \frac{x}{l}\delta l - y\delta\beta\sin 1''$$

$$\delta y = \delta l\sin\beta + l\delta\beta\,.\sin 1''\cos\beta = \frac{y}{l}\,\delta l + x\delta\beta\sin 1''.$$

where β is the bearing of the line and $\delta\beta$ the correction to the bearing.

The normal equations from which the correlatives k_1, k_2, k_3 . . . k_{g+2} are found are of the form :—

$$\left\{\left[Q^2\right]+\left[\frac{px^2}{l^2}\right]\right\}k_1+\left\{\left[\frac{pxy}{l^2}\right]-\left[PQ\right]\right\}k_2-\left[Q\right]_a k_3-\left[Q\right]_b k_4$$
$$-\ \dots\ -\left[Q\right]_g k_{g+2}-\delta X=0$$

$$\left\{\left[P^2\right]+\left[\frac{py^2}{l^2}\right]\right\}k_2+\left[P\right]_a k_3+\left[P\right]_b k_4+\ \dots\ +\left[P\right]_g k_{g+2}$$
$$-\delta Y=0$$

$$n_a k_3+0+\dots\dots\dots+0-\delta\alpha_1=0$$
$$n_b k_4+0+\dots\dots\dots+0-\delta\alpha_2=0$$
$$\dots\dots\dots\dots\dots$$
$$n_g k_{g+2}-\delta\alpha_g=0$$

in which $\left[P^2\right]$, $\left[Q^2\right]$, $\left[PQ\right]$, $\left[\frac{px^2}{l^2}\right]$, $\left[\frac{py^2}{l^2}\right]$, $\left[\frac{pxy}{l^2}\right]$ are summations over the whole traverse, $\left[Q\right]_a$, $\left[Q\right]_b$ . . . $\left[Q\right]_g$ are the sums of the Q's in sections a, b . . . g; $\left[P\right]_a$, $\left[P\right]_b$, . . . $\left[P\right]_g$ are similar summations for the P's and n_a, n_b . . . n_g are the number of angular stations, including one azimuth station, in the different sections. Instead of using these formulæ, however, it is better to form the normal equations in the usual way, taking full advantage of all the usual summation tests.

The above is the general statement of the problem for a long traverse in which there are more sections than one, or more than two control or fixed azimuths, this being the case that occurs very frequently in actual practice. The formulæ have been derived directly from the fundamental formulæ for ordinary plane rectangular co-ordinates, but, in practice, they will still hold for any other system of rectangular co-ordinates—Transverse Mercator, Lambert, Cassini, etc.—because the first terms in all of these are of the form $x = l \,.\cos\beta$, $y = l \,.\sin\beta$, and these are much greater in magnitude than any of the succeeding terms. Jolly, in the *Empire Survey Review*, Vol. IV, No. 28, has discussed the special case of a comparatively short traverse, with control or fixed azimuths at the end stations only, and he assumes that the factor p is the same for all legs, which is equivalent to assuming that the probable error in the measurement of a length is proportional to the length of the line. In this case the normal equations become :—

$$\left\{\left[Q^2\right]+p\left[\frac{x^2}{l^2}\right]\right\}k_1+\left\{p\left[\frac{xy}{l^2}\right]-\left[PQ\right]\right\}k_2-\left[Q\right]k_3-\delta X=0$$
$$\left\{\left[P^2\right]+p\left[\frac{y^2}{l^2}\right]\right\}k_2+\left[P\right]k_3-\delta Y=0$$
$$nk_3-\delta\alpha=0.$$

He then uses a modification of the formula to make the third correlative disappear. Before the co-ordinates are computed, the closing error in bearing is distributed equally among the angles in the way that is usual in the adjustment of the angles in an ordinary traverse, so that the corrections obtained from the solution are to be applied to the angles after this preliminary adjustment has been made.* This gives zero as the absolute term in the third normal equation. The co-ordinates $X_1, X_2, X_3, \ldots X_{n+1}$; $Y_1, Y_2, Y_3, \ldots Y_{n+1}$ used in obtaining the P's and Q's are now referred to the "centre of gravity" of the traverse, that is to a point whose co-ordinates are $\frac{1}{2}(X_1 + X_{n+1})$, $\frac{1}{2}(Y_1 + Y_{n+1})$. When this has been done, $[P]$ and $[Q]$ vanish and the third correlative becomes zero. Hence, the number of correlatives and normal equations is reduced from three to two and the work is correspondingly simplified and reduced.

Adjustment of Angles and Lengths when the Traverse is Computed in Terms of Geographical Co-ordinates. When the traverse is computed directly in terms of geographical co-ordinates, the first terms in the relevant formulæ are (page 333) :—

$$d\phi'' = B \cdot l \cos \alpha$$
$$dL'' = A \sec \phi \cdot l \sin \alpha$$

in which $A = \dfrac{1}{N' \sin 1''}$, $B = \dfrac{1}{R \sin 1''}$, and $d\phi''$ and dL'' are the differences in latitude and longitude between the ends of the line. Hence, the changes in $d\phi$ and dL, due to changes δl and $\delta\alpha$ in α, are :—

$$\delta\phi = B \cdot \delta l \cos \alpha - B\delta\alpha \cdot \sin 1'' \cdot l \sin \alpha$$
$$= \frac{\delta l}{l} d\phi - \frac{B}{A} \delta\alpha \sin 1'' \cdot \frac{dL}{\sec \phi}.$$
$$\delta L = A \sec \phi \delta l \sin \alpha + A \sec \phi \cdot \delta\alpha \sin 1'' \, l \cos \alpha$$
$$= \frac{\delta l}{l} dL + \frac{A}{B} \sec \phi \cdot d\phi \cdot \delta\alpha \cdot \sin 1''.$$

α being the azimuth of the line. Proceeding as in the case of rectangular co-ordinates, we get the two conditional equations of position :—

$$\left[p_1 \frac{\delta l_1}{l_1} d\phi_1 + p_2 \frac{\delta l_2}{l_2} d\phi_2 + p_3 \frac{\delta l_3}{l_3} d\phi_3 + \ldots + p_n \frac{\delta l_n}{l_n} d\phi_n\right] -$$
$$\left[\delta\theta_1 \frac{\sin 1''}{\sec \phi_1} \frac{B_1}{A_1} (L_{n+1} - L_1) + \delta\theta_2 \frac{\sin 1''}{\sec \phi_2} \frac{B_2}{A_2} (L_{n+} - L_2) + \delta\theta_3 \frac{\sin 1''}{\sec \phi_3} \frac{B_3}{A_3}\right.$$
$$\left.(L_{n+1} - L_3) + \ldots + \delta\theta_n \frac{\sin 1''}{\sec \phi_n} \cdot \frac{B_n}{A_n} (L_{n+1} - L_n)\right] - \delta\phi = 0.$$

* It is easy to show that this preliminary adjustment does not affect the values obtained for the finally adjusted angles.

$$\left[p_1\frac{\delta l_1}{l_1}dL_1 + p_2\frac{\delta l_2}{l_2}dL_2 + p_3\frac{\delta l_3}{l_3}dL_3 + \ldots + p_n\frac{\delta l_n}{l_n}dL_n\right] +$$

$$\left[\delta\theta_1 \sin 1'' \sec\phi_1\frac{A_1}{B_1}(\phi_{n+1} - \phi_1) + \delta\theta_2 \sin 1'' \sec\phi_2\frac{A_2}{B_2}(\phi_{n+1} - \phi_2) + \right.$$

$$\left.\delta\theta_3 \sin 1'' \sec\phi_3\frac{A_3}{B_3}(\phi_{n+1} - \phi_3) + \ldots + \delta\theta_n \sin 1'' \sec\phi_n \cdot \frac{A_n}{B_n}(\phi_{n+1} - \phi_n)\right]$$

$$- \delta L = 0.$$

and, as before, we have the angular conditional equations for closure on fixed azimuths :—

$$\begin{array}{llll}
\delta\theta_1 & + \delta\theta_2 + \delta\theta_3 + & \ldots + \delta\theta_a - \delta\alpha_1 = 0 \\
\delta\theta_{a+1} & + \delta\theta_{a+2} & + \ldots + \delta\theta_b - \delta\alpha_2 = 0 \\
\delta\theta_{b+1} & + \delta\theta_{b+2} & + \ldots + \delta\theta_c - \delta\alpha_3 = 0 \\
\ldots & \ldots & \ldots \\
\delta\theta_{f+1} & + \delta\theta_{f+2} & + \ldots + \delta\theta_n - \delta\alpha_n = 0.
\end{array}$$

If, as is usually done, the angular closing error in azimuth is distributed evenly among the angles before the preliminary computation of co-ordinates, the absolute terms in these angular conditional equations become zero and the corrections have to be applied to the preliminary corrected angles.

In the above equations $(d\phi_1, dL_1)$; $(d\phi_2, dL_2)$; $(d\phi_3, dL_3)$; . . . $(d\phi_{n+1}, dL_{n+1})$ are the differences in latitude and longitude between the terminal points of the respective legs and correspond to the latitudes and departures in plane rectangular co-ordinates. (ϕ_1, L_1); (ϕ_2, L_2); (ϕ_3, L_3) . . . (ϕ_{n+1}, L_{n+1}) are the latitudes and longitudes of the different stations of the traverse as obtained from the unadjusted solution. The factors $A_1, A_2, A_3 \ldots A_n$; $B_1, B_2, B_3 \ldots B_n$ vary slightly with latitude, but, as the numerical value of each term is small, they may be taken out for the mean latitude of each traverse section, or, if the traverse runs more or less in an east and west direction, for the mean latitude of the whole traverse. In the latter case, the quantities $\frac{B \sin 1''}{A}$ and $\frac{A \sin 1''}{B}$ may be removed to outside the bracket, thus simplifying the formulæ.

When a traverse consists of one section only, the number of correlatives and of normal equations can be reduced from three to two in a manner similar to that described for the case of the adjustment when rectangular co-ordinates are used.

Instead of computing corrections to the actual measured lengths, the United States Coast and Geodetic Survey compute corrections to the logarithms of the lengths and in this way eliminate the varying denominators in l in the expressions in the first bracket in each equation. Since $d(\log_e l) = \frac{dl}{l}$, we can substitute $\frac{v_1}{M \times 10^6}, \frac{v_2}{M \times 10^6}, \frac{v_3}{M \times 10^6} \ldots \frac{v_n}{M \times 10^6}$ for $\frac{\delta l_1}{l_1}, \frac{\delta l_2}{l_2}, \frac{\delta l_3}{l_3} \ldots \frac{\delta l_n}{l_n}$, where $v_1, v_2, v_3 \ldots v_n$ are corrections in the sixth

place of the logarithms of the first, second, third . . . nth legs and M is the modulus of the common logarithms. This substitution can, of course, also be made in the formula for rectangular co-ordinates.

Having obtained the corrections to the lengths, and having computed the corrected azimuths from the adjusted angles, the correction to each difference in latitude and longitude can be computed from :—

$$\delta\phi = \frac{\delta l}{l} d\phi'' - \delta\alpha \frac{dL''}{\sec\phi} . \frac{B}{A} \sin 1''$$

$$\delta L = \frac{\delta l}{l} dL'' + \delta\alpha . d\phi'' . \sec\phi . \frac{A}{B} . \sin 1''.$$

Weights to be Assigned in the Adjustment of Lengths and Angles. As already stated on page 301, the most difficult problem to decide in the adjustment of a traverse is the assignment of suitable relative weights to the linear and angular observations. The practice in the United States Coast and Geodetic Survey is to make a preliminary solution in which unit weight is given both to the angles and to the ratio of the probable error of the linear measurements to the length of the line. In this case, all the p's in the formulæ given above are replaced by unity, the assumption being that the probable error in the measurement of an angle is independent of the length of the line and that in the measurement of a line is proportional to the length of the line. If the preliminary solution shows that the corrections to the angles are disproportionate to the corrections to the lengths, one or more additional solutions may be tried in which a system of weighting, as suggested by the previous solution, is adopted, and the process may be repeated until the corrections to lengths and angles are not considered to be excessive or out of proportion.

If the traverse is a fairly straight one, the direction of the closing error may suggest suitable weights. Here the component of the closing error in the direction of the traverse may be considered to be due mainly to the linear errors, but, in apportioning the total linear closing error among the different legs, it must not be forgotten that the expression for the probable error in taping is of the form $\sqrt{al + bl^2}$, in which a and b are constants, and hence the probable error is not strictly proportional either to the length of the line or to the square root of the length. If a steel tape is used for the linear measures, the probable error will tend to be proportional to the length rather than to the square root of the length; if an invar tape is used, and great care is taken to eliminate cumulative errors, the tendency will be for the probable error of the taping to follow a square root law.

When the component of the closing error at right angles to the direction of the traverse is taken as a basis for estimating the probable error of an angle, the probable displacement at the end of the traverse, supposed straight, is not simply the length of the traverse multiplied by the probable error of an angle expressed in radians. If the legs of the traverse are reasonably equal in length, r_a may be estimated from :—

$$d^2 = \frac{r_a^2 \sin^2 1''}{6} \left[L_1^2 \frac{(n_1 + 1)(2n_1 + 1)}{n_1} + L_2^2 \frac{(n_2 + 1)(2n^2 + 1)}{n_2} + \ldots \right]$$

or, in abbreviated approximate form :—

$$d^2 = \frac{r_{\prime\prime}^2 \sin^2 1''}{6}\left[L_1^2(2n_1 + 3) + L_2^2(2n_2 + 3) + \ldots\right]$$

where d is the component of the closing error at right angles to the line of the traverse, $L_1, L_2 \ldots$ are the lengths of the different sections, and $n_1, n_2 \ldots$ are the number of the legs in each section. This formula is only to be used when the angles and bearings receive no preliminary adjustment. If the closing error in bearing of each section has been evenly distributed throughout the angles before the closing error in position was computed, the formula becomes :—

$$d^2 = \frac{r_{\prime\prime}^2 \sin^2 1''}{12}\left[L_1^2\frac{(n_1 + 1)(n_1 + 2)}{n_1} + L_2^2\frac{(n_2 + 1)(n_2 + 2)}{n_2} + \ldots\right]$$

or, in abbreviated approximate form :—

$$d^2 = \frac{r_{\prime\prime}^2 \sin^2 1''}{12}\left[L_1^2(n_1 + 3) + L_2^2(n_2 + 3) + \ldots\right]$$

An alternative method, which can either be used as a rough check on the values obtained from the above reasoning or else as an independent method that can be used for traverses of any shape, is to compute the probable error of an angle from the closing errors of the bearings or azimuths at the ends of the different sections, using the formula given on page 252, viz. :—

$$r_{\prime\prime} = \pm\, 0{\cdot}6745\left[\frac{\dfrac{e_1^2}{n_1} + \dfrac{e_2^2}{n_2} + \dfrac{e_3^2}{n_3} + \ldots}{N}\right]^{\frac{1}{2}}$$

where $e_1, e_2, e_3 \ldots$ are the closing errors at the ends of the sections; $n_1, n_2, n_3 \ldots$ the number of angular stations in each section and N is the number of sections. The probable error of the linear measurements can then be estimated in the manner outlined on pages 248 to 251 by estimating the probable errors of the various factors contributing to the error in the measurement of a line. If the legs of the traverse are very unequal in length, a short table, giving the probable errors of measurement of lines of different lengths, can be computed and this table used to assign rough weights to different lengths of leg.

It will be seen from the above that any system of adjustment of a traverse by correcting the observed lengths and angles by least squares means a good deal of labour when legs are short and numerous, and it should thus only be undertaken when the utmost accuracy is desired and the accuracy of the field work justifies such a procedure. For most traverses that are likely to be required for engineering purposes, a simple adjustment by the Bowditch rule will be all that is necessary.

Adjustment of a Network of Traverses. The adjustment of a network of precise traverses is a problem that will normally concern an official survey department rather than the ordinary engineer, but the problem may sometimes arise in city or similar surveys, in which case the traverses forming the network will usually be short ones.

A complete adjustment of a network would involve equations to include corrections to every single angle and length, but this would be a most laborious process. The usual practice therefore is to form equations involving corrections to the assumed approximate co-ordinates of the different junction points of the network only, and then, holding the corrected co-ordinates fixed, to adjust the individual traverses.

In determining the corrections to the co-ordinates of the junction points, the difficulty again is to assign proper weights to the individual traverses or traverse sections that form the network. For most purposes it may be assumed that the weight to be assigned to any traverse or traverse section will be inversely proportional to the length of the section, which is equivalent to the assumption that the probable error of displacement will be proportional to the square root of the length. Alternatively, probable displacements may be estimated by estimating separately the probable displacements due to the linear and angular errors, following the methods already described, and then combining the results by the formula :—

$$R_T = \sqrt{R_l^2 + R_a^2}$$

where R_T is the total probable displacement and R_l and R_a the probable displacements due to the linear and angular errors. The weight to be assigned to any section will then be inversely as the square of the probable displacement. An example of an adjustment by such a method, and a very complete discussion of the whole problem of the probable errors of traversing and of weighting, is given by F. Yates in Vol. III of *Records of the Gold Coast Survey Department*.

Having decided on the weights to be used, condition equations for closure between fixed points and for closure of closed circuits are formed. There will be two sets of such condition equations, one for closure in X, or in latitude if geographical co-ordinates are used, and the second for the closure in Y, or in longitude. These two sets of equations will be identical except in the absolute terms, and they will lead to two sets of normal equations which, of course, will also be identical except in the absolute terms. Solution of these equations will yield two sets of correlatives, one for obtaining the corrections in X, and the other for obtaining the corrections in Y. Application of the corrections will give the adjusted co-ordinates of the different junction points of the network.

Note. In theory, the method outlined above, although sufficiently accurate for most purposes, only holds for the case where the probable error of position is the same in all directions. In a more rigorous treatment of the subject, Yates has remarked that, if the probable error in displacement is resolved in two directions at right angles to one another so as to give the "probable resolved displacements" along those directions, the resulting components are generally not independent of each other but are more or less correlated. In these circumstances, he shows that, in order to resolve a probable error of position in any particular direction, say along the direction of the main axes of co-ordinates, it is not sufficient to know only the probable displacements in *any* two directions at right angles to one another. A complete and rigorous solution demands that the directions of certain rectangular axes, known as the "axes of error," or the "principal axes," along which the probable displacements are a maximum and a minimum, as well as the values of these probable displacements, should be known or determined. These axes can often be chosen from principles of symmetry, and those taken for R_x and R_y in the formulæ given in

Vol. 1, page 238, are the principal axes. For further information on this point, and its application to the adjustment of traverse networks, see Yates' paper mentioned above.

EXAMPLES

1. Observations of three angles and their sum give :—

	°	′	″	
$a =$	40	32	12·4,	weight 1,
$b =$	49	07	50·6,	1,
$c =$	60	22	36·3,	2,
$a + b + c =$	150	02	38·0,	2.

Adjust the angles.

2. Adjust the following station observations closing the horizon :—

	°	′	″	
$a =$	65	18	30·2,	weight 2,
$b =$	78	37	12·3,	1,
$c =$	72	48	02·5,	1,
$d =$	59	00	49·2,	3,
$e =$	84	15	22·7,	1.

3. At a station O in a triangulation survey the following results were obtained :

Angle	Observed Value °	′	″	Weight
AOB	67	14	32·4	1·1
BOC	75	36	21·5	1·0
COD	59	56	02·0	·7
DOE	83	24	17·1	·8
EOA	73	48	45·0	1·4

The weights are proportional to the reciprocals of the squares of the probable errors. Adjust the angles. (R.T.C., 1915.)

4. Adjust the following station observations :

	°	′	″	
$a =$	54	12	40·7,	weight 2,
$b =$	46	31	15·4,	2,
$a + b =$	100	43	53·8,	1,
$c =$	69	22	31·2,	1.
$b + c =$	115	53	49·0,	1.

5. Find the most probable values of the angles a, b, and c from the following observations of equal weight :

	°	′	″
$a =$	48	27	11·04,
$b =$	56	40	30·22,
$a + c =$	105	07	40·65,
$b + c =$	96	36	59·87,
$c =$	39	56	28·24,
$a + b + c =$	145	04	11·33

Find the probable error of an angle.

6. Neglecting spherical excess, adjust the angles of a triangle of which the observed values are :

	°	′	″	
$a =$	49	17	23·2,	weight 3,
$b =$	75	32	46·7,	1.
$c =$	55	09	53·1,	3.

7. Eight measures of an angle give, in the seconds, 10·33, 5·10, 7·23, 9·90, 6·17, 10·07, 6·43, 9·97. Compute the probable error of the arithmetic mean and of a single measure.

8. From the following results, obtained in the measurement of a base line six times by means of a standardised steel tape on the prepared surface of the ground, calculate the probable error of the mean :

3050·53, 3050·26, 3050·48, 3050·19, 3050·22, 3050·36 feet.

State the precautions which would have to be taken to ensure this degree of consistency in the results. (R.T.C., 1914.)

9. A secondary angle is measured by A and B each eight times. A's observations in the seconds are 2·4, 5·0, 3·6, 0·8, 7·5, 8·7, 3·2, 9·6; and B obtains 6·8, 3·0, 5·4, 9·6, 2·8, 10·4, 7·6, 8·0. Compare the weights of their results, and compute the most probable value.

10. Latitude observations are taken at intervals in a long route traverse running approximately north and south. If the linear error of the traverse is of the order $\frac{1}{400}$, at what intervals should a check be made by a latitude observation if the probable error of the latter is $\pm$ 3″ ?

1″ of meridian arc may be taken as equivalent to 101 ft.

11. An observer A measures an angle 32 times with a mean result of 54° 27′ 41″·62, and B measures it 16 times with the result 54° 27′ 40″·05. The sums of the squares of their residual errors are respectively 331·50 and 125·35 in seconds units. Find the most probable value of the angle.

12. In measuring a base line two field tapes were used. No. 1 was suspended 105 times and No. 2 86 times. The probable error in length of No. 1 tape when reduced to the field temperature was ·00012 ft., and that of No. 2 was ·00015 ft. Calculate the probable error in the length of the base due to uncertainty in the length of the tapes.

13. Measurement of the angles of two triangles having a common side BC gives:

	°	′	″		°	′	″
$A =$	54	17	28·2,	$C_2 =$	71	25	14·1,
$B_1 =$	65	02	36·0,	$B =$	111	14	38·2,
$C_1 =$	60	39	57·2,	$C =$	132	05	13·4,
$B_2 =$	46	12	03·8,	$D =$	62	22	44·0.

Adjust the angles.

14. Adjust by the method of correlatives the following observed values of the angles of a geodetic quadrilateral (Fig. 95):

	°	′	″		°	′	″
$a =$	54	30	02·7,	$e =$	60	19	22·8,
$b =$	42	23	34·2,	$f =$	48	11	17·8,
$c =$	37	40	12·5,	$g =$	31	52	31·5,
$d =$	39	36	46·6,	$h =$	45	26	08·3.

Calculate the probable error of an angle and from this, and given that the probable error of the length of the side AB is $\pm$ 0·39 ft. and that the length of AB is 123481·64 ft., find the probable error of the side CD.

15. Make an approximate adjustment of the preceding case.

16. Compute the unknown sides of the triangle ABC of which the side AC = 41,345·30 m. and the observed values of the angles are: A = 51° 17′ 11″·4, B = 58° 50′ 20″·6, C = 69° 52′ 35″·9. Take the weight of the measure of C as twice that of the others.

(For references on Survey Adjustment see references at end of next chapter.)

CHAPTER V

GEODETIC COMPUTATION

CALCULATION OF TRIANGULATION

Computation of Lengths of Sides of the Triangles. Legendre's Theorem. Adjustment of the angles of the triangles is followed by the calculation of the lengths of the sides from the base outwards. The triangles, of course, are triangles on the sphere, or rather on the spheroid (page 317), and not on the plane. Hence, it might be thought that methods of plane trigonometry could not be used and that it would be necessary to solve the triangles as spherical triangles, with some modifications perhaps for the spheroid. However, any triangle, even the largest that is ever observed in ordinary geodetic surveying, is very small when compared with the total area of the earth, and the sides subtend very small angles at the earth's centre. In such circumstances, Legendre has shown that the solution of the observed triangles can be effected on the sphere by the methods and formulæ of ordinary plane trigonometry if the following rule, called "Legendre's Rule," is used :—

If one-third of the spherical excess of the triangle is deducted from each angle, the triangle can be solved, in terms of the linear lengths of the sides, by the ordinary rules of plane trigonometry.

It may also be shown that the errors due to applying the rule to small triangles on the spheroidal earth are negligible.

To prove this formula for the sphere replace the linear values of the sides by the angles which they subtend at the earth's centre, and find the expression for $\cos A$ in terms of the sides of the spherical triangle. Thus :—

$$\cos A = \frac{\cos\frac{a}{r} - \cos\frac{b}{r}\,.\,\cos\frac{c}{r}}{\sin\frac{b}{r}\,.\,\sin\frac{c}{r}}$$

$$= \frac{\left(1-\frac{a^2}{2r^2}+\frac{a^4}{24r^4}-\ldots\right)-\left(1-\frac{b^2}{2r^2}+\frac{b^4}{24r^4}-\ldots\right)\left(1-\frac{c^2}{2r^2}+\frac{c^4}{24r^4}-\ldots\right)}{\left(\frac{b}{r}-\frac{b^3}{6r^3}+\ldots\right)\left(\frac{c}{r}-\frac{c^3}{6r^3}+\ldots\right)}$$

which, after simplification and neglect of powers higher than the fourth, gives :—

$$\cos A = \left[\frac{b^2+c^2-a^2}{2bc} - \frac{1}{6}\,\frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4r^2bc}\right].$$

In the plane triangle with sides a, b and c,

$$\cos A' = \frac{b^2+c^2-a^2}{2bc}$$

$$\therefore\ \sin^2 A' = 1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2$$

$$= \frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4b^2c^2}$$

Hence,
$$\cos A = \cos A' - \frac{bc}{6r^2}\sin^2 A'.$$

Let $A = A' + x$, where x is a small angle. Then

$$\cos(A' + x) = \cos A' - x \sin A' \quad . \ . \ . \ . = \cos A$$

$$\therefore \ \cos A' - x \sin A' = \cos A' - \frac{bc}{6r^2}\sin^2 A'$$

$$\therefore \ x = \tfrac{1}{6}\frac{bc}{r^2}\sin A'$$

$= \frac{1}{3}\epsilon$, where ϵ is the spherical excess of the triangle.

$$\text{Hence, } A = A' + \tfrac{1}{3}\epsilon,$$
$$A' = A - \tfrac{1}{3}\epsilon.$$

In practice, as noted on page 282, the value of r will vary with latitude owing to the spheroidal form of the earth, and the one to be used when calculating ϵ may be assumed to be the mean radius of curvature at the latitude of the centre of the triangle, which may be taken to be $\sqrt{RN}$, where R is the radius of curvature of the meridian at this latitude and N is the length of the corresponding normal to this meridian, the value of $\sqrt{RN}$ or of $1/RN \, . \sin 1''$ being found from special tables. Hence, for ϵ expressed in seconds, we have as in page 283,

$$\epsilon'' = \tfrac{1}{2} \cdot \frac{bc}{RN \sin 1''} \cdot \sin A,$$

with

$$A = A' + \tfrac{1}{3}\epsilon'',$$
$$A' = A - \tfrac{1}{3}\epsilon''.$$

If higher order spherical terms are taken into account it can be shown that $A - A'$ is given by

$$= \frac{bc \sin A}{6r^2 \sin 1''}\left[1 + \frac{(7b^2 + 7c^2 + a^2)}{120r^2}\right],$$

with similar expressions for the differences at B and C. Here, the second term on the right is very small. Thus, Jordan gives an example in which one side is about 65 miles in length, and the spherical excess of the triangle is about 15″ of arc. In this triangle, the greatest discrepancy between the approximate (*i.e.* computed from the simple formula with one term only) and the exact (*i.e.* including the second term) values of a plane angle is 0″·000136.*

By adding the differences for all three angles we note that the spherical excess of the triangle when the second term is taken into account becomes

$$\epsilon'' = \tfrac{1}{2}\frac{bc \sin A}{r^2 \sin 1''}\left[1 + \frac{a^2 + b^2 + c^2}{24r^2}\right],$$

the second term being negligible in ordinary practical work.

As regards the difference between the true spheroidal and the spherical angles, Clarke, in his *Geodesy*,† takes the case of a triangle having a side

* *Handbuch der Vermessungskunde*, Vol. III, 7th (Enlarged) Edit., 1923, page 260.

† *Geodesy*, 1880 Edition, page 122. It is to be noted here that the "spheroidal angles" used by Clarke are the angles between the tangents to the "plane curves" or "curves of normal section" at each station (pages 326 and 387), as these curves are related to the lines of sight. The comparison in the footnote on page 316 between the spherical and the geodesic angles refers to angles between the tangents to the geodesics, or curves of shortest distance (page 388), at each station. For ordinary lines that can be sighted over, the difference between these angles is negligible.

of over 200 miles in length and with a spherical excess of 1′ 36″·426 and obtains the following spheroidal and spherical values :—

	Spheroidal	Spherical	Diff.
A =	98° 44′ 37″·0965	98° 44′ 37″·1899	− 0″·0934
B =	58 16 46 ·5994	58 16 46 ·4737	+ 0 .1257
C =	23 00 12 ·7303	23 00 12 ·7634	− 0 ·0331
ϵ =	1 36 ·4262	1 36 ·4270	

These examples indicate that the theorem is sufficiently accurate for all cases that can occur in ordinary practical trigonometrical work where the sizes of triangles are limited by the distances over which visual observations are possible, and that it is only necessary to use the first term in the formula for spherical excess.

It may be noted here that, for ordinary triangles, when the plane angles of a triangle are given and the spherical angles are required, these are found by adding one-third of the spherical excess of the triangle to each plane angle.

When the observed angles of the triangle have been reduced to plane angles by deducting one-third of the spherical excess from each and then adjusted for closure, the lengths of the sides b and c are computed from the known side a by the ordinary sine rule :—

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Seven-figure logarithms are used in computing the lengths of the sides of primary triangulation, and six-figure logarithms for minor systems. A computing machine can, of course, be used if a good table of natural sines is available. Eight-figure logarithms, however, are required if distances over about 60 to 70 miles are involved.

Example. The adjusted angles of an observed triangle, whose spherical excess is 4″·26, are as follows :—

A =	37° 05′ 38″·06
B =	37 42 17 ·49
C =	105 12 08 ·71
Sum =	180 00 04 ·26 = 180° plus spherical excess.

The logarithm of the side AC is 5·138 1130. Find the logarithms of the other two sides.

The adjusted angles after deducting one-third of the spherical excess from each, are :—

A =	37° 05′ 36″·64
B =	37 42 16 ·07
C =	105 12 07 ·29
Sum =	180 00 00

Hence, adopting the scheme of solution given in Vol. I, page 257 :—

log BC	= 5·132 0557
log sin A	= 9·780 4021
log cosec B	= 0·213 5406
log AC	= 5·138 1130
log sin C	= 9·984 5305
log AB	= 5·336 1841

If the system consists of more than a chain of single triangles, and the individual figures have not been adjusted by least squares, each side can be computed from two or more triangles. In general, the conditional equations for closure of the sides will not be fulfilled so that different results for the same length will be obtained. In such a case it is usual to adopt the mean value for each side.

Computation of the Angles of Triangles when the Lengths of the Sides are Measured by Radar or Other Means. In radar triangulation the lengths of the sides, and not the angles, are measured. In this case the angles of the plane triangle with sides equal in length to the measured sides can be computed from the formula

$$\cos A' = \frac{b^2 + c^2 - a^2}{2\,bc},$$

or, when logarithms are used,

$$\tan \tfrac{1}{2} A' = \frac{p}{(s - a)},$$

where

$$s = \tfrac{1}{2}(a + b + c),$$

and

$$p = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}},$$

with similar expressions for B' and C' and the check $C' = 180° - (A' + B')$.

These formulae will give the angles of the plane triangle, but, in order to turn these plane angles into angles on the spheroid which can be used for calculating azimuths and geographical positions, we must apply a correction similar to the Legendre correction in ordinary triangulation for the difference between the plane and the spherical angles. For ordinary sized triangles this means adding one-third of the spherical excess of the triangle, as computed by the simple formula, to each computed angle, but in radar triangulation the triangles are very large, with very long sides, and here the correction consists of two parts :—

(1) A correction for the difference between the computed plane and the spherical angles.

(2) A correction for the difference between the spherical angles and the geodesic angles, or angles measured between tangents to the geodesics (page 388) to the distant points.

The first correction is given by the full formula involving spherical excess given on page 312, *viz.*

$$(A - A')'' = \frac{bc \sin A}{6RN \sin 1''}\left[1 + \frac{(7b^2 + 7c^2 + a^2)}{120RN}\right],$$

where A is the spherical angle and A' the plane angle at the apex A.

The second correction is given by

$$(A_G - A)'' = \frac{\epsilon'' e^2}{6} \sin(\phi_m + \phi_1)\sin(\phi_m - \phi_1),^*$$

* See "The Clarke Formulæ for Latitude, Longitude and Reverse Azimuths, with Corrective Terms for Use on Very Long Lines," by H. F. Rainsford. *Empire Survey Review*, Vol. VIII, No. 56, page 60.

where A_G is the geodesic angle ; ϵ'' is the spherical excess of the triangle in seconds of arc ; e is the earth's eccentricity, $[(a^2 - b^2)/a^2]^{\frac{1}{2}}$ (page 317) ; ϕ_m is the mean latitude of the triangle and ϕ_A is the latitude of the point A. This second correction is very small numerically even for triangles with 300 to 500 mile sides, and the second term in the formula for $(A - A')$ is also very small.* Hence these terms are not appreciable with triangles of ordinary size but may have to be taken into account in the case of the large triangles used in radar work.

Station Positions. The calculation of the triangle sides gives the position of each station relatively to those adjacent. These data are now applied to computing the co-ordinate positions of the points in order to facilitate their plotting for mapping purposes and to form a record of the results for geodetic and other purposes.

In large surveys the station positions are expressed as geodetic or absolute positions in terms of the co-ordinates, latitude and longitude, as well as the azimuths of the lines joining them. The minimum data required, in addition to the relative positions, are the latitude and longitude of one station, the azimuth of one line from that station, and the dimensions of the earth. By proceeding from the known station, and computing from station to station by means of the formulæ to be given, the required quantities are obtained for the whole system.

If the survey forms the control for large-scale mapping, it is almost essential to have the positions expressed in terms of some system of linear co-ordinates. Innumerable forms of these may be devised but the most commonly used are "rectangular spherical," or "Cassini-Soldner," co-ordinates (page 341), and a modification of these known as the "Transverse Mercator" system (page 354). Rectangular spherical co-ordinates may be employed, to the exclusion of geographical co-ordinates, for isolated surveys extending to several thousand square miles. They have, however, certain disadvantages which those on the Transverse Mercator system do not have. This latter system has accordingly come much into favour during recent years and is at present in use in the United Kingdom, in South Africa, in many of the Crown Colonies, and in parts of the United States. It is more particularly suited to an area which has a considerable extent in latitude but a more limited extent in longitude. A third system, known as the "conical orthomorphic" or as "Lambert's second," is also used in France, the United States, Canada and other countries. It is particularly suitable for countries with a considerable extent in longitude but a limited extent in latitude.

Since the calculation of station positions involves a knowledge of the form and dimensions of the earth, the principal facts relating thereto will first be given.

The Figure of the Earth. By the figure of the earth is meant the form of the equipotential surface corresponding to mean sea level. This is nearly, but not quite, a sphere, the principal departure from an exactly

* For an equilateral triangle of 300-mile sides in latitude 45° ($\epsilon'' = 2' \; 50''\cdot63$ approximately) $(A_G - A)$ only amounts to $0''\cdot022$ and the second term in the formula for $(A - A')$ only amounts to $0''\cdot0012$.

spherical form occurring in a flattening at the poles. This circumstance was deduced by Newton, and announced in the *Principia* in 1687. Verification of Newton's hypothesis by measurement was first achieved by the French Academy, who despatched two expeditions, one to Peru in 1735, and the other to Lapland in 1736, to determine the length of a degree of latitude near the equator and the arctic circle respectively. The results showed that the northern degree was the greater. It was not, however, until the close of the eighteenth century that instruments and methods attained a degree of refinement sufficient for the needs of figure determination. Since then much geodetic work has been accomplished in Europe, America, India, and Africa, and knowledge of the dimensions of the earth has steadily grown.

The results of geodetic measurements show that the earth very closely approximates to an oblate spheroid, which is the solid generated by rotation of an ellipse about its minor axis. The actual figure deviates slightly and irregularly from a true spheroid, and this is recognised by giving it the name "geoid." Since, however, geodetic computations can be made with sufficient precision on the assumption of a spheroidal form, figure determinations are directed to ascertaining the dimensions of the spheroid which most nearly coincides with the actual figure.

Properties of the Spheroid. Let the ellipse of Fig. 116 represent a meridian section of the earth, and let A be a station on the meridian PQP′E. The major axis, or equatorial diameter ($2a$) is represented by EQ, and the minor or polar axis ($2b$) by PP′. The ratio $\dfrac{a-b}{a}$ is called the compression c. ABD is the normal to the ellipse at A, and coincides with the direction of the plumb line at A, if the latter is free from local deviation. The angle ϕ between the normal and the equator is the geodetic latitude of A, and its geocentric latitude is represented by the angle ψ between the radius vector CA and the equator. The radius of curvature R of the ellipse at A lies along AD, the centre of curvature being situated between B and D.

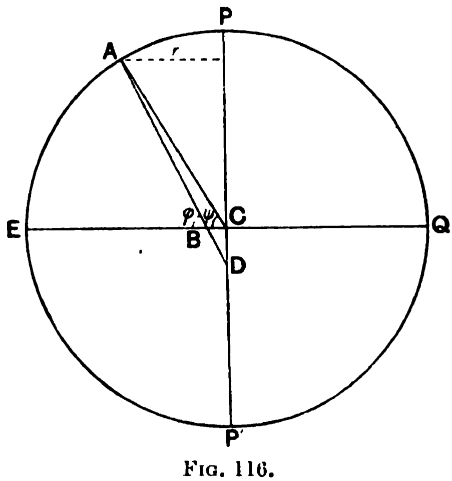


FIG. 116.

Let e = the eccentricity $= \sqrt{\dfrac{a^2 - b^2}{a^2}}$,

N = AD,

n = AB,

r = the radius of the parallel of latitude through A

then for a meridian section,

$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}};$$

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}};$$

$$n = (1 - e^2)N;$$

$$r = N \cos \phi;$$

$$\psi = \tan^{-1} (1 - e^2) \tan \phi.$$

All of the above formulæ can easily be derived from the equation of the ellipse :—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If any section of the spheroid, other than the meridian section, be made by a vertical plane passing through A, the radius of curvature R_A at A for the given azimuth is given by Euler's formula :

$$R_A = \frac{NR}{N \cos^2 A + R \sin^2 A},$$

where R and N = the quantities formulated above,
A = the azimuth of the vertical plane through A.

In particular, if $A = 90°$ or $270°$, $R_A = N$ = the radius of curvature of the prime vertical section. The mean radius of curvature at A, or the mean of the radii for all azimuths, is $\sqrt{NR}$.

R and N are both functions of the latitude of the place and hence vary for different latitudes. They are quantities that enter very frequently into geodetic computations. Hence, most geodetic tables give values for them, generally in the forms $\log \frac{1}{R \sin 1''}$, $\log \frac{1}{N \sin 1''}$, $\log \frac{1}{2RN \sin 1''}$. In these tables, and in many text-books, R is often denoted by the Greek ρ (rho) and N by the Greek ν (nu).

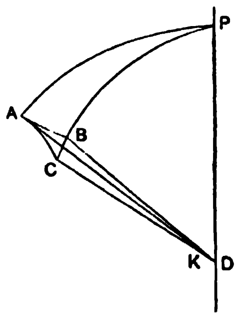
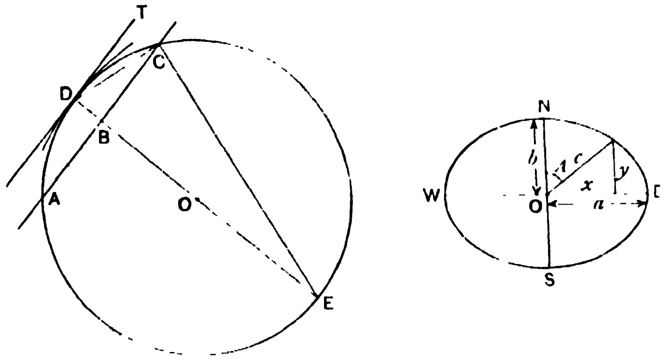


Fig. 117.

To show that the normal N is also the radius of curvature of the prime vertical section, consider Fig. 117, in which A is any point on the meridian AP and let AC be a very small portion of the curve traced out on the surface of the spheroid by a prime vertical section at A. The normals in this vertical section to the curve at the points A and C intersect one another at some point K on DA. Through C draw the meridian section CBP and through A the parallel of latitude AB to cut CBP at B. Then the normals to the meridians at A and B intersect the minor (polar) axis at D. CK, although normal to the curve AC, is not necessarily normal to the surface of the spheroid at C, nor does CK intersect BD, the normal to the surface at B, as may seem to be the case in the figure. Let C move closer and closer to A. As it does so, C and B approach each other, and at the same time the intersection of the normals to the curve at A and C approaches the centre of curvature of the curve. The arc CB is very small compared with the arc AC, and, in the limit, C coincides with B, and the normal KC, which now becomes normal to the surface, coincides with BD. Hence, D becomes the centre of curvature of the curve, and AD is the radius of curvature of the prime vertical section at A.

It is also easy to prove the formulæ for the radius of curvature in any azimuth. In Fig. 118 (a) let D be any point of the surface of the spheroid and at D draw the tangent plane. Let DT be the line where this plane intersects the plane of the paper. Take a point B on the normal at D infinitesimally close to D but just below



(a) (b)

FIG. 118.

the surface of the spheroid, and through B draw a plane parallel to the tangent plane and intersecting the plane of the paper in the line ABC. This plane will cut the surface of the spheroid along an ellipse (the *indicatrix*) as shown in Fig. 118 (b). Now let the plane of the paper represent the plane through the meridian at D. Let O be the centre of curvature of the meridian at D, and, with O as centre, draw the circle of curvature ADCE through the point D. The plane drawn through B will intersect this circle in the points A and C. Join DC and CE. Then, from the similar triangles DBC and CBE,

$$\frac{DB}{BC} = \frac{BC}{BE}$$

$$\therefore DB = \frac{BC^2}{BE} = \frac{BC^2}{2R} \text{ approximately.}$$

Since D and B are very close together, and ADCE is the circle of curvature at D, the arc ADC of the circle will, in the limit, coincide with the arc of the meridional section at D, and BC will be the semi-axis minor, b, of the small ellipse WNES, Fig. 118 (b), traced out by the plane already drawn through ABC. Hence :—

$$DB = \frac{b^2}{2R}.$$

Similarly, if a prime vertical section is taken at D, and the circle of curvature drawn with radius N, this circle will intersect the plane through ABC in the points W and E, Fig. 118 (b), in which OE $= a$, the semi-axis major of the ellipse WNES and we shall have as before :—

$$DB = \frac{a^2}{2N}.$$

Similarly, for a section in azimuth A, we would get :—

$$DB = \frac{c^2}{2R_A}.$$

Hence,

$$\frac{c^2}{b^2} = \frac{R_A}{R} \text{ and } \frac{c^2}{a^2} = \frac{R_A}{N}.$$

But, from Fig. 118 (b), $x = c \cdot \sin A$ and $y = c \cdot \cos A$, and, from the equation of an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence,

$$\frac{c^2 \sin^2 A}{a^2} + \frac{c^2 \cos^2 A}{b^2} = 1.$$

$$\therefore \frac{R_A \sin^2 A}{N} + \frac{R_A \cos^2 A}{R} = 1.$$

$$\therefore R_A = \frac{NR}{R \sin^2 A + N \cos^2 A}.$$

The mean R_A is given by :—

$$R_M = \frac{1}{2\pi}\int_0^{2\pi} R_A dA$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{c^2}{b^2} . RdA.$$

But

$$\int_0^{2\pi} c^2 dA = 2 \times (\text{area of ellipse}) = 2\pi ab$$

$$\therefore R_M = \frac{a}{b} . R.$$

$$= \sqrt{\frac{N}{R}} . R . = \sqrt{NR}$$

Distances along a Meridian. In many geodetic calculations it is necessary to know the linear distance between two latitudes as measured along the arc of a meridian. This distance may be found from the formula :—

$$s = a(1 - e^2)[A(\phi_2 - \phi_1) - \tfrac{1}{2}B (\sin 2\phi_2 - \sin 2\phi_1) + \tfrac{1}{4}C(\sin 4\phi_2 - \sin 4\phi_1) - \tfrac{1}{6}D . (\sin 6\phi_2 - \sin 6\phi_1) + \ldots]$$

where $\phi_2 - \phi_1$ is in radians and

$$A = 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \ldots = 1{\cdot}005\ 1093,$$

$$B = \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \ldots = 0{\cdot}005\ 1202,$$

$$C = \frac{15}{64} e^4 + \frac{105}{256} e^6 + \ldots = 0{\cdot}000\ 0108,$$

$$D = \frac{35}{512} e^6 + \ldots = 0{\cdot}000\ 0000,$$

the numerical values being for the Clarke (1866) figure of the earth.

The proof of this formula is as follows :—

$$ds = Rd\phi$$

$$= \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} . d\phi$$

$$= a(1 - e^2)(1 + \frac{3}{2}e^2 \sin^2 \phi + \frac{15}{8}e^4 \sin^4 \phi + \frac{35}{16} e^6 \sin^6 \phi + \frac{315}{128} e^8 \sin^8 \phi + \ldots)d\phi.$$

But

$$\sin^2 \phi = \frac{1}{2}(1 - \cos 2\phi)$$

$$\sin^4 \phi = \frac{1}{4}(1 - \cos 2\phi)^2$$

$$= \frac{1}{4}(1 - 2\cos 2\phi + \cos^2 2\phi)$$

$$= \frac{1}{8}(3 - 4\cos 2\phi + \cos 4\phi)$$

$$\sin^6 \phi = \sin^2 \phi \sin^4 \phi$$

$$= \frac{1}{32}(10 - 15\cos 2\phi + 6\cos 4\phi - \cos 6\phi)$$

which gives :—

$$s = a(1 - e^2)\int_{\phi_1}^{\phi_2}(A - B\cos 2\phi + C\cos 4\phi - D\cos 6\phi + \ldots)d\phi$$

$$= a(1 - e^2)\Big[A\phi - \frac{1}{2}B\sin 2\phi + \frac{C}{4}\sin 4\phi - \frac{D}{6}\sin 6\phi + \ldots\Big]_{\phi_1}^{\phi_2}$$

Geodetic tables usually include a table of meridional distances. Table IV in the first edition of the War Office publication *Survey Computations* gives these in feet for 10′ intervals of latitude from 0° to 70°. These tables are for the Clarke 1858 figure of the earth (page 323), and give the meridional distances to the nearest decimal place of a foot, but, in the second edition of the book, this table has been omitted and H.M. Stationery Office now publishes one which gives these distances, at 10′ intervals of latitude and to the second decimal place of a foot, for the Clarke 1880 figure.* The Royal Geographical Society also publishes geodetic tables for the Clarke 1880 and Madrid 1924 figures, meridional distances in these, however, being in metres.† These tables are complete up to 90° in latitude.

Figure Determination. The results of geodetic surveys are applied to determining the dimensions of the spheroid (1) by comparing the linear and angular values of arcs, (2) by discussing areas. The form, but not the dimensions, of the spheroid may also be deduced by studying the variation of gravity by means of pendulum observations.

Determinations have usually been made from arcs. The length of meridian between two stations of a chain of triangles lying approximately north and south may be computed by projecting the appropriate intervening triangle sides on to the meridian. Astronomical observations will give the difference in latitude. Hence, the expression for the length of a meridional arc given on page 320 yields one equation in the two unknowns, a and e.

If the arc is a very short one, the linear distance divided by the angular

* *Geodetic Tables for the Clarke 1880 Figure in Feet for Latitudes 0° to 70°.*

† *New Geodetic Tables for Clarke's Figure of 1880 with Transformation to Madrid 1924 and other Figures.*

distance in circular measure will give the radius of curvature for the mean latitude of the stations, or :—

$$\frac{s}{d\phi \sin 1''} = R = \frac{a(1-e^2)}{(1-e^2 \sin^2 \phi_M)^{\frac{3}{2}}},$$

where s = the linear distance between the stations,
$d\phi$ = their difference of latitude in seconds,
ϕ_M = their mean latitude.

The dimensions of the spheroid are known if a and e are obtained. A second meridian arc, preferably in a considerably different latitude, must therefore be treated in the same way, and a and e are derived by simultaneous solution. The method is extended to utilise chains of triangulation oblique to the meridian.

In place of relying upon two arcs, several should be discussed by least squares to afford the most probable values of a and e. It is found that the results derived from different arcs are much more discrepant than can be accounted for by the small errors in the triangulation or the astronomical observations and, further, that the curvature of arcs of parallel is not constant. The disagreements are evidence that the actual figure is not exactly a spheroid, and indicate local deviations of the plumb line from the normal to the mathematical figure. These deviations are due to the irregular distribution of mass, not only at the surface of the earth but in the crust.

As already pointed out on pages 47 to 49, the results of astronomical observations are dependent upon the direction of the plumb line and therefore of the horizon at the place. By regarding the errors of primary triangulation and of astronomical observation as negligible, the discrepancies are utilised in discussing the form of the geoid and in determining the spheroid which most nearly resembles it over any portion of the earth.

In utilising the data of a survey for this purpose, a tentative spheroid is adopted for reference. Starting from the position of one of the stations as obtained astronomically, the geodetic positions of the other stations on the adopted spheroid are computed. The astronomical positions of several of these stations will have been observed, and a list of differences between astronomical and geodetic positions, or station errors, as well as those between observed and computed azimuths can be drawn up. The effects which would be produced upon these differences by applying corrections to the elements of the assumed spheroid and to the assumed position of the initial station are computed, and the most probable values of the corrections are deduced by least squares. The spheroid which best fits the area under discussion and the positions of the stations on that spheroid have now been determined. The station errors remaining after adjustment are the local deflections of the plumb line. They serve for the determination of the geoid, the surface of which is everywhere normal to gravity, and measure the inclination of its surface to that of the spheroid. In modern geodetic survey allowances may be made for the influence of local topographical features on the deflection of the plumb line, and the investigation may be facilitated, and its scientific value increased, by pendulum or torsion balance observations.

Results of Figure Determinations. As geodetic surveys have been extended, the amount of data available for figure determination has increased, and the earlier figures are subject to revision. In consequence, the more recent determinations are probably the more reliable, but different national surveys use those figures which best conform to the part of the surface with which they are concerned.

The results of some of the more important determinations of the spheroid are, in metres,

			a	b	c
1.	Everest,	1830	6,377.304	6,356,103	1/300·8
2.	Bessel,	1841	6,377,397	6,356,079	1/299·2
3.	Clarke,	1858	6,378,293	6,356,619	1/294·3
4.	Clarke,	1866	6,378,206	6,356,584	1/295·0
5.	Clarke,	1880	6,378,249	6,356,515	1/293 5
6.	Helmert,	1906	6.378,200	6,356,818	1/298 3
7.	Hayford,	1910	6,378,388	6,356,909	1/297·0

1 is still employed by the Survey of India. It was based only on an Indian arc and the French arc, and is known to be erroneous because of large irregularities of gravity in India. 2 has been extensively used in Europe. 4 has been employed in the United States Survey since about 1880, and subsequently was adopted by Canada. 6 is based on European, Indian, and African surveys; and 7 on work in the United States and on special investigations elsewhere. Their similarity shows that there is no great difference in the form of the eastern and western hemispheres.

In 1924, the International Union of Geodesy and Geophysics, meeting at Madrid, recommended the Hayford 1910 figure for general use, and this figure is now sometimes called the Madrid 1924 figure. It is unlikely however, that any country which has already done a considerable amount of geodetic work will alter the figures for the dimensions of the earth which it is now using.

Geodetic Tables. In order to carry out geodetic calculations expeditiously and conveniently it is necessary to have special tables based on the dimensions of the spheroid used in the country in which the survey is to be made. The fundamental quantities that are required are :—

(1) A table of meridional distances.

(2) A table giving values for R and N for different latitudes, or, in the form usually given, $\dfrac{1}{R\sin 1''}$, $\dfrac{1}{N\sin 1''}$ and $\dfrac{1}{2RN\sin 1''}$.

For the sake of convenience, the interval of tabulation should be so close that second-order differences will not normally be required. It is also desirable that the table should give the logarithms of functions of R and N as well as, or instead of, their natural values.

Among the tables available the following may be mentioned : "Auxiliary Tables of the Survey of India" (Everest Figure); Tables in Close and Winterbotham's *Text-book of Topographical Surveying* (Clarke's 1858 Figure); "Geodetic Tables for the Clarke 1880 Figure in Feet for Latitudes 0° to 70°" (H.M. Stationery Office); "New Geodetic Tables for Clarke's Figure of 1880 with Transformations to Madrid 1924 and Other Figures" (Royal Geographical Society); "Formulæ and Tables for the

Computation of Geodetic Positions" (United States Coast and Geodetic Survey).

In "Geodetic Tables for the Clarke 1880 Figure in Feet for Latitudes 0° to 70°," log R, log $\frac{1}{R \sin 1''}$, log N, log $\frac{1}{N \sin 1''}$, log $\frac{1}{2RN \sin 1''}$, $\frac{N}{R}$ and meridional distances are tabulated together on the same folio for every 10′ of latitude. Second differences are negligible except in the list of meridional distances, where the maximum second difference is 1·82 ft. at latitude 45°. The tables published by the Royal Geographical Society give meridional distances in metres, log $\frac{1}{R \sin 1''}$, log $\frac{1}{N \sin 1''}$ and log $\frac{1}{2RN \sin 1''}$, also in metres, for both the Clarke 1880 and the Madrid 1924 figures. The interval of tabulation is 10′ of latitude, and the tables are complete from 0° to 90° in latitude. Meridional distances are given to the third decimal place of a metre, and the logarithms of $\frac{1}{R \sin 1''}$ and $\frac{1}{N \sin 1''}$ to ten places and of $\frac{1}{2RN \sin 1''}$ to six places. The introduction contains formulæ for reduction from the Clarke 1880 to the Airy, Everest, Clarke 1858 and Bessel figures.

Calculation of Geodetic Positions. From the known latitude and longitude of a station A and the azimuth and distance from A to B, the co-ordinates of B are obtained by computing the differences of latitude and longitude to be applied to those of A.

The calculation of reverse azimuth is similarly made by computing the difference of azimuth to be applied to the azimuth from A to B to give that from B to A. This azimuth difference represents the convergence of the meridians or the angle between the meridian at A and that at B. Any two meridians are parallel at the equator, and, as they are traced towards the poles, the angle between them increases until at the poles it equals their difference of longitude. Because of convergence, the azimuth of any great circle arc other than the equator or a meridian varies throughout its length.

In computing positions, each should be obtained from two others as a check on the calculation. The computed azimuths of two lines from a station should differ by the angle between the lines, and the azimuths of any other lines from the station are obtained by application of the included angles. These angles are the spherical angles and not those used in computing the sides.

There are four methods in fairly common use for computing latitude and longitude and reverse azimuths from azimuth and distance. These are :—

(1) Clarke's formulæ for long lines.
(2) Clarke's formulæ for medium and short lines.
(3) Puissant's formulæ and modifications of them.
(4) The Mid-Latitude formulæ.

Of these, Clarke's formulæ for long lines are suitable for use with the

longest lines met with in ordinary geodetic surveying but they have the disadvantage that, for the particular conditions for which they are most suitable, they require the use of 8- to 10-figure logarithms. They are accurate for distances up to about 200 miles. The formulæ for medium and short lines are suitable for lengths up to about 70 miles and they are extensively employed in British practice.

Puissant's formulæ are convenient to use and may be adopted for lines up to about 70 or 80 miles. They are used extensively in India and in the United States for all but the longest sides.

The mid-latitude formulæ are simpler in some respects than any of the others, and, unless great accuracy is needed, may be used for ordinary work in latitudes less than 60° for lines not exceeding about 20 to 25 miles in length. Their main disadvantage is that the formulæ for the difference between the forward and reverse azimuths, the convergence, involves the tangent of a small angle, with resulting troublesome interpolation.

The notation to be used is as follows :

ϕ, L = known latitude and longitude of A,
ϕ', L' = latitude and longitude of B,
$d\phi = \phi' - \phi$,
$\phi_M = \dfrac{\phi + \phi'}{2}$ or $\dfrac{\phi + \phi_c}{2}$.
ϕ_c = latitude of C, the foot of the perpendicular drawn from B to the meridian through A,
$k = 90° - \phi$,
$dL = L' - L$,
A = known azimuth from A to B, reckoned clockwise from north,*
A' = azimuth from B to A, ,, ,, ,,
$dA = A' - (A + 180°)$,
s = linear distance from A to B,
R = radius of meridian section at A,
R_M = ,, ,, ,, at latitude ϕ_M,
N = normal at A,
N' = normal at B.
N_c = normal at C.

Clarke's Formulæ for Long Lines. The formulæ given by Colonel Clarke † are suitable for computing positions over the longest lines it is possible to observe visually from and to ground stations and were used for the longer lines of the Ordnance Survey and other surveys.

Let β and $\beta' \pm \zeta$ be the interior angles at A and B of the spherical

* In the first two editions of this book the convention adopted with regard to azimuths in the formulæ for the computation of latitudes and longitudes and reverse azimuths was to reckon them clockwise from south. This convention is that generally given in text-books in connection with these formulæ, and it is very commonly used in all classes of geodetic work. Most engineers and surveyors, however, are accustomed to reckoning azimuth clockwise from north, and, moreover, the practice of doing so is becoming more general, even for geodetic work, in the British Colonies and elsewhere. Consequently, it appears desirable to adopt this convention here.

† *Geodesy*, 1880.

triangle PAB (Fig. 119), where P is the earth's pole, and let θ be the angle, in seconds of arc, subtended by s at D (Fig. 116) and given by :—

$$\theta = \frac{s}{N \sin 1''} + \frac{e^2\theta^3 \sin^2 1'' \cos^2 \phi \cos^2 \beta}{6(1 - e^2)},$$

Also let ζ be a small correction computed from :—

$$\zeta = \frac{e^2\theta^2 \sin 1'' \cos^2 \phi \sin 2\beta}{4(1 - e^2)}.$$

Then,

$$\tan \tfrac{1}{2}(\beta' \pm \zeta - dL) = \frac{\sin \frac{1}{2}(k - \theta)}{\sin \frac{1}{2}(k + \theta)} \cot \frac{\beta}{2},$$

$$\tan \tfrac{1}{2}(\beta' \pm \zeta + dL) = \frac{\cos \frac{1}{2}(k - \theta)}{\cos \frac{1}{2}(k + \theta)} \cot \frac{\beta}{2},$$

$$d\phi = \frac{s \sin \frac{1}{2}(\beta' \pm \zeta - \beta)}{R_M \sin 1'' \sin \frac{1}{2}(\beta' \pm \zeta + \beta)}\left[1 + \frac{\theta^2 \sin^2 1''}{12} \cos^2 \tfrac{1}{2}(\beta' - \beta)\right].$$

Since the mid-latitude is unknown, R_M is obtained from a preliminary solution to determine an approximate mid-latitude.

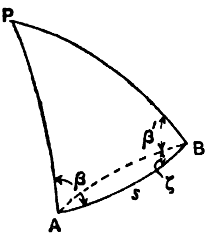


FIG. 119.

From the form of the expression which gives its numerical value, it is obvious that the angle ζ is due to the spheroidal form of the earth and that it would vanish for a spherical earth. It is, in fact, the angle at B between the line of sight from A to B and that from B to A, which, in the spheroid, do not coincide. The line of sight from A to B is the curve traced out on the surface of the spheroid by a plane which passes through B and contains the normal to the surface at A. This plane does not contain the normal to the surface at B. Similarly, the line of sight from B to A

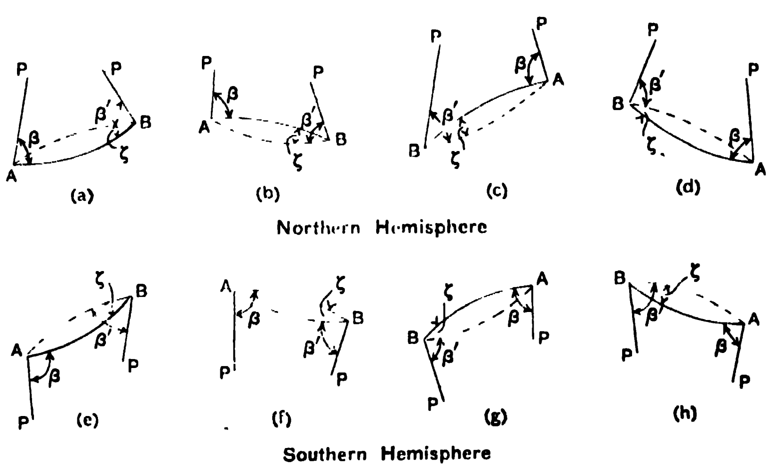


FIG. 120.

is the curve traced out on the surface of the spheroid by a plane which passes through A and contains the normal to the surface at B. This plane does not contain the normal to the surface at A. In Fig. 120 the line of sight from A to B is shown by the full line and that from B to A by the dotted line, and the angle at B between the two curves is the small angle ζ.

The relationship between the two curves and the angles β' and ζ for each of the four quadrants in the northern and southern hemispheres will be understood by reference to Fig. 120 (a) to (h). In each diagram the line of sight from A to B is shown by the full line and that from B to A by the dotted line. In each case P shows the position of the pole, and the angle $\beta' \pm \zeta$ at B to be used in solving the triangle is the angle between BP and the full line BA.

Reckoning azimuths in both hemispheres clockwise through east from north, we can form the following table for deriving the azimuths and for determining the signs to be given to ζ and to $d\phi$ when applying the above formulæ :—

Azimuth from A to B between.	Northern Hemisphere				Southern Hemisphere			
	$A =$	$A' =$	Sign of ζ	Sign of $d\phi$	$A =$	$A' =$	Sign of ζ	Sign of $d\phi$
0° and 90°	β	$360° - \beta'$	+	+	$180° - \beta$	$180° + \beta'$	−	−
90° and 180°	β	$360° - \beta'$	−	−	$180° - \beta$	$180° + \beta'$	+	+
180° and 270°	$360° - \beta$	β'	−	−	$180° + \beta$	$180° - \beta'$	+	+
270° and 360°	$360° - \beta$	β'	+	+	$180° + \beta$	$180° - \beta'$	−	−

Thus, the signs of ζ are identical with those of $d\phi$.

In all cases, dL is the difference in longitude as indicated by the angle subtended by the line AB at the pole P, and its sign is always to be taken as positive when using the formulæ for the solution of the spherical triangle PAB. It is easy to see in which way to apply it to the longitude of A to give the longitude of B. The latitude ϕ, whether north or south, is to be taken as positive, so that a positive $d\phi$ increases the numerical value of ϕ.

The small angle ζ is a maximum when A is 45°, 135°, 225° and 315°. Its value for a line in latitude 0°, whose azimuth is 45° and length is 100 miles, is only 0″·2243, and, for the same azimuth, it varies approximately as the square of the length of the line and as the square of the cosine of the latitude.

The proof of the formulæ giving the spheroidal terms is rather difficult and will not be given here. It will be found in books such as Clarke's or Hosmer's *Geodesy*. For the sphere, the formulæ are derived at once from the rules of ordinary spherical trigonometry.

Clarke's Formulæ for Medium and Short Lines. These formulæ were used for the computation of the shorter sides of the Ordnance Survey

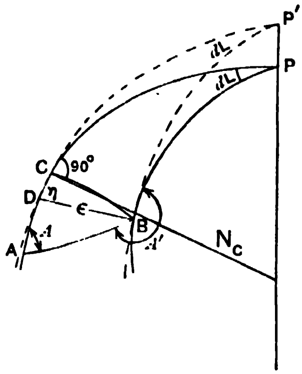


FIG. 121.

Triangulation, and are well known to, and extensively used by, British surveyors all over the world. They ignore the small angle ζ between the lines of sight from the two ends of the line, which is taken into account in the formulæ for long lines, as this angle is inappreciable in the cases to which the formulæ are intended to apply.

In Fig. 121, P is the earth's pole, CB is the curve traced out by a prime vertical section which passes through B and is perpendicular to the meridian AP at C. PD is equal to PB, η is the distance CD,* and ϵ is the spherical excess of the right-angled spherical triangle ACB. Clarke's formulæ for medium and short lines then become :—

$$\epsilon = \frac{s^2 \sin A \cos A}{2R_M \,.\, N_M \sin 1''} \quad \dots \dots \dots \quad (1)$$

$$\eta = \epsilon \tan A \tan \phi_C \quad \dots \dots \dots \quad (2)$$

$$d\phi = \pm \frac{s \cos (A \mp \tfrac{2}{3}\epsilon)}{R_M \sin 1''} - \eta \quad \dots \dots \dots \quad (3)$$

$$dL = \pm \frac{s \sin (A \mp \tfrac{1}{3}\epsilon)}{N_C \sin 1''} \,.\, \sec (\phi' + \tfrac{1}{3}\eta) \quad \dots \dots \dots \quad (4)$$

$$dA = A' - (180^\circ + A) = \pm\, dL \sin (\phi' + \tfrac{2}{3}\eta) \mp \epsilon \quad \dots \quad (5)$$

in which R_M is the R for $\frac{1}{2}(\phi + \phi_C)$.

In order to use these equations it is necessary to compute preliminary values for ϕ_C and ϕ_M. These can be obtained from :—

$$\phi_C = \phi + \frac{s \cos A}{R \sin 1''},$$

$$\phi_M = \tfrac{1}{2}(\phi + \phi_C).$$

When azimuths in both hemispheres are reckoned clockwise from north and dL is taken positive eastwards and negative westwards, we have, after taking into account the signs of the trigonometrical functions involved, the following rules for the $\pm$ and $\mp$ signs :—

Term	A			
	0°–90°	90°–180°	180°–270°	270°–360°
First term in $d\phi$	+	−	−	+
ϵ	−	+	−	+
dL	+	+	−	−
First term in dA	+	+	−	[illegible]

* McCaw has shown that η is also the spherical excess of a triangle supplementary to the triangle ACB and formed between vertices which are the poles of the arcs AP and BP and of a great circle through B perpendicular to BC (see page 330).

In the southern hemisphere, the same conventions will hold if southern latitudes are given a negative sign, except that, as $\tan \phi'$ and $\sin (\phi' + \frac{2}{3}\eta)$ will then be negative, the signs of η and of the first term in dA must be reversed.

In formulæ (1) to (5) the upper of the $\pm$ or $\mp$ signs is the sign for the generalised formulæ with A in the first quadrant. The signs for the other quadrants as given in the table follow by giving the appropriate signs to the trigonometrical functions of A, or of the angles containing A.

It will be noted that the sign of η does not change with A. This follows from the fact that the signs of ϵ and $\tan A$ are always the same.

To prove the formulæ, describe an auxiliary sphere of radius N_c tangential to the spheroid at C (Fig. 121). As the triangle ACB is very small, the points A, D and B on the spheroid will be practically coincident with the equivalent points on the sphere, and the *linear* measures of the arcs AB, AD and DB will be approximately equal on the two surfaces. Let P′ be the point where the minor axis of the spheroid meets the surface of the sphere. In the figure, the full lines PA and PB are the meridians through A and B on the spheroid and the pecked lines P′A and P′B the meridians on the sphere.

Using Legendre's theorem to solve the spherical triangle ACB as a plane triangle :—

Plane angle at A $= A - \frac{1}{3}\epsilon$.

" " " C $= 90^\circ - \frac{1}{3}\epsilon$.

" " " B $= 180^\circ - (A - \frac{1}{3}\epsilon) - (90^\circ - \frac{1}{3}\epsilon) = 90^\circ - (A - \frac{2}{3}\epsilon)$.

where ϵ is the spherical excess of the triangle.

$$\therefore \quad \frac{AC}{\sin\left\{90^\circ - (A - \frac{2}{3}\epsilon)\right\}} = \frac{s}{\sin (90^\circ - \frac{1}{3}\epsilon)}.$$

$$\therefore \quad AC = \frac{s \cos (A - \frac{2}{3}\epsilon)}{\cos \frac{1}{3}\epsilon}.$$

But $\frac{1}{3}\epsilon$ is a very small angle so that we can write $\cos \frac{1}{3}\epsilon = 1$. Hence,

$$AC = s \cos (A - \frac{2}{3}\epsilon).$$

Similarly,

$$BC = s \sin (A - \frac{1}{3}\epsilon).$$

We now use a theorem * in spherical trigonometry which states that, if PCB is a spherical triangle, right angled at C, in which the angles P and $90^\circ - B$ are very small, then :—

$$\eta = c - b = \frac{p^2}{2} \cot b - \frac{p^4}{24} \cot b\,(1 + 3 \cot^2 b) + \ldots$$

$$P = p \operatorname{cosec} (b + \tfrac{2}{3}\eta) \text{ approximately.}$$

$$\frac{\pi}{2} - B = P \cos (b + \tfrac{1}{3}\eta) \text{ approximately.}$$

Applying these formulæ to the right-angled spherical triangle P′CB, and neglecting the second term in the expression for $c - b$, we have :—

$$P' = dL$$

$$b = 90^\circ - \phi_c = 90^\circ - \phi' - \eta.$$

$$p = \frac{s}{N_c} \sin (A - \tfrac{1}{3}\epsilon).$$

$$\therefore \quad CD \text{ (radians)} = \frac{s^2}{2N_c^2} \sin^2 (A - \tfrac{1}{3}\epsilon) \tan \phi_c,$$

or

$$CD \text{ (seconds)} = \frac{s^2}{2N_c^2 \sin 1''} \sin^2 A \tan \phi_c,$$

in which, as CD and ϵ are small quantities, we have written $\sin^2 A$ for $\sin^2 (A - \frac{1}{3}\epsilon)$.

* For a proof of this theorem see Appendix III, page 533.

Although the linear measures of AC and CD on the spheroid will be practically identical with their linear values on the sphere of radius N_C, their angular values will be slightly different on the two surfaces since the angular values are to be taken as angles on the meridian of the spheroid. Hence, on the meridian through A :—

$$\text{AC (seconds)} = \frac{s}{R_M \sin 1''} \cos (A - \tfrac{2}{3}\epsilon),$$

$$\text{CD (seconds)} = \frac{N_C}{R_M} \cdot \frac{s^2}{2N_C^2 \sin 1''} \sin^2 A \tan \phi_C$$

$$= \frac{s^2 \sin^2 A}{2R_M N_C \sin 1''} \tan \phi_C.$$

But, as CD is small, we may take $N_C = N_M$, and, as $\epsilon'' = \dfrac{s^2 \sin A \cos A}{2R_M N_M \sin 1''}$, we have :—

$$\text{CD (seconds)} = \eta = \epsilon \tan A \tan \phi_C \text{ approximately,}$$

$$\therefore\ \phi' - \phi = d\phi = \text{AD} = \text{AC} - \text{DC}$$

$$= \frac{s}{R_M \sin 1''} \cdot \cos (A - \tfrac{2}{3}\epsilon) - \eta$$

In addition :—

$$dL \text{ (seconds)} = \frac{s \sin (A - \tfrac{1}{3}\epsilon)}{N_C \sin 1''} \operatorname{cosec} (90^\circ - \phi' - \eta + \tfrac{2}{3}\eta)$$

$$= \frac{s \sin (A - \tfrac{1}{3}\epsilon)}{N_C \sin 1''} \sec (\phi' + \tfrac{1}{3}\eta),$$

and

$$\begin{aligned} A' &= 360^\circ - \text{P'BA} \\ &= 360^\circ - \text{ABC} - \text{P'BC} \\ &= 360^\circ - \left\{ 180^\circ + \epsilon - A - 90^\circ \right\} - \left\{ 90^\circ - dL \cos (90^\circ - \phi' - \eta + \tfrac{1}{3}\eta) \right\} \\ &= 180^\circ + A + dL \sin (\phi' + \tfrac{2}{3}\eta) - \epsilon. \\ \therefore\ dA &= A' - (180^\circ + A) \\ &= dL \sin (\phi' + \tfrac{2}{3}\eta) - \epsilon. \end{aligned}$$

The mathematically inclined reader may be interested in the following alternative and very elegant proof of Clarke's formulæ for medium and short lines which is due to Captain G. T. McCaw.

In Fig. 122, A′ is the pole of the circle of which AP forms an arc, B′ is the pole of the arc BP and P′ is the pole of the arc AB. Then the triangle P′A′B′ is a triangle on the sphere supplemental to PAB, and PAB is supplemental to P′A′B′, so that P is the pole of A′B′, A of A′P′ and B of B′P′. The poles of the circles PA and PB must lie on the equator and it is easily seen that A′B′ is equal to GH, which is equal to dL. Side A′P′ is the angle between the poles of the circles AP and AB and must therefore be equal to the angle PAB, or A. Similarly, B′P′ = 180° − ABP = 180° − (360° − A') = A' − 180° where A' is the azimuth of A from B. Further, the side AP is the angle between the planes containing A′B′ and A′P′, and hence angle HA′P′ = 90° − ϕ. In a similar manner it can be shown that angle A′B′P′ is 90 − ϕ'.

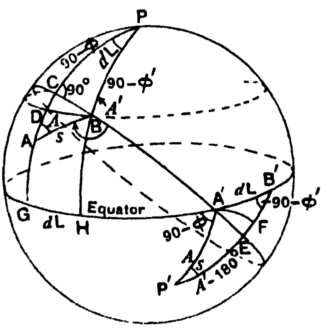


FIG. 122.

Through B draw the great circle BC perpendicular to AP at C. Then CB produced will pass through A′, the pole of AP. B is the pole of B′P′ and therefore, if BA′ is produced to cut B′P′ at E, the angle A′EB′ is a right angle. Also, CA′ = 90° and EB = 90°. Consequently, CB = A′E.

Through B draw the parallel of latitude BD and through A′ the small circle A′F whose pole is P′.

Angle EA′B′ = CA′G = ϕ_C where ϕ_C is the latitude of C. Hence spherical excess of triangle EA′B′ = 90° + ϕ_C + 90° − ϕ' − 180° = $\phi_C - \phi'$ = CD = η.

Similarly, in triangle P′BE (not shown), arc P′E = angle P′BE. But ABP′ = 90°. Hence ABC = 180° − 90° − P′E = 90° − P′E.

Again, ϵ = spherical excess of triangle ABC = A + ABC + 90° − 180° = A − P′E = P′A′ − P′E = P′F − P′E = EF.

Consequently, the arc-versine in the one figure becomes the spherical excess in the other.

In both cases, the triangles ACB and A′EB′ are small so that they can be solved by Legendre's theorem.

$$\mathrm{CB} = s \sin (A - \tfrac{1}{3}\epsilon)$$
$$\mathrm{A'E} = dL \sin (90° - \phi' - \tfrac{1}{3}\eta)$$
$$\therefore\ dL \cos (\phi' + \tfrac{1}{3}\eta) = s \sin (A - \tfrac{1}{3}\epsilon)$$
$$\therefore\ dL = s \sin (A - \tfrac{1}{3}\epsilon) \sec (\phi' + \tfrac{1}{3}\eta)$$

or, with s in linear measure and dL in seconds of arc,

$$dL = \frac{s}{N_C \sin 1''} \sin (A - \tfrac{1}{3}\epsilon) \sec (\phi' + \tfrac{1}{3}\eta)$$

where N_C is the radius of the sphere.

$$\mathrm{AC} = \mathrm{AB} \sin (180° + \epsilon - 90° - A - \tfrac{1}{3}\epsilon) = s \cos (A - \tfrac{2}{3}\epsilon)$$
$$\phi' = \mathrm{GD} = \mathrm{GC} - \mathrm{CD}$$
$$= \mathrm{GA} + \mathrm{AC} - \mathrm{CD}$$
$$= \phi + s \cos (A - \tfrac{2}{3}\epsilon) - \eta.$$

With ϕ and ϕ' in angular measure this becomes, after the second two terms on the right have been reduced to seconds of arc on the meridian,

$$\phi' = \phi + \frac{s}{R_C \sin 1''} \cos (A - \tfrac{2}{3}\epsilon) - \eta''.$$

$$A' - 180° = \mathrm{B'P'} = \mathrm{P'F} + \mathrm{FB'}$$
$$= A + \mathrm{EB'} - \mathrm{EF}$$
$$= A + \mathrm{A'B'} \sin \left\{ 180° + \eta - 90° - (90 - \phi') - \tfrac{1}{3}\eta \right\} - \epsilon.$$
$$\therefore\ 180° + A + dA - 180° = A + dL \sin (\phi' + \tfrac{2}{3}\eta) - \epsilon.$$
$$\therefore\ dA = dL \sin (\phi' + \tfrac{2}{3}\eta) - \epsilon,$$

where dA, dL and ϵ are all in seconds of arc.

$$\eta = \text{spherical excess of triangle A'EB'}$$
$$= \tfrac{1}{2}\mathrm{A'E} \,.\, \mathrm{B'E}.$$

But B′E = A′E tan ϕ' and A′E = BC = s sin A approximately.

$$\therefore\ \eta = \tfrac{1}{2}\frac{s^2}{N_C^2} \sin^2 A \tan \phi'.$$

The angular value of η is to be reckoned on the meridian and not on the sphere (page 329). Hence, the value to be used on the meridian is :—

$$\eta'' = \frac{N_C}{R_M} \frac{s^2}{2N_C^2 \sin'' 1} \sin^2 A \tan \phi'$$
$$= \frac{s^2}{2R_M N_C \sin 1''} \sin^2 A \tan \phi'$$
$$\epsilon'' = \text{spherical excess of triangle ACB.}$$
$$= \frac{s^2}{2R_M N_M \sin 1''} \,.\, \sin A \,.\, \cos A.$$
$$= \frac{s^2}{2R_M N_C \sin 1''} \sin A \cos A \text{ approximately}$$
$$\therefore\ \eta'' = \epsilon'' \tan A \tan \phi'.$$

Example. Given that the latitude and longitude of the point A are ϕ = 7° 10′ 47″·239N, L = 0° 21′ 05″·398E and that the azimuth and logarithm of the

distance from A to B are 257° 12′ 12″·96 and 5·530 1955 respectively, find the latitude and longitude of the point B and the azimuth B to A.

(1) Approximate $d\phi$

$$\begin{aligned}
\log s &= 5{\cdot}530\ 1955\\
\log\cos A &= \bar{1}{\cdot}345\ 3487\\
\log\frac{1}{R\sin 1''} &= \bar{3}{\cdot}996\ 6194\\
&\overline{\ \ 2{\cdot}872\ 1636\ \ }\\
\therefore\ d\phi &= \quad -\ 12'\ 25''{\cdot}012\\
\phi &= +\ 7^\circ\ 10\ \ 47''{\cdot}239\\
\phi_c &= +\ 6\ \ 58\ \ 22''{\cdot}227\\
\phi_m &= \quad 7\ \ 04\ \ 34''{\cdot}733
\end{aligned}$$

(2) ϵ and η

$$\begin{aligned}
2\log s &= 11{\cdot}060\ 39\\
\log\cos A &= \bar{1}{\cdot}345\ 35\\
\log\sin A &= \bar{1}{\cdot}989\ 08\\
\log\frac{1}{2R_M N_e\sin 1''} &= \overline{10}{\cdot}374\ 88\\
\log\epsilon &= 0{\cdot}769\ 70\\
\log\tan A &= 0{\cdot}643\ 73\\
\log\tan\phi' &= \bar{1}{\cdot}087\ 44\\
\log\eta &= 0{\cdot}500\ 87\\
\epsilon &= 5''{\cdot}884\\
\eta &= 3''{\cdot}169
\end{aligned}$$

(3)

$$\begin{aligned}
A &= 257^\circ\ 12'\ 12''{\cdot}96\\
\tfrac{1}{3}\epsilon &= \quad -\quad 1\ {\cdot}96\\
A \pm \tfrac{1}{3}\epsilon &= 257\ \ 12\ \ 11\ {\cdot}00\\
\tfrac{1}{3}\epsilon &= \quad -\quad 1\ {\cdot}96\\
A \pm \tfrac{2}{3}\epsilon &= 257^\circ\ 12\ \ 09\ {\cdot}04
\end{aligned}$$

(4) $d\phi$ and ϕ'

$$\begin{aligned}
\log s &= 5{\cdot}530\ 1955\\
\log\cos(A \pm \tfrac{2}{3}\epsilon) &= \bar{1}{\cdot}345\ 3850\\
\log\frac{1}{R_M\sin 1''} &= \bar{3}{\cdot}996\ 6214\\
&\overline{\ \ 2{\cdot}872\ 2019\ \ }\\
d\phi &= -\ \ 12\ \ 25{\cdot}078\\
\phi &= 7\ \ 10\ \ 47{\cdot}239\\
\phi_c &= 6\ \ 58\ \ 22{\cdot}161\\
-\eta &= \quad -\quad 3{\cdot}169\\
\phi' &= 6\ \ 58\ \ 18{\cdot}992\\
\tfrac{1}{3}\eta &= \quad +\quad 1{\cdot}056\\
\phi' + \tfrac{1}{3}\eta &= 6\ \ 58\ \ 20{\cdot}048\\
\tfrac{1}{3}\eta &= \quad +\quad 1{\cdot}056\\
\phi' + \tfrac{2}{3}\eta &= 6\ \ 58\ \ 21{\cdot}104
\end{aligned}$$

(5) dL, L', dA and A'

$$\begin{aligned}
\log s &= 5{\cdot}530\ 1955\\
\log\sin(A \pm \tfrac{1}{3}\epsilon) &= \bar{1}{\cdot}989\ 0764\\
\log\sec(\phi' + \tfrac{1}{3}\eta) &= 0{\cdot}003\ 2235\\
\log\frac{1}{N_t\sin 1''} &= \bar{3}{\cdot}993\ 7100\\
\log dL &= 3{\cdot}516\ 2054\\
\log\sin(\phi' + \tfrac{2}{3}\eta) &= \bar{1}{\cdot}084\ 1952\\
\log dA &= 2{\cdot}600\ 4006\\
dL &= -\ 0\ \ 54\ \ 42{\cdot}504\\
L &= +\ 0\ \ 21\ \ 05{\cdot}398\\
dL' &= -\ 0\ \ 33\ \ 37{\cdot}106\\
dA &= \quad -\ \ 6\ \ 38{\cdot}47\\
A \pm 180^\circ &= 77\ \ 12\ \ 12{\cdot}96\\
&\quad 77\ \ 05\ \ 31{\cdot}49\\
&\qquad\quad -\ 5{\cdot}88\\
&\quad 77\ \ 05\ \ 28{\cdot}61
\end{aligned}$$

$\phi' = 6^\circ\ 58'\ 18''{\cdot}992N$, $L' = 0^\circ\ 33'\ 37''{\cdot}106W$, $A' = 77^\circ\ 05'\ 28''{\cdot}61$.

Note. The geodetic factors used in this example are for the Clarke 1858 figure of the earth and are taken from the tables in Close and Winterbotham's *Text-book of Topographical Surveying*, Third Edition, 1925. The arrangement of the work closely follows a standard War Office computation form as given in *Survey Computations*, Second Edition, 1932.

Puissant's Formulæ. The formulæ given by Puissant in his *Traité de Géodésie*, Vol. I, have been expressed in different ways, and appropriate factors, depending upon the figure of the earth adopted and the known latitude, are tabulated by various surveys.

It may be shown that, for $d\phi$, dL and dA in radians,

$$d\phi = \frac{s\cos A}{R_M} - \frac{s^2\sin^2 A\tan\phi}{2NR_M} - \frac{s^3\sin^2 A\cos A\,(1+3\tan^2\phi)}{6N^2R_M} + \ldots$$

$$\sin dL = \sin\frac{s}{N}\cdot\frac{\sin A}{\cos\phi'},$$

$$dA = dL\sin\phi_M\sec\tfrac{1}{2}d\phi + \tfrac{1}{12}dL^3(\sin\phi_M\sec\tfrac{1}{2}d\phi - \sin^3\phi_M\sec^3\tfrac{1}{2}d\phi).$$

These formulæ are used by the United States Coast and Geodetic Survey for distances up to about 70 miles in the following modified form:

$$d\phi'' = s\cos A\,.\,\mathrm{B} - s^2\sin^2 A\,.\,\mathrm{C} - (\delta\phi'')^2\,.\,\mathrm{D} - hs^2\sin^2 A\,.\,\mathrm{E},$$

$dL'' = s\sin A\sec\phi'\,.\,\mathrm{A}$ (subject to a correction for the difference between the arc and sine of dL and of the angular value of s),

$$dA'' = dL''\sin\phi_M\sec\tfrac{1}{2}d\phi + (dL'')^3\,.\,\mathrm{F},$$

where $\mathrm{A} = \dfrac{1}{N'\sin 1''}$, $\mathrm{B} = \dfrac{1}{R\sin 1''}$, $\mathrm{C} = \dfrac{\tan\phi}{2NR\sin 1''}$,

$$\delta\phi'' = s\cos A\,.\,\mathrm{B} - s^2\sin^2 A\,.\,\mathrm{C} - hs^2\sin^2 A\,.\,\mathrm{E},$$

$$\mathrm{D} = \frac{3\,e^2\sin\phi\cos\phi\sin 1''}{2(1 - e^2\sin^2\phi)},\quad h = s\cos A\,.\,\mathrm{B},$$

$$\mathrm{E} = \frac{1 + 3\tan^2\phi}{6N^2},\quad \mathrm{F} = \tfrac{1}{12}\sin\phi_M\cos^2\phi_M\sin^2 1''.$$

The logarithms of A, B, C, D, E, and F, based on the 1866 Clarke spheroid and metric units, are published* for latitudes 0° to 72°.

In performing the computation, $d\phi$ is first obtained, B, C, D and E being taken from the tables with ϕ as argument. The algebraic sum of the 1st, 2nd and 4th terms gives $\delta\phi''$, an approximate value of $d\phi''$. $(\delta\phi'')^2\,.\,\mathrm{D}$ is a corrective term which allows for the difference between R_M and R. In computing dL, the value of A is obtained with ϕ' as argument. To the resulting value of dL is applied the algebraic sum of the tabulated corrections for $\log s$ and $\log dL$. The evaluation of dA is straightforward. In all cases care must be exercised with the signs of the functions of A.

When dL'' is calculated from the formula $dL'' = s\sin A\,.\sec\phi'\,.\,\mathrm{A}$, the correction for the difference between the arc and sine of dL'' and of the angular value of s is generally applied direct to the logarithm. The logarithm of the angle itself, when the latter is expressed in circular measure, is always greater than the logarithm of its sine, and, for the angle θ expressed in circular measure, we have :—

$$\log\theta - \log\sin\theta = \frac{M\theta^2}{6}\ \text{approximately},$$

where M is the modulus of the common logarithms. Applying this result to $\dfrac{s}{N'}$, we have :—

$$\log\sin\frac{s}{N'} = \log\frac{s}{N'} - \frac{Ms^2}{6N'^2}.$$

Similarly,

$$\log dL'' = \log\sin dL'' + \frac{M(dL''\sin 1'')^2}{6}$$

Hence, we have :—

$$\log dL'' = \log(s\sin A\,.\,\sec\phi'\,.\,\mathrm{A}) + \frac{M(dL''\sin 1'')^2}{6} - \frac{Ms^2}{6N'^2},$$

in which $\log\dfrac{M}{6} = \bar{2}\cdot 8596$ and $\log\dfrac{M\sin^2 1''}{6} = \overline{12}\cdot 2308$.

* United States Coast and Geodetic Survey Report, 1894, Appendix No. 9, and 1901, Appendix No. 4.

If a computing machine is used, the formula for dL is best put in the form :—

$$dL'' = \frac{s}{N' \sin 1''} \sin A \,.\, \sec \phi' \left\{ 1 - \frac{s^2}{6N'^2} (1 - \sin^2 A \sec^2 \phi') \right\}$$

When s is less than about 12 miles, and in minor triangulation, the formulæ may be simplified to

$$\begin{aligned} d\phi'' &= s \cos A \,.\, \mathrm{B} - s^2 \sin^2 A \,.\, \mathrm{C} - h^2 \,.\, \mathrm{D}, \\ dL'' &= s \sin A \sec \phi' \,.\, \mathrm{A}, \\ dA'' &= dL'' \sin \phi_M. \end{aligned}$$

The above formulæ will hold for all quadrants in the northern hemisphere provided azimuths are reckoned clockwise from north, the proper signs are given to the trigonometrical functions of A, latitudes are taken with a plus sign and longitudes are reckoned positive eastwards and negative westwards. In the southern hemisphere, the formulæ will hold under similar conditions except that the latitudes must now be taken as negative and the negative sign given, when necessary, to their trigonometrical functions.

In the method of expressing Puissant's formulæ devised by Col. Everest, each term is derived from those preceding. Everest employed four terms of the series for primary work. The constants, published in the Survey of India Auxiliary Tables were, however, computed for Everest's figure, which is unsuitable for general use. The logarithms of the factors P, Q, R, S, T, for the first two terms, based on Clarke's figure, are given in earlier editions of Sir C. F. Arden-Close's *Text-book of Topographical Surveying* but have been omitted in the third edition. These are sufficient for all minor work.

The two-term formulæ are :

$$\begin{aligned} d\phi &= d_1\phi + d_2\phi, \\ dL &= d_1L + d_2L, \\ dA &= d_1A + d_2A, \end{aligned}$$

where

$$\begin{aligned} d_1\phi &= \mathrm{P}\, s \cos A, & d_2\phi &= d_1A\, \mathrm{R}\, s \sin A, \\ d_1L &= d_1\phi\, \mathrm{Q} \sec \phi \tan A, & d_2L &= d_2\phi\, \mathrm{S} \cot A, \\ d_1A &= d_1L \sin \phi, & d_2A &= d_2L\, \mathrm{T}, \end{aligned}$$

the tabulated quantities for angles in seconds being,

$$\mathrm{P} = \frac{1}{R \sin 1''},\ \mathrm{Q} = \frac{R}{N},\ \mathrm{R} = \frac{1}{2R},\ \mathrm{S} = \frac{2R \sec \phi}{N},$$

$$\mathrm{T} = \tfrac{1}{2}\left(2 \tan^2 \phi + \frac{N}{R}\right) \cot \phi \cos \phi.$$

Taking north latitudes and east longitudes as positive, and reckoning azimuth clockwise from north, the signs of the terms are as follows, when the known station A is in north latitude.

Term	A			
	0–90°	90°–180°	180°–270°	270°–360°
$d_1\phi$	+	−	−	+
d_1L	+	+	−	−
d_1A	+	+	−	−
$d_2\phi$	−	−	−	−
d_2L	+	−	+	−
d_2A	+	−	+	−

When the latitude is south, it is to be given a negative sign, with longitudes still reckoned positive eastwards and negative westwards. In this case, after allowing for the signs of the trigonometrical functions of the negative ϕ, the signs of d_1A, $d_2\phi$, d_2L and d_2A in the above table must be reversed.

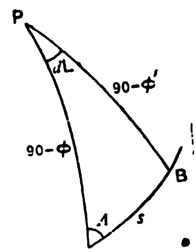


FIG. 123.

The Puissant formulæ may be proved as follows :—

Let PAB (Fig. 123) be the spherical triangle of radius N, tangential to the parallel of latitude through A. Then :—

$$\sin\phi' = \sin\phi\cos AB + \cos\phi\sin AB\cos PAB$$

$$\therefore\ \sin(\phi + d\phi) = \sin\phi\cos\frac{s}{N} + \cos\phi\sin\frac{s}{N}\cos A.$$

Expanding $\sin(\phi + d\phi)$ in ascending powers of $d\phi$ and replacing $\cos\frac{s}{N}$ and $\sin\frac{s}{N}$ by series in powers of $\frac{s}{N}$ this gives :—

$$\sin\phi + \cos\phi\, d\phi - \sin\phi\frac{d\phi^2}{2} - \cos\phi\frac{d\phi^3}{6} + \ldots$$

$$= \sin\phi + \frac{s}{N}\cos A\cos\phi - \frac{s^2}{2N^2}\sin\phi - \frac{s^3}{6N^3}\cos A\cos\phi + \ldots$$

By three successive approximations, commencing with $d\phi = \frac{s}{N}\cos A$, we get :—

$$d\phi = \frac{s}{N}\cos A - \frac{s^2}{2N^2}\sin^2 A\tan\phi - \frac{s^3}{6N^3}\cos A\sin^2 A\left\{1 + 3\tan^2\phi\right\}.$$

This gives the value of $d\phi$ in circular measure on a sphere of radius N. $d\phi$, however, is to be measured along a meridian on the spheroid. Hence, to convert the above value into seconds of arc as measured on the meridian, we must multiply the right-hand side of the equation by $\frac{N}{R_M\sin 1''}$. When this has been done we obtain :—

$$d\phi'' = \frac{s}{R_M\sin 1''}\cos A - \frac{s^2\sin^2 A\tan\phi}{2R_M N\sin 1''}$$

$$- \frac{s^3\sin^2 A\cos A}{6R_M N^2\sin 1''}\left\{1 + 3\tan^2\phi\right\}.$$

For computing purposes we want to use R instead of R_M and this means using a corrective term.

Now
$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}.$$
$$\therefore \frac{dR}{d\phi} = a(1 - e^2)\frac{3e^2 \sin \phi \cos \phi}{(1 - e^2 \sin^2 \phi)^{\frac{5}{2}}}$$
$$= R \cdot \frac{3e^2 \sin \phi \cos \phi}{1 - e^2 \sin^2 \phi}.$$

Hence, if we put
$$\delta\phi'' = \frac{s}{R \sin 1''} \cos A - \frac{s^2}{2RN \sin 1''} \sin^2 A \tan \phi - \frac{s^3}{6RN^2 \sin 1''} \sin^2 A \cos A \left\{1 + 3 \tan^2 \phi\right\},$$
we have for the approximate mean latitude,
$$\phi_M = \phi + \frac{\delta\phi''}{2}.$$
and R_M will be given by
$$R_M = R + \frac{dR}{d\phi} \cdot \frac{\delta\phi''}{2} \sin 1'' + \ldots$$
$$= R + \tfrac{3}{2} R \cdot \frac{e^2 \sin \phi \cos \phi}{(1 - e^2 \sin^2 \phi)} \delta\phi'' \sin 1'' \text{ approximately}$$
$$\therefore \frac{1}{R_M} = \frac{1}{R}\left(1 - \tfrac{3}{2} \frac{e^2 \sin \phi \cos \phi}{(1 - e^2 \sin^2 \phi)} \delta\phi'' \sin 1''\right) \text{ approximately.}$$
Hence,
$$d\phi'' = \frac{s}{R \sin 1''} \cos A - \frac{s^2}{2RN \sin 1''} \sin^2 A \tan \phi - \frac{s^3}{6RN^2 \sin 1''} \sin^2 A \cos A \left\{1 + 3 \tan^2 \phi\right\}$$
$$- \tfrac{3}{2} \frac{e^2 \sin \phi \cos \phi}{(1 - e^2 \sin^2 \phi)} \sin 1'' (\delta\phi'')^2.$$

By the sine rule :—
$$\sin dL = \sin \text{AB} \cdot \frac{\sin A}{\sin (90^\circ - \phi')}$$
$$= \sin \text{AB} \cdot \frac{\sin A}{\cos \phi'}$$

Since the latitude involved is now ϕ', and not ϕ, we must consider the triangle PAB as being on an auxiliary sphere of radius N', and not of radius N. Consequently, the angular value of the side AB in radians is $\frac{s}{N'}$, and we have :—
$$\sin dL = \sin \frac{s}{N'} \sin A \sec \phi'$$
$$= \left(\frac{s}{N'} - \frac{s^3}{6 N'^3} + \ldots\right) \sin A \sec \phi'.$$

Hence, using the expression for an angle in terms of its sine (page 8),
$$dL = \frac{s}{N'} \sin A \sec \phi' - \frac{s^3}{6N'^3} \sin A \sec \phi'(1 - \sin^2 A \sec^2 \phi'),$$
or,
$$dL'' = \frac{s}{N' \sin 1''} \sin A \sec \phi' \left\{1 - \frac{s^2}{6N'^2}(1 - \sin^2 A \sec^2 \phi')\right\}.$$

In triangle PBA,
$$\text{angle PBA} = 360^\circ - A',$$
$$\tfrac{1}{2}(\text{PAB} + \text{PBA}) = \tfrac{1}{2}(A + 360^\circ - A')$$
$$= \tfrac{1}{2}(A + 360^\circ - 180^\circ - A - dA)$$
$$= 90^\circ - \frac{dA}{2}$$

Accordingly, by formula (iv) on page 4.

$$\tan\left(90° - \frac{dA}{2}\right) = \frac{\cos\frac{1}{2}(90° - \phi' - 90° + \phi)}{\cos\frac{1}{2}(90° - \phi' + 90° - \phi)} \cot\frac{dL}{2}$$

$$\therefore \tan\frac{dA}{2} = \frac{\cos\left(90° - \frac{\phi + \phi'}{2}\right)}{\cos\frac{1}{2}(\phi - \phi')} \tan\frac{dL}{2}$$

$$= \frac{\sin\frac{1}{2}(\phi + \phi')}{\cos\frac{d\phi}{2}} \tan\frac{dL}{2},$$

or,

$$\tan\frac{dA}{2} = \sin\phi_M \sec\frac{d\phi}{2}\left\{\frac{dL}{2} + \frac{dL^3}{24} + \ldots\right\},$$

which gives :—

$$dA = dL\sin\phi_M \sec\frac{d\phi}{2} + \frac{dL^3}{12}\left(\sin\phi_M \sec\frac{d\phi}{2} - \sin^3\phi_M \sec^3\frac{d\phi}{2}\right).$$

In the above, we have proved the formula for dL in the form of a series in ascending powers of s but we have previously explained that, when logarithms are used, it is usual to compute from :—

$$dL'' = \frac{s}{N' \sin 1''} \sin A \sec\phi',$$

and then introduce a correction for the difference between the logarithms of an angle and its sine. To do this we used the formula

$$\log\theta - \log\sin\theta = \frac{M\theta^2}{6},$$

which can be derived as follows :—

$$\log_e \sin\theta = \log_e\left(\theta - \frac{\theta^3}{6} + \ldots\right)$$

$$= \log_e \theta . \left(1 - \frac{\theta^2}{6} + \ldots\right)$$

$$= \log_e \theta + \log_e\left(1 - \frac{\theta^2}{6} + \ldots\right)$$

$$= \log_e \theta - \left(\frac{\theta^2}{6} - \ldots\right) - \frac{1}{2}\left(\frac{\theta^2}{6} - \ldots\right)^2 -$$

$$\therefore \log_e \theta = \log_e \sin\theta + \frac{\theta^2}{6} \text{ approximately}$$

$$\therefore \log_{10} \theta = \log_{10} \sin\theta + \frac{M\theta^2}{6} \text{ approximately.}$$

where M is the modulus of the common logarithms.

Mid-Latitude Formulæ. In addition to Clarke's and Puissant's formulæ, which have been given above, mention should be made of the Mid-Latitude formulæ. These are simpler to use than either Clarke's or Puissant's, but do not give such accurate results. They may, however, be used for lines not exceeding 25 miles in length, in latitudes less than 60°, without introducing errors greater than 0″·01.

They are :—

$$\tan\tfrac{1}{2}dA = \tan\tfrac{1}{2}dL \sin(\phi + \tfrac{1}{2}d\phi) \sec\tfrac{1}{2}d\phi \qquad (1)$$

$$d\phi = \frac{s}{R_M \sin 1''} \cos(A + \tfrac{1}{2}dA) \qquad (2)$$

$$dL = \frac{s}{N_M \sin 1''} \cdot \frac{\sin(A + \tfrac{1}{2}dA)}{\cos(\phi + \tfrac{1}{2}d\phi)} \qquad (3)$$

The notation is that shown on page 325, with the addition of the symbol N_M, which is the normal at the mean latitude ϕ_M.

In using these formulæ it is first necessary to find approximate values for $d\phi$ and dL to use in equation (1). An approximate value for $d\phi$ may be found from equation (2) by substituting R for R_M, and $\cos A$ for $\cos (A + \frac{1}{2}dA)$. The approximate value for $d\phi$ is then used in equation (3) and $\sin A$ is used for $\sin (A + \frac{1}{2}dA)$. In this way an approximate value is found for dL.

To determine the algebraic signs of the quantities dA, $d\phi$ and dL a diagram may be drawn with the points A and B in their relative positions, and the meridians through each converging towards the pole. The signs may then be obtained by inspection of the diagram.

The proof of the formulæ follows directly from simple spherical trigonometry and can be left as an exercise for the reader.

Formulæ for Very Long Lines. All the formulæ given above will hold with sufficient precision, and within the limits stated for each, only for lines whose lengths do not exceed distances which can be observed visually from and to stations fixed on the earth's surface. The development of radar and flare triangulation, and also certain developments in ballistics, necessitates other formulæ which are accurate for much longer lines. Several sets of formulæ are available. Those given for reference purposes in Appendix V are now being used by the Geodetic Survey of Canada for the computation of radar triangulation measured with the Shoran apparatus. These formulæ give accurate results up to about 1,500 miles when computed by eight-figure logarithms and are of interest as an example of the solution of a spheroidal triangle by the use of a special auxiliary spherical triangle.

Inverse Cases. (1) It is frequently required to determine s, A, and A' from the given quantities ϕ, ϕ', L, and L'.

For the most precise results Clarke's formulæ for long lines may be used indirectly. Close approximations to s and A are first obtained from the Puissant formulæ by the method below. These values are substituted in the Clarke formulæ, and ϕ' and L' are computed. The small discrepancies between the results and the known values of ϕ' and L' indicate the nature of the errors in the assumed values of s and A, and by successive approximations these are eliminated.

For medium lines, the shorter Clarke formulæ may be used in a modified form which may be written :—

$$\eta = \frac{N'(dL)^2 \sin\phi' \cos\phi' \sin 1''}{2R'} \qquad (1)$$

$$\phi_c = \phi' + \eta \qquad (2)$$

$$\epsilon = \tfrac{1}{2}(\phi_c - \phi)dL \cos\phi' \sin 1'' \qquad (3)$$

$$\log\tan A = \log \frac{N_c dL \cos(\phi' + \frac{1}{3}\eta)}{R_M(\phi_c - \phi)} + \tfrac{1}{3}M \sin 1''.\epsilon.\cot A(1 + 2\tan^2 A) \qquad (4)$$

$$s = \frac{dL\cos(\phi' + \frac{1}{3}\eta)N_c \sin 1''}{\sin(A - \frac{1}{3}\epsilon)} = \frac{(\phi_c - \phi)R_M \sin 1''}{\cos(A - \frac{2}{3}\epsilon)} \qquad (5)$$

$$A' = A \pm 180° + dL \sin(\phi' + \tfrac{2}{3}\eta) - \epsilon \qquad (6)$$

in which ϕ_c is an approximate latitude (the latitude of C in Fig. 121), N_c is the N for ϕ_c and M is the modulus of the common logarithms. Here, $\log \frac{1}{3}M \sin 1'' = \bar{7}{\cdot}846\ 23$.

If a computing machine is used instead of logarithms, A can be obtained from :—

$$\tan A = \frac{N_c dL \cos(\phi' + \frac{1}{3}\eta)}{R_M(\phi_c - \phi)} + \tfrac{1}{3}\epsilon \sin 1''(1 + 2\tan^2 A) \quad . \quad . \quad (7)$$

In both of the expressions for obtaining A, a preliminary approximate value must be computed for use in the small second term.

The above formulæ hold if azimuths are measured clockwise from north, longitudes are positive eastwards and negative westwards, north latitudes are positive and south latitudes negative, and, with these conventions, the proper signs are given to the trigonometrical functions.

An example of a logarithmic computation will be found in *Survey Computations*, second edition, page 78.

In using the Puissant formulæ, the equations for $d\phi$ and dL are combined to yield s and A. First of all, we determine $s \sin A$ from

$$\log s \sin A = \log(dL \cos \phi' N' \sin 1'') - \frac{M(dL \sin 1'')^2}{6}.$$

This value is then substituted in the formula for $d\phi$ given on page 332, the result being :—

$$s \cos A = d\phi R_M \sin 1'' + \frac{s^2 \sin^2 A}{2N} \tan \phi$$
$$+ \frac{s^2 \sin^2 A}{6N^2} \,.\, s \cos A \left\{1 + 3\tan^2 \phi\right\},$$

or,

$$s \cos A = \frac{d\phi}{\mathrm{B}} + s^2 \sin^2 A \frac{\mathrm{C}}{\mathrm{B}} + (\delta\phi'')^2 \,.\, \frac{\mathrm{D}}{\mathrm{B}} + h \sin^2 A \,.\, \frac{\mathrm{E}}{\mathrm{B}}.$$

Here, the third term on the right in the first equatic... and the last two terms in the second equation contain $s \cos A$, but the first two terms can first be calculated to obtain an approximate value of $s \cos A$, and this approximate value can then be inserted in the remaining terms.

Since $s \sin A$ and $s \cos A$ are now known, A is obtained from $\tan A = \dfrac{s \sin A}{s \cos A}$ and s can be derived from either of the above equations. The Everest formulæ can be used in the same way.

Sufficiently accurate results for mapping are obtained by the use of fewer terms in the evaluation of $s \cos A$. Other forms of approximate solutions are given by Close.

The mid-latitude formulæ will provide a direct solution to be used for minor work only. The value of dA is obtained from the first of these equations and the third divided by the second gives :—

$$\tan(A + \tfrac{1}{2}dA) = \frac{N_M}{R_M} \cdot \frac{dL}{d\phi} \cos(\phi + \tfrac{1}{2}d\phi).$$

When the values of A and dA have been found, the value of s can be obtained from the second and third equations.

(2) A similar case occurs when ϕ, ϕ', L and A are known, and s, A', L' are required.

If Clarke's formulæ are used we have :—

$$\epsilon = \tfrac{1}{2}\frac{R_M}{N_M}(\phi' - \phi)^2 \tan A \sin 1'',$$
$$\eta = \epsilon \tan A \tan \phi',$$
$$s = (\phi' - \phi + \eta)R_M \sin 1'' \sec (A - \tfrac{2}{3}\epsilon),$$
$$dL = \frac{s \sin (A - \tfrac{1}{3}\epsilon) \sec (\phi' + \tfrac{1}{3}\eta)}{N_c \sin 1''},$$
$$dA = dL \sin (\phi' + \tfrac{2}{3}\epsilon) - \epsilon.$$

The Puissant expression for $d\phi$ can be written in the form :—

$$s = \frac{\phi' - \phi}{\cos A} R_M \sin 1'' + \frac{s^2 \sin^2 A \tan \phi}{2N \cos A} + \frac{s^3 \sin^2 A}{6N^2}(1 + 3 \tan^2 \phi),$$

and this can be solved by successive approximations for s beginning with $s = \dfrac{(\phi' - \phi)R_M \sin 1''}{\cos A}$. After obtaining s, dL and dA can be calculated in the ordinary way.

In applying the calculation to latitude and azimuth traverse (page 445) refined computation is not required, and, since such traverses should run nearly north and south, it is usually sufficient to employ only the first terms of the formulæ, so that

$$d\phi'' = \frac{s \cos A}{R \sin 1''},$$
$$\text{or } s = d\phi R \sin 1'' \sec A,$$

$\dfrac{1}{R \sin 1''}$ being the constant B of the American tables, or P of the Indian.

When lines are short, and the use of the mid-latitude formulæ is justifiable, obtain an approximate value for dL by substituting A for $(A + \tfrac{1}{2}dA)$ in the third of the mid-latitude equations. By inserting the value of dL so found in the first of these equations the value of dA is found with sufficient accuracy. The values of s and dL can then be obtained from the second and third equations respectively.

All of the Clarke formulæ for the inverse problem follow easily by substitution in the original equations with the exception, perhaps, of that for $\tan A$. Both the ordinary and the logarithmic expansions are derived from the equations :—

$$s \sin (A - \tfrac{1}{3}\epsilon) = dLN_c \sin 1'' \cos (\phi' + \tfrac{1}{3}\eta),$$
$$s \cos (A - \tfrac{2}{3}\epsilon) = R_M \sin 1'' (\phi' - \phi) + R_M \sin 1'' . \eta$$
$$= R_M \sin 1'' (\phi_c - \phi).$$

Expanding the terms on the left, and remembering that ϵ in seconds must be converted to radians in the expansions, we have :—

$$s \sin A - \tfrac{1}{3}\epsilon \sin 1'' s \cos A = dLN_c \sin 1'' \cos (\phi' + \tfrac{1}{3}\eta),$$
$$s \cos A + \tfrac{2}{3}\epsilon \sin 1'' . s \sin A = R_M \sin 1'' (\phi_c - \phi).$$

Hence, by simple division,

$$\tan A = \frac{dLN_c \sin 1'' \cos (\phi' + \tfrac{1}{3}\eta)}{R_M \sin 1''(\phi_c - \phi)} + \tfrac{1}{3}\epsilon \sin 1''(1 + 2 \tan^2 A).$$

For the logarithmic solution, by applying Taylor's theorem to expand $\log \sin(A - \tfrac{1}{3}\epsilon)$ and $\log \cos (A - \tfrac{2}{3}\epsilon)$, we have :—

$$\log_e \sin (A - \tfrac{1}{3}\epsilon) = \log_e \sin A - \tfrac{1}{3}\epsilon \frac{\cos A}{\sin A},$$

$$\log_e \cos (A - \tfrac{2}{3}\epsilon) = \log_e \cos A + \tfrac{2}{3}\epsilon \frac{\sin A}{\cos A},$$

$$\therefore \log_{10} \sin (A - \tfrac{1}{3}\epsilon) = \log_{10} \sin A - \frac{M}{3}\epsilon \cot A$$

$$\log_{10} \cos (A - \tfrac{2}{3}\epsilon) = \log_{10} \cos A + \frac{2M}{3}\epsilon \tan A.$$

$$\therefore \log_{10} \sin A - \log_{10} \cos A = \log_{10} \frac{dLN_c \sin 1'' \cos (\phi' + \tfrac{1}{3}\eta)}{R_M \sin 1''(\phi_c - \phi)} + \frac{M}{3}\epsilon(\cot A + 2 \tan A).$$

$$\therefore \log \tan A = \log \frac{dLN_c \cos (\phi + \tfrac{1}{3}\eta)}{R_M(\phi_c - \phi)} + \frac{M}{3}\epsilon \cot A(1 + 2 \tan^2 A).$$

Rectangular Spherical or Cassini Co-ordinates.* In Vol. I we have discussed ordinary plane rectangular co-ordinates and have considered various problems connected with them. These co-ordinates take no account of the curvature of the earth, and are based on the assumption that the survey is carried out on a flat surface—in this case a plane tangential to the earth at some point in the region covered by the survey. This assumption is valid only if no great degree of accuracy is aimed at, or the area covered by the survey is very limited. For accurate work, which covers a large area, the curvature of the earth must be taken into account in the computation of position, as the measurements are made on the curved surface of the earth and not on a plane. Hence, it is natural to modify the system of ordinary plane rectangular co-ordinates by introducing extra terms involving allowance for curvature. There are a number of ways of doing this but perhaps the simplest and most obvious is the system known as "rectangular spherical," or "Cassini," co-ordinates which we now proceed to describe. They were originally devised by Cassini for use in France and much of the older work of the Ordnance Survey is based on them, though in Great Britain they are now being replaced by co-ordinates on the "Transverse Mercator" system. They have also been employed in many other parts of the world, and are still in use in many places, but, as they have certain disadvantages which will be discussed later, the modern tendency is to replace them by Transverse Mercator co-ordinates in countries which are suitable for the use of the latter system.

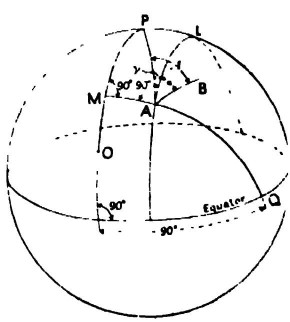


FIG. 124.

In Fig. 124, O is a point in the centre of the area under survey whose latitude and longitude are known and which is chosen as origin of co-ordinates. A is a point whose position with respect to O is to be defined.

Assuming the earth to be spherical and P its pole, through O draw the meridian OP and from A draw a great circle AM perpendicular

* When the formulæ used include terms to allow for the spheroidal shape of the earth the name "rectangular spheroidal" may be used instead of "rectangular spherical."

to OP at M. Then the X co-ordinate of A is defined as the arc OM and the Y co-ordinate as the arc MA, both of these arcs being expressed in linear units.

Through A draw the small circle AL cut out by a plane parallel to the plane of the meridian OP. The arc MA produced will cut the equator at Q, the pole of the meridian OMP, and the angle LAQ = LAM will be a right angle. Then the azimuth from A of the point B will be defined by the angle PAB, measured clockwise from the meridian AP, which the line of sight from A to B makes with AP. The bearing of AB referred to the small circle AL will be the angle LAB and the angle PAL is called the "convergence" at A.

The convergence may therefore be defined as the angle between the meridian at A and the small circle perpendicular at A to the rectangular spherical Y co-ordinate. Similarly, the "rectangular spherical bearing" at any point may be defined as a bearing referred to the small circle perpendicular at that point to its rectangular spherical Y co-ordinate.

If bearings, like azimuths, are reckoned clockwise from north, and A is the azimuth, γ the convergence and α the bearing, it can easily be seen by drawing a suitable diagram that the following rules will hold with regard to signs :—

For Northern Hemisphere—

$\alpha = A - \gamma$ when the point lies east of the central meridian.

$\alpha = A + \gamma$ when the point lies west of the central meridian.

For Southern Hemisphere—

$\alpha = A + \gamma$ when the point lies east of the central meridian.

$\alpha = A - \gamma$ when the point lies west of the central meridian.

These rules are, of course, valid for points south of the origin of co-ordinates O as well as for those north of it.

Conversion of Rectangular Spherical Co-ordinates into Latitudes and Longitudes and the Inverse Problem. It is obvious from the definition of rectangular spherical co-ordinates that, if the latitude and longitude of the origin are known, there must be some relationship between the rectangular spherical and the geographical co-ordinates of a point and that it must be possible to convert one set of co-ordinates into the other. This conversion is often of some importance in practice. Thus, if a point is to be used as a station for astronomical observations for azimuth, it is, in general, necessary to know its latitude. If the rectangular spherical co-ordinates are known, the latitude, longitude and convergence are easily determined.

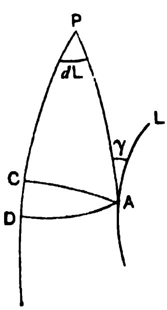


FIG. 125.

In Fig. 125 OC is the X co-ordinate and CA is the Y co-ordinate of the point A with respect to the origin O. Let PAC be the spherical triangle described on the auxiliary sphere of radius N_c and tangential to the spheroid at the point C, P being the point or pole where the earth's minor axis meets the sphere. Then, if OC and CA are small in comparison with N_c, the points O and A on the spheroid will be practically coincident with their projection on the sphere, and the linear measures of the arcs OC and CA will be the same for both surfaces.

Through A draw the parallel of latitude AD cutting the meridian OCP at D. From A draw the small circle AL parallel to the meridian ODCP. Then angle PAL $= \gamma =$ the convergence at A.

Since the length OC is defined by the X co-ordinate, and is known, and the latitude and longitude of O are also known, the value of the arc OC, in degrees, minutes and seconds, can be found from a table of meridional distances Hence, the latitude of C can be found from

$$\phi_c = \phi_o + X^\circ$$

where ϕ_c is the latitude of C, ϕ_o is the latitude of O and X° is the linear distance X converted into angular measure.

From the figure it follows that latitude of A $=$ latitude of D $= \phi_c - \eta$ where η is the arc CD.

Similarly, for the inverse problem, the latitude of A is given and hence we know the latitude of D. Then, from a table of meridional distances, we can find the linear distance$_D$ corresponding to OD or to $\phi_A - \phi_O$. The distance η can be computed and we then have :—

$$X_A = X_D + \eta.$$

For the conversion of rectangular spherical co-ordinates into geographical co-ordinates the formulæ are :—

$$\phi_c = \phi_o + X^\circ \quad \ldots \quad (1)$$

$$\eta = \frac{Y^2 \tan \phi_c}{2R_c N_c \sin 1''} - \frac{Y^4 \tan \phi_c (1 + 3 \tan^2 \phi_c)}{24 R_c N_c^3 \sin 1''} \quad \ldots \quad (2)$$

$$\phi = \phi_c - \eta \quad \ldots \quad (3)$$

$$dL = \frac{Y \sec \phi_c}{N_c \sin 1''} - \tfrac{1}{3}\frac{Y^3 \sec \phi_c \tan^2 \phi_c}{N_c^3 \sin 1''} + \tfrac{1}{15}\frac{Y^5 \sec \phi_c \tan^2 \phi_c (1 + 3 \tan^2 \phi_c)}{N_c^5 \sin 1''} \quad (4)$$

$$\gamma = \frac{Y \tan \phi_c}{N_c \sin 1''} - \tfrac{1}{6}\frac{Y^3 \tan \phi_c (1 + 2 \tan^2 \phi_c)}{N_c^3 \sin 1''} + \frac{Y^5 \tan \phi_c}{120 N_c^5 \sin 1''}(1 + 20 \tan^2 \phi_c + 24 \tan^4 \phi_c) \quad . \quad . \quad (5)$$

If the proper signs are given to the trigonometrical functions, the above relations will be valid for all cases provided the following conventions are observed :—

(1) North latitudes are positive and south latitudes are negative.

(2) X co-ordinates are positive if north of the origin and negative if south of it.

(3) Y co-ordinates are positive if east of the central meridian and negative if west of it.

(4) dL is positive for points east of the central meridian and negative for points west of it.

(5) The sign of γ depends on the sign of Y, but, when signs are otherwise ignored, the following relations hold between azimuth A, bearing α and convergence γ, azimuths and bearings being reckoned clockwise from north :—

(*a*) *In northern latitudes* :—

$A = \alpha + \gamma$ when the point is east of the central meridian.

$A = \alpha - \gamma$ when the point is west of the central meridian.

(*b*) *In southern latitudes* :—

$A = \alpha - \gamma$ when the point is east of the central meridian.

$A = \alpha + \gamma$ when the point is west of the central meridian.

For the inverse problem, that is to determine X and Y from the given latitude and longitude of the point and of the origin of co-ordinates,

$$X = X_p + \eta \quad \ldots \quad (1)$$

$$= X_p + \tfrac{1}{2}N(dL\cos\phi\sin 1'')^2\tan\phi + \tfrac{1}{24}N(dL\cos\phi\sin 1'')^4\tan\phi(5 - \tan^2\phi) \quad \ldots \quad (2)$$

$$Y = N(dL\cos\phi\sin 1'') - \tfrac{1}{6}N(dL\cos\phi\sin 1'')^3\tan^2\phi - \tfrac{1}{120}N(dL\cos\phi\sin 1'')^5\tan^2\phi(8 - \tan^2\phi) \quad \ldots \quad (3)$$

$$\gamma = dL\sin\phi + \tfrac{1}{3}(dL\cos\phi\sin 1'')^3\tan\phi\operatorname{cosec} 1'' + \tfrac{1}{15}(dL\cos\phi\sin 1'')^5\tan\phi(2 - \tan^2\phi)\operatorname{cosec} 1'' \quad \ldots \quad (4)$$

Note here that ϕ is the latitude of the given point A, and N is the radius of curvature of the prime vertical section or the normal for that latitude, whereas for the first problem the quantities involved are ϕ_c and N_c.

In deriving η in terms of Y we may use the theorem in spherical trigonometry already employed on page 329 in deriving the Clarke formulæ for medium and short lines, but this theorem in itself is not sufficient to give the third term in the expressions for dL and Y. This is because the rectangular spherical Y co-ordinate will often be much longer than the distance s used in the Clarke formulæ, and, in deriving the theorem, we neglected powers of p higher than the third and of η higher than the first.

In the case of latitude, substitution of $\frac{Y}{N_c}$ for p in the theorem at once gives :—

$$\eta = \frac{Y^2}{2N_c^2}\tan\phi_c - \frac{Y^4}{24N_c^4}\tan\phi_c(1 + 3\tan^2\phi_c).$$

Here, η is in radians on a sphere of radius N_c, but, in determining latitude, the angular value of η is to be taken as the angle measured on the meridian. Hence, as η is very small, we may take R_c as the mean radius of curvature of the meridian and the angular value of η in seconds on the meridian then becomes $\frac{N_c}{R_c\sin 1''}$ of the above value, so that :—

$$\phi = \phi_0 + X^0 - \frac{Y^2\tan\phi_c}{2R_cN_c\sin 1''} + \frac{Y^4}{24R_cN_c^3\sin 1''}\tan\phi_c(1 + 3\tan^2\phi_c).$$

To obtain dL we have :—

$$\sin dL = \frac{\sin\frac{Y}{N_c}}{\sin(90^\circ - \phi)}$$

$$= \sin\frac{Y}{N_c}\sec(\phi_c - \eta).$$

$$= \sin\frac{Y}{N_c}\left\{\sec\phi_c - \eta\sec\phi_c\tan\phi_c + \frac{\eta^2}{2}\sec\phi_c(1 + 2\tan^2\phi_c) - \ldots\right\}$$

$$= \sin\frac{Y}{N_c}\left\{\sec\phi_c - \frac{Y^2}{2N_c^2}\sec\phi_c\tan^2\phi_c + \frac{Y^4}{24N_c^4}\sec\phi_c\tan^2\phi_c(4 + 9\tan^2\phi_c) + \ldots\right\}$$

$$= \left\{\frac{Y}{N_c} - \frac{Y^3}{6N_c^3} + \frac{Y^5}{120N_c^5} - \ldots\right\}\left\{\sec\phi_c - \frac{Y^2}{2N_c^2}\sec\phi_c\tan^2\phi_c + \frac{Y^4}{24N_c^4}\sec\phi_c\tan^2\phi_c(4 + 9\tan^2\phi_c) - \ldots\right\}$$

$$= \frac{Y}{N_c}\sec\phi_c - \frac{Y^3}{6N_c^3}\sec\phi_c(1 + 3\tan^2\phi_c) + \frac{Y^5}{120N_c^5}\sec\phi_c(1 + 30\tan^2\phi_c + 45\tan^4\phi - \ldots$$

But $dL = \sin dL + \frac{1}{6}\sin^3 dL + \frac{3}{40}\sin^5 dL + \ldots$

Substituting above value for $\sin dL$ in this expression, and dividing by $\sin 1''$ to bring to seconds of arc, we get :—

$$dL'' = \frac{Y}{N_c \sin 1''}\sec\phi_c - \frac{Y^3}{3N_c^3 \sin 1''}\sec\phi_c \tan^2\phi_c$$
$$+ \frac{Y^5}{15N_c^5 \sin 1''}\sec\phi_c \tan^2\phi_c(1 + 3\tan^2\phi_c) - \ldots$$

In Fig. 125 γ is the angle 90° −PAC. Hence, in the right-angled spherical triangle PAC,

cot PAC = sin AC cot PC.

$$\therefore \tan\gamma = \sin\frac{Y}{N_c}\tan\phi_c$$
$$= \tan\phi_c\left(\frac{Y}{N_c} - \frac{Y^3}{6N_c^3} + \frac{Y^5}{120N_c^5} - \ldots\right).$$

But
$$\gamma = \tan\gamma - \tfrac{1}{3}\tan^3\gamma + \tfrac{1}{5}\tan^5\gamma$$
$$= \tan\phi_c\left(\frac{Y}{N_c} - \frac{Y^3}{6N_c^3} + \frac{Y^5}{120N_c^5} - \ldots\right)$$
$$- \tfrac{1}{3}\tan^3\phi_c\left(Y - \frac{Y^3}{6N_c^3} + \ldots\right)^3 + \tfrac{1}{5}\tan^5\phi_c(Y - \ldots)^5$$

$$\therefore \gamma'' = \frac{Y}{N_c \sin 1''}\tan\phi_c - \frac{Y^3}{6N_c^3 \sin 1''}\tan\phi_c(1 + 2\tan^2\phi_c)$$
$$+ \frac{Y^5}{120N_c^5 \sin 1''}\tan\phi_c(1 + 20\tan^2\phi_c + 24\tan^4\phi_c).$$

In finding the formulæ for the reverse process—the conversion of geographicals into rectangular spherical co-ordinates—the latitude ϕ is known from the beginning and it is therefore convenient to express the formulæ in terms of functions of it instead of ϕ_c. In this case, the auxiliary sphere will be one of radius N and not N_c, although, from the point of view of practical computing, it hardly matters which is used. As the theorem we have already employed involves ϕ_c, and not ϕ, we cannot use it directly and we therefore proceed by other means :—

$$\sin\frac{Y}{N} = \sin(90° - \phi)\sin dL$$
$$= \cos\phi\left\{dL - \frac{dL^3}{6} + \frac{dL^5}{120} - \ldots\right\}.$$
$$\frac{Y}{N} = \sin\frac{Y}{N} + \tfrac{1}{6}\sin^3\frac{Y}{N} + \tfrac{3}{40}\sin^5\frac{Y}{N}$$
$$= \cos\phi\left\{dL - \frac{dL^3}{6} + \frac{dL^5}{120} + \ldots\right\} + \tfrac{1}{6}\cos^3\phi\left\{dL - \frac{dL^3}{6} + \ldots\right\}^3$$
$$+ \tfrac{3}{40}\cos^5\left\{dL - \ldots\right\}^5,$$

which gives, for dL expressed in seconds of arc :—

$$Y = N\,dL\sin 1''\cos\phi - \frac{N}{6}(dL\sin 1''\cos\phi)^3\tan^2\phi$$
$$- \frac{N}{120}(dL\sin 1''\cos\phi)^5\tan^2\phi(8 - \tan^2\phi).$$

For η we have :—

$$\cos(90° - \phi) = \cos(90° - \phi - \eta)\cos\frac{Y}{N}.$$
$$\therefore \sin(\phi + \eta) = \sin\phi\sec\frac{Y}{N}$$
$$= \sin\phi\left\{1 + \frac{Y^2}{2N^2} + \frac{5Y^4}{24N^4} + \ldots\right\}$$

Substituting the value already obtained for $\frac{Y}{N}$ we finally obtain :—

$$\eta \text{ (radians)} = \frac{(dL \cos \phi)^2}{2} \tan \phi + \frac{(dL \cos \phi)^4}{24} \tan \phi \, (5 - \tan^2 \phi).$$

Hence, linear value of η with dL in seconds of arc is :—

$$\eta = \frac{N}{2}(dL \cos \phi \sin 1'')^2 \tan \phi + \frac{N}{24} (dL \cos \phi \sin 1'')^4 \tan \phi \, (5 - \tan^2 \phi).$$

For γ we have :—

$$\begin{aligned} \cot \text{PAC} &= \cos \text{PA} \tan \text{CPA}. \\ \therefore \tan \gamma &= \sin \phi \tan dL \\ &= \sin \phi \, (dL + \tfrac{1}{3}dL^3 + \tfrac{2}{15}\, dL^5 + \ . \quad .). \\ \gamma &= \tan \gamma - \tfrac{1}{3} \tan^3 \gamma + \tfrac{1}{5} \tan^5 \gamma \\ &= \sin \phi \, (dL + \tfrac{1}{3}dL^3 + \tfrac{2}{15}\, dL^5) \\ &\qquad - \tfrac{1}{3} \sin^3 \phi \, (dL + \tfrac{1}{3}dL^3 + \ . \ . \ .)^3 + \tfrac{1}{5} \sin^5 \phi \, (dL + \ . \ . \ .) \\ &= dL \sin \phi + \tfrac{1}{3}dL^3 \sin \phi \cos^2 \phi + \frac{dL^5 \sin \phi \cos^2 \phi}{15} (2 - 3 \sin^2 \phi). \\ \therefore \gamma'' &= dL'' \sin \phi + \frac{(dL \cos \phi \sin 1'')^3}{3} \tan \phi \operatorname{cosec} 1'' \\ &\qquad + \frac{(dL \cos \phi \sin 1'')^5}{15} \tan \phi \, (2 - \tan^2 \phi) \operatorname{cosec} 1''. \end{aligned}$$

Examples of these two computations will be found in the second edition of *Survey Computations*, pages 109 and 110.

Computation of Rectangular Spherical Co-ordinates from Bearing and Distance. In Fig. 126 (a) O is the origin of co-ordinates, P the earth's pole, the earth here being considered to be a sphere of radius r, and A a point whose rectangular spherical co-ordinates are given. Draw great

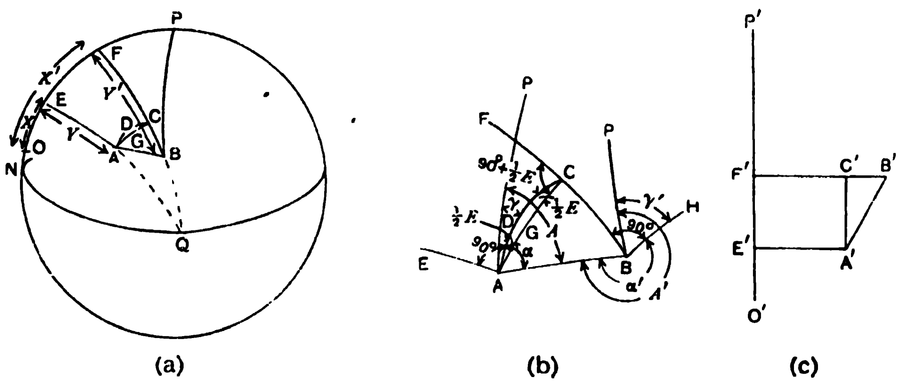


FIG. 126.

circle AE perpendicular to the central meridian OP. Then the co-ordinates of A are OE $= X$ and EA $= Y$. Let B be some other point whose distance s and bearing α from A are known. Draw the great circle BF perpendicular to the meridian OP. Then the rectangular spherical co-ordinates of B are OF $= X'$ and FB $= Y'$. The great circles EA and FB will intersect at a point Q on the earth's equator NQ such that NQ $= 90°$.

From FB cut off an arc FC = EA and describe a great circle AGC through A and C. Then FC = Y, and, in the triangle ACB, CB = $(Y' - Y)$. AC, however, will be less than $(X' - X)$ as the latter is equal to EF, an arc on the central meridian.

From A draw the small circle ADC parallel to the meridian OEFP and hence perpendicular to EA at A. The bearing of B from A will then be the angle DAB = α, and not the angle GAB.

Fig. 126 (b) shows, on an exaggerated scale, the angles, azimuths and bearings at A and B respectively. Here, AP and BP are the meridians at A and B formed by drawing great circles through A and P and through B and P. AD and BH are the arcs of small circles at A and B perpendicular to the arcs AE and BF. Hence the bearings at A and B are α and α' reckoned clockwise from AD and BH, and the angles PAD = γ and PBH = γ' are the convergencies at the points A and B.

Consider the figure AGCFEA in Fig. 126 (a). EA = FC, and EQ and FQ are each equal to 90°. Hence, QA = QC and the angles EAG and FCG are equal. The angles AEF and CFE are each equal to 90°, and, as the figure is made up of arcs of great circles on a spherical surface, the four interior angles must add up to $360° + E$, where E is the spherical excess of the figure. From this it follows that EAG = FCG = $90° + \frac{1}{2}E$. Further, in the triangle AGCB, the angle GAB = $\alpha - \frac{1}{2}E$, angle GCB = $90° - \frac{1}{2}E$ and hence CBA = $180° + \epsilon -$ GAB $-$ GCB = $180° + \epsilon - (\alpha - \frac{1}{2}E) - (90° - \frac{1}{2}E) = 90° - \alpha + E + \epsilon$, where ϵ is the spherical excess of the triangle ACB.

In Fig. 126 (c), the rectangular spherical co-ordinates are plotted as plane rectangular co-ordinates. O′E′ = OE = X, O′F′ = OF = X', F′B′ = FB = Y', F′C′ = E′A′ = EA = Y, B′C′ = BC = $Y' - Y$. Then it is obvious that, although C′B′ on the plane is equal to the distance CB on the sphere, the distance A′C′, being equal to EF, is greater than the distance AC on the sphere. It therefore follows that, if rectangular spherical co-ordinates are plotted as plane rectangular co-ordinates, or if they are treated as plane rectangular co-ordinates and bearings and distances are computed from them by the ordinary methods of plane geometry, there will be no distortion in the Y co-ordinates but there will be distortion in the X co-ordinates and in the computed bearings and distances.

In computing rectangular spherical co-ordinates from bearings and distances three methods may be used. The first is to compute the spherical excesses E and ϵ of the figures ACFE and ACB and then to use these values to obtain modified angles or bearings; the second is to use a development in series of the first method and the third is to compute corrections to the observed distances and bearings and then to compute the co-ordinates by the formulæ for ordinary plane rectangular co-ordinates.

The first method is :—

Compute E from

$$E'' = \frac{Ys \cos \alpha}{r^2 \sin 1''} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Compute ϵ from

$$\epsilon'' = \frac{s^2 \cos \alpha \sin \alpha}{2r^2 \sin 1''} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Compute EF − AGC = dX from

$$dX = \frac{Y^2 s \cos\alpha}{2r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then

$$X' = X + s\cos\{\alpha - (E + \tfrac{2}{3}\epsilon)\} + dX \quad . \quad . \quad . \quad (4)$$

$$Y' = Y + s\sin\{\alpha - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)\} \quad . \quad . \quad . \quad (5)$$

$$\alpha' = \alpha \pm 180° - (E + \epsilon) \quad . \quad . \quad . \quad . \quad (6)$$

These formulæ are of the same type as Clarke's formulæ for computing latitudes and longitudes when lines are short or of medium length (page 328). They hold for all quadrants, in both hemispheres, provided the proper signs are given throughout (including the computation of E, ϵ and dX) to the trigonometrical functions of α, bearings are reckoned clockwise from north, X co-ordinates north of the origin are taken as positive and those south of it as negative, and Y co-ordinates east of the central meridian are taken as positive and those west of it as negative.

The value of r to be used may be taken with sufficient accuracy as $\sqrt{RN}$ for the latitude of A.

The second set of formulæ corresponds to Puissant's formulæ for computing latitudes and longitudes. It is :—

$$Y' = Y + s\sin\alpha - \frac{s^3}{6r^2}\sin\alpha\cos^2\alpha - \frac{Ys^2\cos^2\alpha}{2r^2} \quad . \quad . \quad (7)$$

$$X' = X + s\cos\alpha + \frac{Y'^2 s\cos\alpha}{2r^2} - \frac{s^3\sin^2\alpha\cos\alpha}{6r^2} \quad . \quad . \quad (8)$$

or,

$$X' = X + s\cos\alpha + \frac{Y^2 s\cos\alpha}{2r^2} + \frac{Ys^2\sin\alpha\cos\alpha}{r^2} + \frac{s^3\sin^2\alpha\cos\alpha}{3r^2} \quad . \quad (9)$$

$$\alpha' = \alpha \pm 180° - \frac{s\cos\alpha\,(Y + Y')}{2r^2\sin 1''} \quad . \quad . \quad . \quad . \quad (10)$$

Of the two formulæ for X, one is in terms of Y' and the other in terms of Y, but the first is to be preferred since it contains one term less than the other.

These formulæ conform to the same sign conventions as the others and are the ones which are most commonly used and quoted in text-books. An example is given in *Survey Computations*, 2nd edition, page 111.

The third method of computing rectangular spherical co-ordinates from distance and bearing is given on page 352. This method is little used although it may sometimes afford a useful independent check on computations carried out by either of the other two methods.

In deriving the formulæ, the first step is to obtain an expression for the difference between EF and AGC. In the spherical triangle AGCQ, AQ = CQ = $90° - \frac{Y}{r}$, angle at Q = $\frac{\text{EF}}{r}$ and angle GCQ = GAQ = $90° - \frac{1}{2}E$. Hence, by the sine rule :—

$$\sin\frac{\text{AC}}{r} = \sin\left(90° - \frac{Y}{r}\right)\sin\frac{\text{EF}}{r}\operatorname{cosec}(90° - \tfrac{1}{2}E)$$

$$= \sin\frac{\text{EF}}{r}\cos\frac{Y}{r}\sec\tfrac{1}{2}E.$$

Since AC and EF are very small compared with r, the angles $\frac{\mathrm{AC}}{r}$ and $\frac{\mathrm{EF}}{r}$ are very small, and so are $\frac{Y}{r}$ and E. Hence, we may write :—

$$\frac{\mathrm{AC}}{r} = \frac{\mathrm{EF}}{r}\left(1 - \frac{Y^2}{2r^2} + \frac{Y^4}{24r^4} - \ldots\right)\left(1 + \frac{E^2}{8} + \ldots\right) - \frac{\mathrm{EF}^3}{6r^3} + \frac{\mathrm{AC}^3}{6r^3} + \ldots$$

$$\therefore\ \mathrm{AC} = \mathrm{EF}\left(1 - \frac{Y^2}{2r^2} + \frac{Y^4}{24r^4} + \frac{E^2}{8} + \ldots\right),^{*}$$

$$\mathrm{EF} = \mathrm{AC}\left(1 + \frac{Y^2}{2r^2} + \frac{5}{24}\frac{Y^4}{r^4} - \frac{E^2}{8} - \ldots\right).$$

$\frac{Y^4}{r^4}$ and E^2 are very small compared with $\frac{Y^2}{r^2}$, E^2 in general being smaller than $\frac{Y^4}{r^4}$. Hence these terms may be neglected and we have :—

$$dX = \mathrm{EF} - \mathrm{AC} = \mathrm{AC}\,.\,\frac{Y^2}{2r^2}.$$

The triangle ABC is a very small triangle which may be solved by Legendre's theorem. The spherical angles are $\alpha - \frac{1}{2}E$, $90^\circ - \frac{1}{2}E$ and $180^\circ + \epsilon - (\alpha - \frac{1}{2}E) - (90^\circ - \frac{1}{2}E) = 90^\circ - \alpha + E + \epsilon$. The plane angles, therefore, are :—

$$A = \alpha - \tfrac{1}{2}E - \tfrac{1}{3}\epsilon$$
$$C = 90^\circ - \tfrac{1}{2}E - \tfrac{1}{3}\epsilon$$
$$B = 90^\circ - \alpha + E + \epsilon - \tfrac{1}{3}\epsilon = 90^\circ - \left\{\alpha - (E + \tfrac{2}{3}\epsilon)\right\}.$$

Hence, by the sine rule for plane triangles :—

$$\mathrm{AC} = s\sin\left\{90^\circ - \alpha + (E + \tfrac{2}{3}\epsilon)\right\}\operatorname{cosec}(90^\circ - \tfrac{1}{2}E - \tfrac{1}{3}\epsilon)$$
$$= s\cos\left\{\alpha - (E + \tfrac{2}{3}\epsilon)\right\}\sec(\tfrac{1}{2}E + \tfrac{1}{3}\epsilon).$$
$$\mathrm{BC} = s\sin\left\{\alpha - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)\right\}\operatorname{cosec}\left\{90^\circ - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)\right\}$$
$$= s\sin\left\{\alpha - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)\right\}\sec(\tfrac{1}{2}E + \tfrac{1}{3}\epsilon).$$
$$\alpha' = 270^\circ - \mathrm{CBA}$$
$$= 270^\circ - 90^\circ + \alpha - E - \epsilon$$
$$= 180^\circ + \alpha - (E + \epsilon).$$

As E and ϵ are small angles we can put the secants in the above expressions equal to unity and we then have :—

$$X' = X + s\cos\left\{\alpha - (E + \tfrac{2}{3}\epsilon)\right\} + \frac{Y^2}{2r^2}s\cos\alpha$$

$$Y' = Y + s\sin\left\{\alpha - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)\right\}.$$

To obtain the second set of formulæ, we have :—

$$Y' - Y = s\sin\left\{\alpha - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)\right\}\sec(\tfrac{1}{2}E + \tfrac{1}{3}\epsilon).$$

E and ϵ are small angles whose approximate values are given by :—

$$E = \frac{Y}{r^2}s\cos\alpha$$

$$\epsilon = \frac{s^2\sin\alpha\cos\alpha}{2r^2}.$$

* Note that the length of the small circle ADC is given by EF . $\cos\frac{Y}{r}$ so that the difference in length between the great circle and the small circle is the small quantity $\frac{E^2}{8}\times$ EF approximately.

Hence, we can write :—

$$Y' - Y = s\sin\alpha - (\tfrac{1}{2}E + \tfrac{1}{3}\epsilon)s\cos\alpha$$
$$= s\sin\alpha - \frac{Ys^2\cos^2\alpha}{2r^2} - \frac{s^3\sin\alpha\cos^2\alpha}{6r^2}$$
$$\therefore\ Y' = Y + s\sin\alpha - \frac{Ys^2\cos^2\alpha}{2r^2} - \frac{s^3\sin\alpha\cos^2\alpha}{6r^2}.$$

For the X we have :—

$$X' = X + s\cos\left\{\alpha - (E + \tfrac{2}{3}\epsilon)\right\}\sec(\tfrac{1}{2}E + \tfrac{1}{3}\epsilon) + \frac{Y^2}{2r^2}s\,.\cos\alpha.$$

On neglecting the secant which occurs in the second term on the right, this term becomes :—

$$s\cos\alpha + (E + \tfrac{2}{3}\epsilon)s\sin\alpha = s\cos\alpha + \frac{Ys^2}{r^2}\cos\alpha\sin\alpha + \frac{s^3\cos\alpha\sin^2\alpha}{3r^2}.$$

Hence,

$$X' = X + s\cos\alpha + \frac{s^3\sin^2\alpha\cos\alpha}{3r^2} + \frac{Ys^2\cos\alpha\sin\alpha}{r^2} + \frac{Y^2s\cos\alpha}{2r^2}.$$

In this put $Y = Y' - s\sin\alpha$ and we have :—

$$X' = X + s\cos\alpha + \frac{s^3\sin^2\alpha\cos\alpha}{3r^2} + \frac{Y's^2\cos\alpha\sin\alpha}{r^2} - \frac{s^3\cos\alpha\sin^2\alpha}{r^2}$$
$$+ \frac{Y'^2s\cos\alpha}{2r^2} - \frac{Y's^2\cos\alpha\sin\alpha}{r^2} + \frac{s^3\sin^2\alpha\cos\alpha}{2r^2}.$$
$$= X + s\cos\alpha + \frac{Y'^2}{2r^2}s\cos\alpha - \frac{s^3\sin^2\alpha\cos\alpha}{6r^2}.$$

For α' we get :—

$$\alpha' = \alpha \pm 180° - (E + \epsilon)$$
$$= \alpha \pm 180° - \text{spherical excess of figure ABFE}$$
$$= \alpha \pm 180° - \frac{(Y' + Y)}{2r^2\sin 1''}\,.\,s\cos\alpha.$$

Distortion of Length and Bearing from Computed or Plotted Co-ordinates. If distance and bearing are computed from rectangular spherical co-ordinates by treating them as rectangular co-ordinates on the plane, we have for the computed or plotted distance S :—

$$S^2 = (X' - X)^2 + (Y' - Y)^2$$
$$= \left[s\cos\alpha + \frac{Y'^2}{2r^2}s\cos\alpha - \frac{s^3\sin^2\alpha\cos\alpha}{6r^2}\right]^2 + \left[s\sin\alpha - \frac{Ys^2\cos^2\alpha}{2r^2} - \frac{s^3\sin\alpha\cos^2\alpha}{6r^2}\right]^2$$
$$= s^2\cos^2\alpha + \frac{Y'^2s^2\cos^2\alpha}{r^2} - \frac{s^4\sin^2\alpha\cos^2\alpha}{3r^2} + s^2\sin^2\alpha - \frac{Ys^3\cos^2\alpha\sin\alpha}{r^2} - \frac{s^4\sin^2\alpha\cos^2\alpha}{3r^2} + \ldots$$
$$= s^2 + \frac{s^2\cos^2\alpha}{r^2}(Y'^2 - Ys\sin\alpha) - \tfrac{2}{3}s^4\sin^2\alpha\cos^2\alpha + \ldots$$

Hence, by the binomial theorem :—

$$S = s + \frac{s\cos^2\alpha}{2r^2}(Y'^2 - Ys\sin\alpha) - \frac{s^3\sin^2\alpha\cos^2\alpha}{3r^2} + \ldots$$

Now, by using the approximate values $s \cos\alpha = (X' - X)$ and $s \sin\alpha = (Y' - Y)$, we have :—

$$S = s + \frac{s\cos^2\alpha}{2r^2}(Y'^2 - Y'Y + Y^2) - \frac{s\cos^2\alpha}{3r^2}(Y' - Y)^2$$

$$= s + \frac{s\cos^2\alpha}{6r^2}(Y'^2 + Y'Y + Y^2).$$

$$\therefore \frac{S}{s} = 1 + \frac{\cos^2\alpha}{6r^2}(Y'^2 + Y'Y + Y^2),$$

and hence :—

$$\frac{s}{S} = 1 - \frac{\cos^2}{6r^2}\alpha(Y'^2 + Y'Y + Y^2).$$

The ratio $\frac{S}{s}$ is generally called the "scale" and the quantity $\frac{\cos^2\alpha}{6r^2}(Y'^2 + Y'Y + Y^2)$ the "scale factor." It will thus be seen that, except when $\alpha = 90°$ or $270°$, all natural distances away from the central meridian are increased when plotted on the plane or computed from rectangular spherical co-ordinates.

When the points are very close together $Y' = Y$ and :—

$$\frac{S}{s} = 1 + \frac{Y^2}{2r^2}\cos^2\alpha.$$

Along the meridian $\alpha = 0°$, and :—

$$\frac{S}{s} = 1 + \frac{Y^2}{2r^2}.$$

Along the Y co-ordinate $\alpha = 90°$ and $\frac{S}{s} = 1$.

In a similar manner, if β is the bearing on the plane, so that β is given by :—

$$\tan\beta = \frac{Y' - Y}{X' - X}.$$

it can be shown that :—

$$\beta = \alpha - \frac{Y'^2 + Y'Y + Y^2}{6r^2 \sin 1''}\sin\alpha\cos\alpha - \frac{(X' - X)(2Y + Y')}{6r^2 \sin 1''},$$

which, for very short lines, becomes :—

$$\beta = \alpha - \frac{Y^2}{2r^2 \sin 1''}\sin\alpha\cos\alpha.$$

For computing purposes we can put :—

$$\frac{Y'^2 + Y'Y + Y^2}{6r^2} = \frac{1}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \frac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right],$$

where $\left(\frac{Y' + Y}{2}\right)^2$ and $\left(\frac{Y' - Y}{2}\right)^2$ can easily be found from a table of

squares or from the Tables of Quarter Squares in Chambers's *Mathematical Tables.* Hence,

$$\frac{S}{s} = 1 + \frac{\cos^2\alpha}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right],$$

and

$$\beta = \alpha - \frac{\sin\alpha\cos\alpha}{2r^2\sin 1''}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right] - \frac{(X' - X)(2Y + Y')}{6r^2\sin 1''}.$$

In addition, it will usually be more convenient to apply a correction to log s to give log S rather than to apply the correction to s itself. We then have :—

$$\log S = \log s + \frac{M\cos^2\alpha}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right],$$

and

$$\log s = \log S - \frac{M\cos^2\alpha}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right],$$

where $M = 0{\cdot}434\ 29$ ($\log M = \bar{1}{\cdot}637\ 78$) is the modulus of the common logarithms.

The above formulæ give the corrections to be applied to the observed spherical distances and bearings to give the distances and bearings on the plane. These distances and bearings on the plane are often called "grid" distances and "grid" bearings and they may be used directly to compute plane co-ordinates, which now also become rectangular spherical co-ordinates. In this case the procedure is to compute an approximate value of Y' from $Y' = Y + s\sin\alpha$ and to use this approximate Y' for computing S and β. The final co-ordinates then become

$$X' = X + S\cos\beta.$$
$$Y' = Y + S\sin\beta.$$

The formula for the correction to bearings is easily obtained by dividing $(Y' - Y)$ by $(X' - X)$. When terms involving powers of r higher than the second are neglected, this gives :—

$$\tan\beta = \frac{Y' - Y}{X' - X} = \tan\alpha - \frac{Y'^2}{2r^2}\tan\alpha - \frac{Y}{2r^2}s\cos\alpha + \frac{s^2\tan\alpha}{6r^2}(\sin^2\alpha - \cos^2\alpha).$$

Put $\beta = \alpha + \delta\alpha$. Then $\tan\beta = \tan(\alpha + \delta\alpha)$, which, by expansion by Taylor's theorem, is equal to $\tan\alpha + \delta\alpha\sec^2\alpha + \ldots$. Hence :—

$$\delta\alpha = -\frac{Y'^2}{2r^2}\sin\alpha\cos\alpha - \frac{Ys\cos^3\alpha}{2r^2} + \frac{s^2\sin\alpha\cos\alpha}{6r^2}(\sin^2\alpha - \cos^2\alpha)$$
$$= -\frac{\cos\alpha}{6r^2}\left[3Y'^2\sin\alpha + 3Ys\cos^2\alpha - s^2\sin\alpha(\sin^2\alpha - \cos^2\alpha)\right].$$

Substituting the approximate values $\frac{(Y'-Y)}{s}$ and $\frac{(X'-X)}{s}$ for the sines and cosines respectively in the terms in the square bracket, we obtain, after a little simplification :—

$$\begin{aligned} \delta\alpha &= -\frac{\cos\alpha}{6r^2 s}\left[(Y'+2Y)\left\{(Y'-Y)^2+(X'-X)^2\right\}+(Y'^3-Y^3)\right] \\ &= -\frac{\cos\alpha}{6r^2 s}\left[(Y'+2Y)s^2+(Y'-Y)(Y'^2+Y'Y+Y^2)\right] \\ &= -\frac{s\cos\alpha}{6r^2}(Y'+2Y)-\frac{\cos\alpha}{6r^2}\cdot\frac{Y'-Y}{s}(Y'^2+Y'Y+Y^2) \\ &= -\frac{(X'-X)(Y'+2Y)}{6r^2}-\frac{\cos\alpha\sin\alpha}{6r^2}(Y'^2+Y'Y+Y^2). \end{aligned}$$

Errors in Distances and Bearings Arising from the Use of a System of Simple Plane Rectangular Co-ordinates. It will be obvious from the foregoing discussion that, if a system of simple plane rectangular co-ordinates is used to compute a survey, serious discrepancies between observed and computed distances and bearings will arise as the survey proceeds to considerable distances outwards in either direction from the central meridian. For short lines, the ratio $\frac{S}{s}$ will be given by

$$\frac{S}{s} = 1 + \frac{Y^2}{2r^2}\cos\alpha$$

which is a maximum, and equal to $1 + \frac{Y^2}{2r^2}$, when $\alpha = 0°$ and is unity when $\alpha = 90°$. At the same time, as $X' - X$ is small for short lines, $\frac{(X'-X)(2Y'+Y)}{6r^2}$ is a small quantity and can be neglected. Hence, in this case, we have :—

$$\beta - \alpha = \frac{Y^2}{2r^2 \sin 1''}\sin\alpha\cos\alpha$$

which is a maximum, and equal to $-\frac{Y^2}{4r^2\sin 1''}$, for a bearing of 45° and is zero when the bearing is 0° or 90°.

The following table gives the approximate maximum linear and angular errors at various distances from the central meridian, the error in distance being expressed as a fraction of the length and that in bearing both by angle and by lateral displacement expressed as a fraction of length.

	Distance in Miles from Central Meridian				
	50	100	150	200	250
Maximum error in distance	$\frac{1}{12,510}$	$\frac{1}{3,142}$	$\frac{1}{1,396}$	$\frac{1}{785}$	$\frac{1}{503}$
Maximum error in bearing	8″·2	32″·8	1′ 13″·9	2′ 11″·3	3′ 25″·2
	$\frac{1}{25,154}$	$\frac{1}{6,289}$	$\frac{1}{2,791}$	$\frac{1}{1,571}$	$\frac{1}{1,005}$

Computation of Bearing and Distance from Rectangular Spherical Co-ordinates. The formulæ just evolved for errors in distance and bearing give a direct method of obtaining distance and bearing from given rectangular spherical co-ordinates. The first step is to compute the plane rectangular or grid bearing and distance from :—

$$\tan \beta = \frac{Y' - Y}{X' - X}$$

$$S = (Y' - Y) \operatorname{cosec} \beta = (X' - X) \sec \beta.$$

In computing the higher order terms containing the trigonometrical functions of α we may assume that $\alpha = \beta$, and, if we put,

$$K = \left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right]$$

we have :—

$$s = S\left[1 - \frac{K \cos^2 \beta}{2r^2}\right]$$

or,

$$\log s = \log S - \frac{MK \cos^2 \beta}{2r^2}$$

and

$$\alpha = \beta + \frac{\sin \beta \cos \beta}{2r^2 \sin 1''} K + \frac{(X' - X)(2Y + Y')}{6r^2 \sin 1''}$$

in which the factor K can be obtained from tables of squares or quarter squares and M is the modulus of the common logarithms.

An alternative procedure is to use the formulæ for obtaining rectangular spherical co-ordinates to obtain $s \, . \sin \alpha$ and $s \, . \cos \alpha$. Inserting the approximate values $(Y' - Y) = s \, . \sin \alpha$ and $(X' - X) = s \cos \alpha$ in the higher order terms we get :—

$$s \sin \alpha = (Y' - Y) + \frac{Y(X' - X)^2}{2r^2} + \frac{(X' - X)^2(Y' - Y)}{6r^2},$$

$$s \cos \alpha = (X' - X) - \frac{Y'^2(X' - X)}{2r^2} + \frac{(Y' - Y)^2(X' - X)}{6r^2}.$$

α is then obtained from :—

$$\tan \alpha = \frac{s \, . \sin \alpha}{s \, . \cos \alpha}$$

and s from :—

$$s = s \, . \sin \alpha \, . \operatorname{cosec} \alpha = s \cos \alpha \, . \sec \alpha.$$

Transverse Mercator Co-ordinates. We have just seen how the use of rectangular spherical co-ordinates produces distortion both in distance and in bearing, and at any single point the amount of these distortions varies with the direction of the line. To the ordinary surveyor, engineer or miner who will have to use the co-ordinates, distortion in bearing may be much more inconvenient than distortion in length. Moreover, it is highly inconvenient, for the short distances with which the ordinary surveyor will normally be concerned, to have both distortions dependent on the direction of the line. It is impossible to avoid distortion of some

kind, but it is possible to devise systems in which, *for short lines*, distortion in distance is independent of direction, and, in this case, the grid bearings computed from the co-ordinates, when the latter are treated as ordinary plane rectangular co-ordinates, will be identical with bearings on the ground, although this will not be so when very long lines are involved. When this condition is satisfied, the system is said to be "orthomorphic" or "conformal," and one result is that small figures retain their true shape, though not their true size or area.

The Transverse Mercator system of co-ordinates is merely a modification of rectangular spherical co-ordinates which has been designed to make the system conformal. It was first invented by Lambert; Gauss suggested its use for the survey of Hanover and of Egypt. He actually applied it in Hanover, and, about 1910, on the initiative of J. I. Craig, the Survey of Egypt adopted it for use in that country. In more recent years, it has been adopted by the Ordnance Survey for application in Great Britain, and most of the British African Colonies are now using it, either in its original or in a modified form. It is also in use in South Africa, Australia, and in certain States in America. Apart from ordinary plane rectangular co-ordinates, which are only applicable to small local surveys or to cases where accuracy is not essential, it may be said that the Transverse Mercator system is now more extensively used than any other.

Relation between Transverse and Rectangular Spherical Co-ordinates. In Fig. 127 (a) ABC is a small differential triangle on the earth's surface in which the rectangular spherical Y co-ordinate of the point C is equal to that of the point A. Fig. 127 (b) shows this triangle plotted on the

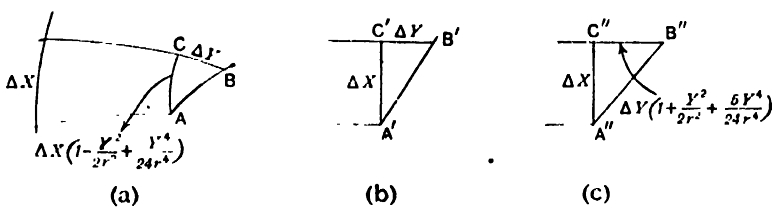


FIG. 127.

plane by treating the rectangular spherical as plane rectangular co-ordinates. Then C′B′ in Fig. 127 (b) is equal to CB in Fig. 127 (a) and this is equal to ΔY. A′C′ = ΔX in Fig. 127 (b) is greater than AC in Fig. 127 (a) by $\mathrm{AC}\left(\frac{Y^2}{2r^2}+\frac{5Y^4}{24r^4}\right)$ (page 349). Hence, in the limit, when they become infinitesimally small, so that ACB becomes a plane triangle on the plane tangential to the earth at the centre of the triangle, the two triangles ACB and A′C′B′ are not similar to one another, and A′C′B′ is greater than ACB in area. In order to make A′C′B′ similar to ACB we could either make C′B′ equal to CB, and at the same time decrease the X co-ordinate so that A′C′ is equal to AC, or we could keep A′C′ as it is and increase the co-ordinate X so that C′B′ becomes $\Delta Y\left(1+\frac{Y^2}{2r^2}+\frac{5Y^4}{24r^4}\right)$.

If we adopted the first procedure, the X co-ordinates would no longer be true distances along the central meridian, or proportional to them, but would be less than true distances and not directly proportional to them. This would be inconvenient, as it is an advantage for the X co-ordinates to be directly proportional to true distances along the meridian. Accordingly, we keep the rectangular spherical X co-ordinate as it is and increase the rectangular spherical Y co-ordinate by the amount $\frac{Y^3}{6r^2} + \frac{Y^5}{24r^4}$. If this is done, the side C′B′ in the triangle A′C′B′ becomes $\Delta Y\left(1 + \frac{Y^2}{2r^2} + \frac{5Y^4}{24r^4}\right)$. Hence, the sides A′C′ and C′B′ in the triangle A′C′B′ are proportional to the sides AC and CB in the triangle ACB, the angle A′C′B′ is a right angle and the angle ACB becomes a right angle when the triangle is very small. The two triangles are therefore now similar in all respects, although A′C′B′ is slightly larger than ACB. Hence angle CAB, which is the spherical bearing of the line AB, is equal to C′A′B′, which is the bearing on the plane, or the grid bearing, of the line A′B′.

Fig. 127 (c) shows A′C′B′ replotted with the increased ΔY. Here,

$$\mathrm{A''C''} = \mathrm{A'C'} = \mathrm{AC}\left(1 + \frac{Y^2}{2r^2} + \frac{5}{24} \cdot \frac{Y^4}{r^4}\right),$$

$$\mathrm{C''B''} = \mathrm{C'B'}\left(1 + \frac{Y^2}{2r^2} + \frac{5}{24}\frac{Y^4}{r^4}\right) = \mathrm{CB}\left(1 + \frac{Y^2}{2r^2} + \frac{5}{24}\frac{Y^4}{r^4}\right).$$

Hence, since the triangles ACB and A″C″B″ are similar in all respects,

$$\mathrm{A''B''} = \mathrm{AB}\left(1 + \frac{Y^2}{2r^2} + \frac{5}{24}\frac{Y^4}{r^4}\right).$$

By retaining the rectangular spherical X and increasing the rectangular spherical Y in the manner described above we get the Transverse Mercator co-ordinates. Hence :—

Transverse Mercator co-ordinates are simply the rectangular spherical co-ordinates with the term $\frac{Y^3}{6r^2} + \frac{Y^5}{24r^4}$ *added to the spherical Y co-ordinate.**

To derive the term $\frac{Y^3}{6r^2} + \frac{Y^5}{24r^4}$, divide the rectangular spherical Y co-ordinate into a number of small equal parts, each of length ΔY. Then we have seen that, in order that figures on the plane should be similar to figures on the sphere, the nth part will have to be increased by $\Delta Y\left(\frac{Y_n^2}{2r^2} + \frac{5}{24}\frac{Y_n^4}{r^4}\right)$, where Y_n is the Y co-ordinate

* This relation is an approximate form of the more general expression $Y_M = r \,.\, \log_e \tan\left(45° + \frac{Y_R}{2r}\right)$, or $\sin\frac{Y_R}{r} = \tanh\frac{Y_M}{r}$, where Y_M is the Transverse Mercator Y and Y_R the rectangular spherical Y. The term $\left(1 + \frac{Y^2}{2r^2} + \frac{5}{24}\frac{Y^4}{r^4}\right)$ is made up of the first three terms of the expansion of $\sec\frac{Y}{r}$ and $\log_e \tan\left(45° + \frac{Y}{2r}\right)$ will be recognised as the integral of $\sec\frac{Y}{r}$.

of the centre of the nth part. Similarly, the $(n-1)$th part will be increased by $\Delta Y\left(\frac{Y_{n-1}^2}{2r^2}+\frac{5}{24}\frac{Y_{n-1}^4}{r^4}\right)$. Hence, the total increment to Y_n is :—

$$\frac{\Delta Y}{2r^2}\left\{Y_1^2+Y_2^2+Y_3^2+\ldots+Y_{n-1}^2+Y_n^2\right\}$$
$$+\frac{5}{24}\frac{\Delta Y}{r^4}\left\{Y_1^4+Y_2^4+Y_3^4+\ldots+Y_{n-1}^4+Y_n^4\right\}.$$

In the limit, as ΔY becomes infinitesimally small and n infinitely great, this becomes :—

$$\int_0^Y\frac{Y^2}{2r^2}dY+\int_0^Y\frac{5}{24}\frac{Y^4}{r^4}dY$$
$$=\frac{Y^3}{6r^2}+\frac{Y^5}{24r^4}.$$

For most practical purposes, the term in $\frac{}{r^4}$ is negligible and the Transverse Mercator Y then becomes the rectangular spherical Y plus $\frac{Y^3}{6r^2}$.

General Properties of Transverse Mercator Co-ordinates. We can now summarise the main characteristics of Transverse Mercator co-ordinates. These are :—

(1) Distances along the central meridian remain meridian distances, or are proportional to them.

(2) Away from the central meridian, the computed or plotted lengths —the grid lengths—of lines are greater than their true lengths on the sphere.

(3) At any single point, the increase in the length of a *short* line, or the ratio of grid length to true length, depends only on the position of the point and is independent of the direction of the line.

(4) For *short* lines, computed or plotted bearings—grid bearings—remain the same as the bearings on the sphere.

(5) The shape of a *small* figure plotted on the plane is the same as the shape of the equivalent figure on the sphere, though its size and area are increased.

(6) The Transverse Mercator X co-ordinate is the same as the rectangular spherical X co-ordinate.*

(7) The Transverse Mercator Y co-ordinate is the rectangular spherical Y co-ordinate plus the term $\frac{Y^3}{6r^2}+\frac{1}{24}\frac{Y^5}{r^4}$.

Formulæ for Transverse Mercator Co-ordinates. The following are the formulæ usually needed in working with Transverse Mercator co-ordinates :—

(1) *Relation between Rectangular Spherical and Transverse co-ordinates.*

$$\left.\begin{aligned}X_M&=X_R\\ Y_M&=Y_R+\frac{Y_R^3}{6R_cN_c}+\frac{Y_R^5}{24R_c^2N_c^2}\end{aligned}\right\}\quad\ldots\ldots\quad(1)$$

* This is not strictly true for the spheroid, although the difference is negligible in practice. See *Some Investigations into the Theory of Map Projections*, by A. E. Young, page 73.

$$\left.\begin{aligned} X_R &= X_M \\ Y_R &= Y_M - \frac{Y_M^3}{6R_cN_c} + \frac{Y_M^5}{24R_c^2N_c^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the R and M suffixes denote spherical rectangular and Transverse Mercator co-ordinates respectively.

(2) *Conversion of Transverse Mercator into Geographical Co-ordinates* :—

$$\phi_c = \phi_o + X^\circ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which X° is the distance X converted into degrees, minutes and seconds by means of a table of meridional distances and ϕ_o is the latitude of the origin

$$\phi = \phi_c - \frac{Y^2 \tan \phi_c}{2R_cN_c \sin 1''} + \frac{Y^4}{24R_cN_c^3 \sin 1''} \tan \phi_c (5 + 3 \tan^2 \phi_c) \quad . \quad . \quad . \quad . \quad (4)$$

$$dL'' = \frac{Y \sec \phi_c}{N_c \sin 1''} - \frac{Y^3}{6N_c^3 \sin 1''} \sec \phi_c \left(\frac{N_c}{R_c} + 2 \tan^2 \phi_c\right) + \frac{Y^5}{120N_c^5 \sin 1''} \sec \phi_c (5 + 28 \tan^2 \phi_c + 24 \tan^4 \phi_c) \quad . \quad . \quad (5)$$

$$\gamma'' = \frac{Y \tan \phi_c}{N_c \sin 1''} - \frac{Y^3}{3N_c^3 \sin^3 1''} \tan \phi_c \left(2 + \tan^2 \phi_c - \frac{N_c}{R_c}\right) \sin^2 1'' + \frac{Y^5}{15N_c^5 \sin^5 1''} \tan \phi_c \sec^2 \phi_c (2 + 3 \tan^2 \phi_c) \sin^4 1'' . \quad . \quad (6)$$

$$\gamma'' = \frac{Y \tan \phi}{N \sin 1''} + \frac{Y^3}{6N^3 \sin 1''} \tan \phi \left(5\frac{N}{R} - 4 + \tan^2 \phi\right) + \frac{Y^5}{120N^5 \sin 1''} \tan \phi \sec^2 \phi (1 + 9 \tan^2 \phi) \quad . \quad . \quad . \quad . \quad . \quad (6\text{A})$$

(3) *Conversion of Geographical into Transverse Mercator Co-ordinates* :—

$$X = X_p + \frac{(dL'' \sin 1'')^2 N \sin \phi \cos \phi}{2} + \frac{(dL'' \sin 1'')^4 N \sin \phi \cos^3 \phi (5 - \tan^2 \phi)}{24} \quad . \quad . \quad . \quad (7)$$

where $X_p = \phi - \phi_o$ converted into feet by means of a table of meridian distances.

$$Y = (dL'' \cos \phi \sin 1'')N + \frac{(dL'' \cos \phi \sin 1'')^3}{6} N\left(\frac{N}{R} - \tan^2 \phi\right) + \frac{(dL'' \cos \phi \sin 1'')^5}{120} N(5 - 18 \tan^2 \phi + \tan^4 \phi) \quad . \quad (8)$$

$$\gamma'' = dL'' \sin \phi + dL''^3 \sin \phi \cos^2 \phi \left(\frac{N}{R} - \frac{2}{3}\right) \sin^2 1'' + \frac{dL''^5 \sin \phi \cos^4 \phi (2 - \tan^2 \phi) \sin^4 1''}{15} \quad . \quad . \quad . \quad . \quad (9)$$

In formulæ (1) to (6) above, R_{ϵ} and N_{ϵ} are to be taken out for the latitude ϕ_{ϵ}, that is for the point $X = X$, $Y = 0$, which is the foot of the perpendicular on the central meridian from the point (X, Y). In formulæ (6A) to (9), R and N are for the given latitude ϕ.

(4) *Co-ordinates from Bearing and Distance* :—

$$Y' = Y + s\sin\alpha - \frac{Ys^2\cos^2\alpha}{2r^2} - \frac{s^2\cos^2\alpha \,.\, s\sin\alpha}{6r^2} + \frac{(Y'^3 - Y^3)}{6r^2} \quad \ldots \quad (10)$$

in which $r = \sqrt{RN}$, this quantity being taken out for the mid-latitude of the line.

$$X' = X + s\cos\alpha + \frac{Y'^2 s\cos\alpha}{2r^2} - \frac{s\cos\alpha \,.\, s^2\sin^2\alpha}{6r^2} \quad \ldots \quad (11)$$

$$\alpha' = \alpha \pm 180^\circ - \frac{s\cos\alpha(Y + Y')}{2r^2\sin 1''} \quad \ldots \quad (12)$$

Alternatively, by grid distances and bearings as follows :—

Calculate approximate values of Y' from $Y' = Y + s\sin\alpha$ and of X' from $X' = X + s\cos\alpha$ and use these values in the following equations :—

$$S = s + \frac{s}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right] \quad \ldots \quad (13)$$

or

$$\log S = \log s + \frac{M}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right] \quad \ldots \quad (13\text{A})$$

where $\log M = \bar{1}{\cdot}637\ 78$, and

$$\beta = \alpha - \frac{(X' - X)(2Y + Y')}{6r^2\sin 1''} \quad \ldots \quad (14)$$

Then :—

$$X' = X + S\cos\beta\,;\quad Y' = Y + S\sin\beta \quad \ldots \quad (15)$$

$$\alpha' = \beta \pm 180^\circ + \frac{(X - X')(2Y' + Y)}{6r^2\sin 1''} \quad \ldots \quad (16)$$

(5) *Bearing and Distance from Co-ordinates* :—

$$s = S - \frac{S}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right] \quad \ldots \quad (17)$$

or,

$$\log s = \log S - \frac{M}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right] \quad \ldots \quad (17\text{A})$$

$$\alpha = \beta + \frac{(X' - X)(2Y + Y')}{6r^2\sin 1''} \quad \ldots \quad (18)$$

$$\alpha' = \beta \pm 180^\circ + \frac{(X - X')(2Y' + Y)}{6r^2\sin 1''} \quad \ldots \quad (19)$$

where,

$$\tan \beta = \frac{(Y' - Y)}{(X' - X)}, \; S = (X' - X) \sec \beta = (Y' - Y) \operatorname{cosec} \beta \quad (20)$$

(6) *Scale and Scale Error.* For *long* lines see (13), (13A), (17) and (17A) above. For *short* lines :—

$$\frac{S}{s} = 1 + \frac{Y^2}{2r^2} + \frac{Y^4}{24r^4} \quad \ldots \quad (21)$$

$$\log S = \log s + \frac{MY^2}{2r^2}\left(1 - \frac{Y^2}{6r^2}\right) \quad \ldots \quad (21\text{A})$$

$$\frac{s}{S} = 1 - \frac{Y^2}{2r^2} + \frac{5}{24}\frac{Y^4}{r^4} \quad \ldots \quad (22)$$

$$\log s = \log S - \frac{MY^2}{2r^2}\left(1 - \frac{Y^2}{6r^2}\right) \quad \ldots \quad (22\text{A})$$

in which $\log M = \bar{1}{\cdot}637\ 78$.

The sign conventions to be used with Transverse Mercator co-ordinates are the same as those given on page 343 for use with ordinary rectangular spherical co-ordinates.

Note that in formulæ (12), (14), (16), (18) and (19) there is no term involving Y^2, such as there is in the equivalent formulæ for rectangular spherical co-ordinates. The last term on the right in each expression becomes inappreciable *for short lines*, and, in this case, the plane or grid bearing is equal to the spherical one and so is independent of the direction of the line. Note also that $\cos^2 \alpha$ does not occur in formulæ (13), (13A) (17), (17A), (21), (21A), (22) and (22A), as it does in the equivalent formulæ for rectangular spherical co-ordinates, so that scale and scale error are independent of the direction of the line. This, of course, should be the case since it represents the main condition which the system was designed to fulfil. When lines are very short, Y' is approximately equal to Y and formulæ (13), (13A), (17) and (17A), which are the formulæ for long lines, reduce to the first two terms of formulæ (21), (21A), (22) and (22A).

In formulæ (5), (6), (8) and (9) the terms $\frac{N_c}{R_c}$ and $\frac{N}{R}$ have been introduced. These are spheroidal terms and are appreciable only when the utmost accuracy is desired and Y is very large. For most purposes it will be sufficient if $\frac{N_c}{R_c}$ and $\frac{N}{R}$ are put equal to unity, in which case the formulæ become those for the Transverse Mercator co-ordinates on the sphere.

It has been shown on page 356 that, if Y_M is the Transverse Mercator and Y_R the rectangular spherical Y, Y_M is given by :—

$$Y_M = Y_R + \frac{Y_R^3}{6r^2} + \frac{Y_R^5}{24r^4}.$$

Hence, by successive approximations, or by the formula for the reversion of series (page 8),

$$Y_R = Y_M - \frac{Y_M^3}{6r^2} + \frac{Y_M^5}{24r^4}.$$

If this value of Y_R is inserted in the formulæ for the conversion of rectangular spherical into geographical co-ordinates given on page 343, and $\sqrt{RN}$ is substituted for r, we get the equivalent formulæ for the Transverse Mercator co-ordinates. These are (4), (5) and (6) above but with N substituted for R in the small terms in Y^4 and Y^5 in cases where this approximation simplifies the expression without appreciable loss of accuracy.*

Equations (7) and (9) are the equivalent of the similar formulæ for rectangular spherical co-ordinates. To obtain (8) we have, from the corresponding formulæ for rectangular sphericals (page 306),

$$Y_R = NdL \sin 1'' \cos\phi - \frac{N}{6}(dL \sin 1'' \cos\phi)^3 \tan^2\phi$$
$$- \frac{N}{120}(dL \sin 1'' \cos\phi)^5 \tan^2\phi\,(8 - \tan^2\phi).$$

In this write $Y_R = Y - \dfrac{Y^3}{6RN} + \dfrac{Y^5}{24R^2N^2}$ on the left-hand side of the equation.

A first approximation gives $Y = NdL \sin 1'' \cos\phi$. A second gives :—

$$Y - \frac{N^2(dL \sin 1'' \cos\phi)^3}{6R} = NdL \sin 1'' \cos\phi$$
$$- \frac{N}{6}(dL \sin 1'' \cos\phi)^3 \tan^2\phi$$

from which :—

$$Y = NdL \sin 1'' \cos\phi + \frac{N}{6}(dL \sin 1'' \cos\phi)^3\left(\frac{N}{R} - \tan^2\phi\right).$$

This value is now substituted in the term $\dfrac{Y^3}{6RN}$, and $(NdL \sin 1'' \cos\phi)^5$ in the term $\dfrac{1}{24}\dfrac{Y^5}{R^2N^2}$, when, after a little reduction, and the substitution of N for R in terms containing Y^5, we obtain :—

$$Y = N(dL \cos\phi \sin 1'') + \tfrac{1}{6}N(dL \cos\phi \sin 1'')^3\left(\frac{N}{R} - \tan^2\phi\right)$$
$$+ \tfrac{1}{120}N(dL \cos\phi \sin 1'')^5(5 - 18\tan^2\phi + \tan^4\phi).$$

Equation (10) is the rectangular spherical Y with the term $\dfrac{(Y'^3 - Y^3)}{6r^2}$ added on to it to convert from the rectangular spherical to the Transverse Mercator system, and equations (11) and (12) are the same as in the rectangular spherical system.

In deriving the expressions for $\dfrac{S}{s}$ and $(\beta - \alpha)$ we may proceed exactly as we did on pages 352 and 353 for rectangular sphericals, but substituting the value for $(Y' - Y)$ from equation (10) instead of the value originally used. The following method of obtaining $\dfrac{S}{s}$ is, however, shorter and neater than the one we used in the previous case.

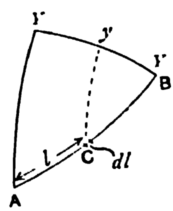


FIG. 128.

In Fig. 128 take the point C on the line AB, the distance AC being l and that of AB being s. Take a small element at C of length dl and let the Y co-ordinate of this element be y, that of A being Y and that of B being Y'. Then the element on the plane corresponding to dl will be given by :—

$$dS = dl\left\{1 + \frac{y^2}{2r^2} + \ldots\right\}.$$

* For the derivation of the full spheroidal formulæ from the spherical formulæ see paper by Captain McCaw in the *Empire Survey Review*, Vol. V, No. 35, January, 1940.

The total length of the line on the plane will therefore be the line integral

$$S = \int_C \left(1 + \frac{y^2}{2r^2} + \ldots\right) dl$$

taken along the curve AB.

But we can easily see that

$$\frac{l}{s} = \frac{y - Y}{Y' - Y} \text{ approximately.}$$

$$\therefore\ dl = \frac{s\,dy}{(Y' - Y)}$$

$$\therefore\ S = \frac{s}{(Y' - Y)} \int_Y^{Y'} \left(1 + \frac{y^2}{2r^2} + \ldots\right) dy$$

$$= s\left[1 + \frac{(Y'^3 - Y^3)}{6r^2(Y' - Y)}\right]$$

$$= s\left[1 + \frac{Y'^2 + Y'Y + Y^2}{6r^2}\right]$$

$$= s\left[1 + \frac{1}{2r^2}\left\{\left(\frac{Y' + Y}{2}\right)^2 + \frac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right\}\right].$$

Reduction of Scale Error. The scale error factor in the Transverse Mercator system, $\frac{Y^2}{2r^2}$, becomes 1/2000 at a distance of about 125 miles from the central meridian and at about 200 miles it has a value of about 1/785. Even a scale error of 1/2000 is inconveniently large in practice. A large-scale error at the outside limits of the area under survey can, however, be reduced by introducing a negative scale error along the central meridian of, say, half the amount of the maximum positive scale error. Thus, if the area to be covered is about 250 miles in width—that is, about 125 miles on either side of the central meridian—the normal scale errors would be zero on the central meridian and + 1/2000 at the extreme edges where Y has the values − 125 miles and + 125 miles. If a negative scale error of 1/4000 is introduced, the scale error becomes − 1/4000 along the central meridian and + 1/4000 along two lines 125 miles on either side of the central meridian. At two lines about 89 miles on either side of the central meridian the scale error will be zero, and at no place will it exceed 1/4000.

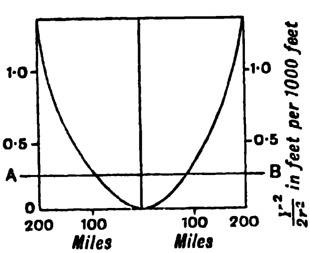


FIG. 129.

This method of introducing a negative scale error along the central meridian, so as to reduce the maximum value of the error, is very commonly adopted in practice.

Fig. 129 shows a plot of the scale error factor $\frac{Y^2}{2r^2}$, abscissæ being distances in miles from the central meridian and ordinates the scale error factor in feet per thousand feet. The curve is a parabola with all points positive and above the horizontal axis. The line AB has been drawn through the point 0·25 ft.

on the vertical axis. This line therefore represents a scale error of 1/4000, and, if it is taken as the new zero line for scale error, there will be a negative scale error for all points on the parabola which lie below this line and a positive scale error for all points which lie above it.

If the greatest distance from the central meridian is d, and if the negative scale error on the central meridian is to be equal to the positive scale error at distance d, the points at which there is zero scale error will be given by :—

$$\frac{Y^2}{2r^2} = \frac{d^2}{2 \times 2r^2},$$

or

$$Y = \pm \frac{d}{\sqrt{2}}.$$

The reduction of the maximum scale error is made by multiplying all measured distances by the factor $\left(1 - \frac{1}{F}\right)$, where $\frac{1}{F}$ is the scale error on the central meridian. In addition, a spheroid which is smaller than the true spheroid by the same proportional amount must be used in the computations. This means that meridian distances and values for R and N must also be multiplied by $\left(1 - \frac{1}{F}\right)$. In this case, it is best to prepare special tables in which this reduction has purposely been made.

Countries which have such a large extent in longitude that even the introduction of a negative scale factor still makes the scale error too large in extreme cases are divided into two or more belts, each with its own origin and its own central meridian. Thus, Nigeria has an extent of over 12° in longitude—about 760 miles in those latitudes—and it is divided into three belts. In Great Britain, there is a single meridian—2° W—so that no part of the country extends much more than about 150 miles from this meridian.

Computation of Triangulation in Terms of Transverse Mercator Co-ordinates. A system of triangulation can be solved directly in terms of Transverse Mercator co-ordinates if the following procedure is adopted :—

(*a*) Assuming that the co-ordinates and spherical bearing at only one end of the initial line are known, compute approximate co-ordinates of the other end by using the true distance and the spherical bearing α in the formulæ :—

$$X' = X + s \cos \alpha \; ; \quad Y' = Y + s \sin \alpha.$$

In addition, compute the reverse bearing by formula (12). If the azimuth of the initial line, and not the spherical bearing, is given, compute the convergence and apply it to the observed azimuth to give α as described in page 342.

(*b*) Use these approximate co-ordinates to find the co-ordinate or grid length S of the initial side. The formula needed is (13) or (13A), page 359.

(*c*) Using this value for the initial side, solve the first triangle by using the observed (spherical) angles, each reduced by one-third of the spherical excess of the triangle.

(*d*) The last process gives approximate values for the length of each side. Knowing the forward and reverse bearings of the line AB, and the observed angles at A and B, the spherical bearings of the lines AC and BC can be found. Use these bearings and the approximate lengths derived from the preliminary solution of the triangle to obtain the approximate co-ordinates of C.

(*e*) Use these approximate co-ordinates to determine the correction $\beta - \alpha$ to be applied to the spherical bearings to give the grid bearing of each side. The formulæ required are (14) and (16), page 359.

(*f*) Apply these corrections to the spherical bearings to obtain grid bearings and from the grid bearings deduce the plane or grid angles.* Assuming that the triangle has previously been adjusted for closure, the sum of the plane angles should, of course, be equal to 180°.

(*g*) Using the plane distance as obtained in (*b*), and the plane or grid angles as obtained in (*f*), solve the triangle as an ordinary plane triangle. This will give the lengths on the plane of the other two sides of the triangle and these values can then be used as the bases for other plane triangles.

(*h*) The final co-ordinates of the points B and C can now be calculated from the grid bearings and grid lengths by means of the relations :—

$$X' = X + S\cos\beta\,; \quad Y' = Y + S\sin\beta.$$

The alternative to the above method of computation is to solve the triangles by Legendre's theorem, using the spherical or natural lengths and angles, and then to work out co-ordinates by the bearing and distance formulæ (10) and (11), page 359.

The example given below will not only illustrate the above procedure but will also serve as an example of various computations in connection with the formulæ given on pages 357 to 360. The computation appears to be a very long one but it must be remembered that it starts from the very beginning, as if the first side were a base line. After the first triangle has been solved, the work for each of the remaining triangles becomes much less because the grid length of each initial line and the bearings and co-ordinates of both ends are now known. Hence, steps (1), (2), (3) and parts of steps (6) and (8) in the example are no longer necessary. Moreover, the work can be simplified and reduced by employing specially computed tables such as are described on page 369.

* The corrections, if taken with their proper signs and applied in the correct order, may be applied directly to the angles by noting that an angle is the difference between two bearings. The sum of the corrections to the three angles should equal the spherical excess of the triangle.

The method of solution by working on the plane with modified lengths and bearings is particularly suitable for computing secondary and tertiary work, or where the adjustment of the figures involves certain sides and directions being held fixed in a least square adjustment, such as occurs when new work has to be joined to old. It is also invaluable in traverse work.

Example. Given the following data with regard to the triangle ABC, find the co-ordinates of the points B and C.

Transverse Mercator co-ordinates of A.

$X = +\ 813\ 687{\cdot}03$ ft. $\qquad Y = +\ 325\ 707{\cdot}28$ ft.

Geographical co-ordinates of A.

$\phi = 6°\ 54'\ 30''{\cdot}907$ N. $\qquad L = 0°\ 06'\ 06''{\cdot}380$ W.

log . AB (length on ground) = 5·142 2852.
Observed azimuth A to B = 104° 52′ 21″·00.
Observed angles of triangle after adjustment.

$$\begin{aligned} A &= 45°\ 48'\ 58''{\cdot}64 \\ B &= 87\ \ 47\ \ 47\ {\cdot}94 \\ C &= 46\ \ 23\ \ 17\ {\cdot}94 \\ \hline & 180°\ 00'\ 04''{\cdot}52 = 180° + \text{spherical excess.} \end{aligned}$$

The central meridian is 1° W, and the point C lies north of A.

(1) Convergence and bearing at A.

$$dL = +\ 53'\ 53''{\cdot}616 = 3233''{\cdot}616$$

$$\begin{aligned} \log dL &= 3{\cdot}509\ 6884 \\ \log \sin\phi &= 9{\cdot}080\ 2136 \\ \hline \log dL \sin\phi &= 2{\cdot}589\ 9020 \end{aligned}$$

$$\begin{aligned} dL \sin\phi &= 388{\cdot}957 \\ (2) &= +\ 0{\cdot}032 \\ \hline & 388{\cdot}989 \end{aligned}$$

$$\begin{aligned} \frac{N}{R} &= 1{\cdot}00675 \\ \left(\frac{N}{R} - \frac{2}{3}\right) &= +\ 0{\cdot}34008 \\ 3 \log dL &= 10{\cdot}529\ 07 \\ \log \sin\phi &= \bar{1}{\cdot}080\ 21 \\ 2 \log \cos\phi &= \bar{1}{\cdot}993\ 67 \\ \log\left(\frac{N}{R} - \frac{2}{3}\right) &= \bar{1}{\cdot}531\ 58 \\ 2 \log \sin 1'' &= \bar{11}{\cdot}371\ 15 \\ \hline & \bar{2}{\cdot}505\ 68 \end{aligned}$$

$$\begin{aligned} \gamma &= +\ 6'\ 28''{\cdot}99 \\ A &= 104°\ 52'\ 21''{\cdot}00 \\ -\gamma &= -\ 6\ \ 28''{\cdot}99 \\ \hline \alpha &= 104°\ 45'\ 52''{\cdot}01 \end{aligned}$$

$$dL^3 \sin\phi \cos^2\phi\left(\frac{N}{R} - \frac{2}{3}\right)\sin^2 1'' = +\ 0''{\cdot}032 \quad . \quad . \quad . \quad . \quad (2)$$

Third term is negligible.

(2) Approximate co-ordinates of B

$$\begin{aligned} \log \Delta X &= 4{\cdot}548\ 5627 \\ \hline \log \cos\alpha &= 9{\cdot}406\ 2775 \\ \log s &= 5{\cdot}142\ 2852 \\ \log \sin\alpha &= 9{\cdot}985\ 4183 \\ \hline \log \Delta Y &= 5{\cdot}127\ 7035 \end{aligned}$$

	X		Y
A	813 687·03		325 707·28
ΔX =	−35 364·11	ΔY =	+ 134 184·87
B	778 322·92		459 892·15

(3) Grid distance and reverse bearing of AB.

$$Y' = 459\ 900$$
$$Y = 325\ 700$$

$$(Y' + Y) = 785\ 600 \qquad (Y' - Y) = 134\ 200$$
$$\left(\frac{Y' + Y}{2}\right)^2 = 154\ 291\ 840\ 000 \qquad \left(\frac{Y' - Y}{2}\right)^2 = 4\ 502\ 410\ 000$$
$$\tfrac{1}{3}\left(\frac{Y' - Y}{2}\right)^2 = 1\ 500\ 800\ 000$$
$$K = 155\ 792\ 640\ 000$$

$$\log K = 11{\cdot}192\ 55$$
$$\log M = \bar{1}{\cdot}637\ 78$$
$$\log \frac{1}{2r^2} = \overline{15}{\cdot}060\ 46$$
$$\log \frac{KM}{2r^2} = 5{\cdot}890\ 79$$

$$\frac{KM}{2r^2} = 0{\cdot}000\ 0777$$
$$\log s = 5{\cdot}142\ 2852$$
$$\log S = 5{\cdot}142\ 3629$$

$$\log (Y' + Y) = 5{\cdot}895\ 20$$
$$\log (X' - X) = 4{\cdot}548\ 56$$
$$\log \frac{1}{2r^2 \sin 1''} = \overline{10}{\cdot}374\ 89$$
$$\log (\alpha' - \alpha) = 0{\cdot}818\ 65$$

$$\alpha = 104°\ 45'\ 52''{\cdot}01$$
$$180$$
$$284\quad 45\quad 52\ {\cdot}01$$
$$\alpha' - \alpha = +\quad 6\ {\cdot}59$$
$$\alpha' = 284°\ 45'\ 58''{\cdot}60$$

(4) Approximate solution of triangle ABC.

	Observed angle	$\frac{1}{3}$ of sph. ex.	Plane angle
A	45° 48′ 58″·64	− 1″·51	45° 48′ 57″·13
B	87 47 47 ·94	− 1 ·51	87 47 46 ·43
C	46 23 17 ·94	− 1 ·50	46 23 16 ·44
	180° 00′ 04″·52	− 4″·52	180° 00′ 00″·00

$$\log BC = 5{\cdot}138\ 1906$$

$$\log \sin A = 9{\cdot}855\ 5820$$
$$\log AB = 5{\cdot}142\ 3629$$
$$\log \operatorname{cosec} C = 0{\cdot}140\ 2457$$
$$\log \sin B = 9{\cdot}999\ 6786$$
$$\log AC = 5{\cdot}282\ 2872$$

(5) Bearings of AC and BC and approximate co-ordinates of C.

$$\alpha = 104°\ 45'\ 52{\cdot}01 \qquad \alpha' = 284\ 45\ 58{\cdot}60$$
$$A = 45\ \ 48\ \ 58{\cdot}64 \qquad B = 87\ 47\ 47{\cdot}94$$
$$\text{BgAC} = 58\ \ 56\ \ 53{\cdot}37 \qquad \text{BgBC} = 12\ 33\ 46{\cdot}54$$

$$\log \Delta X = 4{\cdot}994\ 7799 \qquad \log \Delta X = 5{\cdot}127\ 6661$$

$$\log \cos AC = 9{\cdot}712\ 4927 \qquad \log \cos BC = 9{\cdot}989\ 4755$$
$$\log AC = 5{\cdot}282\ 2872 \qquad \log BC = 5{\cdot}138\ 1906$$
$$\log \sin AC = 9{\cdot}932\ 8293 \qquad \log \sin BC = 9{\cdot}337\ 4827$$

$$\log \Delta Y = 5{\cdot}215\ 1165 \qquad \log \Delta Y = 4{\cdot}475\ 6733$$

	X	Y		X	Y
A	813 687	325 707	B	778 323	459 892
Δ	+ 98 805	+ 164 103	Δ	+ 134 173	+ 29 900
C	912 492	489 810	C	912 496	489 792

(6) Corrections to bearings $(\beta - \alpha)$.

AB and BA

$$
\begin{aligned}
Y' &= 459\ 892 & 2Y' &= 919\ 784 & \log \frac{1}{2r^2 \sin 1''} &= \overline{10}{\cdot}374\ 89 \\
2Y &= 651\ 414 & Y &= 325\ 707 & \log 3 &= 0{\cdot}477\ 12 \\
(2Y + Y') &= 1{,}111{,}306 & (2Y' + Y) &= 1{,}245{,}491 & \log \frac{1}{6r^2 \sin 1''} &= \overline{11}{\cdot}897\ 77
\end{aligned}
$$

$$
\begin{aligned}
\log \delta\alpha &= 0{\cdot}492\ 16 & \log (X' - X) &= 4{\cdot}548\ 56 \\
\log (2Y + Y') &= 6{\cdot}045\ 83 & \log \frac{1}{6r^2 \sin 1''} &= \overline{11}{\cdot}897\ 77 \\
\log \frac{(X' - X)}{6r^2 \sin 1''} &= \bar{6}{\cdot}446\ 33 & \log \frac{(X' - X)}{6r^2 \sin 1''} &= \bar{6}{\cdot}446\ 33 \\
\log (Y + 2Y') &= 6{\cdot}095\ 36 & & \\
\log \delta\alpha' &= 0{\cdot}541\ 69 & &
\end{aligned}
$$

$$
\begin{aligned}
-\delta\alpha &= +3''{\cdot}11 & -\delta\alpha' &= -3''{\cdot}48 \\
\alpha(AB) &= 104\ \ 45\ \ 52\ {\cdot}01 & \alpha'(BA) &= 284\ \ 45\ \ 58''{\cdot}60 \\
\beta(AB) &= 104^\circ\ 45'\ 55''{\cdot}12 & \beta'(BA) &= 284^\circ\ 45'\ 55''{\cdot}12
\end{aligned}
$$

AC and CA

$$
\begin{aligned}
Y' &= 489\ 801 & Y &= 325\ 707 \\
2Y &= 651\ 414 & 2Y' &= 979\ 602 \\
(2Y + Y') &= 1{,}141\ 215 & (2Y' + Y) &= 1{,}305\ 309 \\
\log \delta\alpha &= 0{\cdot}949\ 92 & \log (X' - X) &= 4{\cdot}994\ 78 \\
\log (2Y + Y') &= 6{\cdot}057\ 37 & \log \frac{1}{6r^2 \sin 1''} &= \overline{11}{\cdot}897\ 77 \\
\log \frac{(X' - X)}{6r^2 \sin 1''} &= \bar{6}{\cdot}892\ 55 & \log \frac{(X' - X)}{6r^2 \sin 1''} &= \bar{6}{\cdot}892\ 55 \\
\log (Y + 2Y') &= 6{\cdot}115\ 71 & & \\
\log \delta\alpha' &= 1{\cdot}008\ 26 & &
\end{aligned}
$$

$$
\begin{aligned}
-\delta\alpha &= -8''{\cdot}91 & +\delta\alpha &= -10''{\cdot}19 \\
\alpha(AC) &= 58\ \ 56\ \ 53\ {\cdot}37 & \beta'(CA) &\ \ 238\ \ 56\ \ 44\ {\cdot}46 \\
\beta(AC) &= 58^\circ\ 56'\ 44''{\cdot}46 & \alpha'(CA) &= 238^\circ\ 56'\ 34''{\cdot}27
\end{aligned}
$$

BC and CB

$$
\begin{aligned}
Y' &= 489\ 801 & Y &= 459\ 892 \\
2Y &= 919\ 784 & 2Y' &= 979\ 602 \\
2Y + Y' &= 1{,}409\ 585 & (2Y' + Y) &= 1{,}439{,}494 \\
\log \delta\alpha &= 1{\cdot}174\ 53 & \log (X' - X) &= 5{\cdot}127\ 67 \\
\log (2Y + Y') &= 6{\cdot}149\ 09 & \log \frac{1}{6r^2 \sin 1''} &= \overline{11}{\cdot}897\ 77 \\
\log \frac{(X' - X)}{6r^2 \sin 1''} &= \bar{5}{\cdot}025\ 44 & \log \frac{(X' - X)}{6r^2 \sin 1''} &= \bar{5}{\cdot}025\ 44 \\
\log (Y + 2Y') &= 6{\cdot}158\ 21 & & \\
\log \delta\alpha' &= 1{\cdot}183\ 65 & &
\end{aligned}
$$

$$
\begin{aligned}
-\delta\alpha &= -14''{\cdot}95 & +\delta\alpha' &= -15''{\cdot}26 \\
\alpha(BC) &= 12\ \ 33\ \ 46\ {\cdot}54 & \beta'(CB) &= 192\ \ 33\ \ 31\ {\cdot}59 \\
\beta(BC) &= 12^\circ\ 33'\ 31''{\cdot}59 & \alpha'(CB) &= 192^\circ\ 33'\ 16''{\cdot}33
\end{aligned}
$$

(7) Final solution of triangle ABC

	Spherical angle	Corrections to bearings		Plane angle
A	45° 48′ 58″·64	+ 3·11 − 8·91	+ 12″·02	45° 49′ 10″·66
B	87 47 47 ·94	− 14·95 − 3·48	− 11 ·47	87 47 36 ·47
C	46 23 17 ·94	+ 10·19 + 15·26	− 5 ·07	46 23 12 ·87
	180° 00′ 04″·52		− 4″·52	180° 00′ 00″·00

$$\log \text{BC} = 5{\cdot}138\ 2254$$

$$\begin{aligned}\log \sin A &= 9{\cdot}855\ 6096\\ \log \text{AB} &= 5{\cdot}142\ 3629\\ \log \operatorname{cosec} C &= 0{\cdot}140\ 2529\\ \log \sin B &= 9{\cdot}999\ 6778\end{aligned}$$

$$\log \text{AC} = 5{\cdot}282\ 2936$$

(8) Final co-ordinates of B and C.

$$\log \Delta X = 4{\cdot}548\ 6652$$

$$\begin{aligned}\log \cos \beta &= 9{\cdot}406\ 3023\\ \log S &= 5{\cdot}142\ 3629\\ \log \sin \beta &= 9{\cdot}985\ 4165\end{aligned}$$

$$\log \Delta Y = 5{\cdot}127\ 7794$$

	X	Y
A	+ 813 687·03	+ 325 707·28
Δ	− 35 372·47	+ 134 208·31
B	+ 778 314·56	+ 459 915·59

$\log \Delta X = 4{\cdot}994\ 8174$	$\log \Delta X = 5{\cdot}127\ 7080$
$\log \cos \beta = 9{\cdot}712\ 5238$	$\log \cos \beta = 9{\cdot}989\ 4826$
$\log S = 5{\cdot}282\ 2936$	$\log S = 5{\cdot}138\ 2254$
$\log \sin \beta = 9{\cdot}932\ 8180$	$\log \sin \beta = 9{\cdot}337\ 3414$
$\log \Delta Y = 5{\cdot}215\ 1116$	$\log \Delta Y = 4{\cdot}475\ 5668$

	X	Y		X	Y
A	+ 813 687·03	+ 325 707·28	B	+ 778 314·56	+ 459 915·59
Δ	+ 98 813·74	+ 164 101·12		+ 134 186·23	+ 29 892·81
C	+ 912 500·77	+ 489 808·40		+ 912 500·79	489 808·40

Co-ordinates of B :—

$X = +\ 778\ 314{\cdot}56$ $\qquad Y = +\ 459\ 915{\cdot}59.$

Mean co-ordinates of C :—

$X = +\ 912\ 500{\cdot}78$ $\qquad Y = +\ 489\ 808{\cdot}40.$

In step (6) above, the back bearings, β', of AC and BC are obtained by adding 180° to the forward bearing. Hence, the plane or grid angle at C could be obtained directly without working out the $(\beta - \alpha)$ corrections to bearings at C. These corrections, however, are needed to give the

spherical bearings, which should always be computed and tabulated as they may be required as a start for other work.

Note also that in step (7) the plane angles have been deduced directly from the $(\beta - \alpha)$ corrections. Thus, the angle at C is given by bearing CA minus bearing CB. Hence, the correction is equal to correction to CA minus correction to CB, that is to $[-(-10.19)] - [-(-15.26)] = -5''\cdot07$. When this correction has been applied to the spherical angle, the resulting plane angle should be the same as that given by subtracting the grid bearing of CB from the grid bearing of CA. Thus, $46° 23' 17''\cdot94 - 5''\cdot07 = 46° 23' 12''\cdot87 = 238° 56' 44''\cdot46 - 192° 33' 31''\cdot59$.

Tables to Assist in Computing Transverse Mercator Co-ordinates. If much computing has to be done in terms of Transverse Mercator Co-ordinates it will save considerable time if tables of the following factors, or their logarithms, are computed at the very beginning. Tables of meridian distances and of the ordinary geodetic factors are assumed to be already available.

Formula	Factor	Argument	Number of Places	Interval of Argument
4	$\frac{1}{24}\frac{(5+3\tan^2\phi_r)}{R_r N_r^3 \sin 1''}$	X or ϕ_r	4	50,000 ft. or 4′ of arc
5	$\frac{1}{6}\frac{\left(\frac{N_r}{R_r}+2\tan^2\phi_r\right)}{N_r^3 \sin 1''}$	X or ϕ_r	5	50,000 ft. or 4′ of arc
5	$\frac{1}{120}\frac{(5+28\tan^2\phi_r+24\tan^4\phi_r)}{N_r^5 \sin^3 1''}$ . .	X or ϕ_r	3	50,000 ft. or 4′ of arc
6	$\frac{1}{2}\frac{\left(2+\tan^2\phi_r-\frac{N_r}{R_r}\right)}{N_r^3 \sin 1''}$	X or ϕ_r	5	50,000 ft. or 4′ of arc
7	$\frac{1}{24} N \sin^4 1'' \sin\phi \cos^3\phi (5-\tan^2\phi)$.	ϕ	5	4′ of arc
8	$\frac{1}{6}(\cos\phi \sin 1'')^3 N\left(\frac{N}{R}-\tan^2\phi\right)$ . .	ϕ	5	4′ of arc
8	$\frac{1}{120} N(\cos\phi \sin 1'')^5(5-18\tan^2\phi+\tan^4\phi)$	ϕ	3	4′ of arc
9	$N \sin\phi (\cos\phi \sin 1'')^2\left(\frac{N}{R}-\frac{2}{3}\right)$ . .	ϕ	5	4′ of arc
9	$\frac{1}{15} \sin\phi (\cos\phi \sin 1'')^4 (2-\tan^2\phi)$.	ϕ	3	4′ of arc
21 & 21A 22 & 22A	$\frac{Y^2}{2r^2}$; $\log\frac{Y^2}{2r^2}$; $\frac{MY^2}{2r^2}$; $\log\frac{MY^2}{2r^2}$ . .	Y	7,4	10,000 ft.
14, 16, 18 & 19	$\frac{(2Y+Y')}{6r^2 \sin 1''}$	$(2Y+Y')$	7	10,000 ft.

When these tables are being constructed, they can first be computed to one or two places more than the number of places required, and at

intervals of four to eight times the final interval. Intermediate values can then be interpolated by means of second and third differences (page 27).

For the smaller quantities that are only required to three decimal places, a wider interval of the argument can be used than that given above. It is, however, convenient to have all the factors relating to a particular computation tabulated together on the same page, and, in this case, it may be just as convenient to tabulate all quantities with the same interval.

The arrangement of the table for $\dfrac{(2Y + Y')}{6r^2 \sin 1''}$ can be for values of $(2Y + Y')$ in thousands of feet and the tabulated quantity is then the factor by which $(X' - X)$ must be multiplied to give $\beta - \alpha$. The table can be fitted on a single page if it is arranged in rows and columns, the interval of tabulation being 100,000 feet between the rows and 10.000 feet between the columns. This same table can also be used to evaluate the factor $\dfrac{(Y + Y')}{2r^2 \sin 1''}$ in formula (12) if it is entered with $(Y + Y')$ as argument instead of $(2Y + Y')$ and the result multiplied by 3. Otherwise, a separate table for $\dfrac{(Y + Y')}{2r^2 \sin 1''}$ can easily be constructed from the original table.

In all the formulæ involving $\dfrac{1}{r^2}$ the value of r to be used when computing the tables is $\sqrt{R \,.\, N}$, taken out for the mid-latitude of the area under survey.

Conical Orthomorphic or Lambert Co-ordinates. Transverse Mercator co-ordinates are best suited to an area or country which has a large extent in a north and south direction and a relatively small extent in an east and west direction, and it is not suited for areas whose principal dimension lies in an east and west direction. For such areas, conical orthomorphic or Lambert co-ordinates * are more suitable than Transverse Mercator ones.

In the United States of America each State now has its own system of co-ordinates, with its own central meridian and origin, and some of these systems are Transverse Mercator and some Lambert, the general rule being that Transverse Mercator co-ordinates are used when the principal dimension is in a north and south direction and Lambert when the principal dimension lies east and west. In all, there are nineteen States which use Transverse Mercator co-ordinates, thirty which use Lambert, and there is one State, Florida, in which both systems are employed.

Conical orthomorphic or Lambert co-ordinates are used by the Geographical Section of the General Staff in Canada in connection with military surveys and the production of military maps and they are also

* There are two systems of Lambert co-ordinates and the one described here is generally known as "Lambert's second." For the sake of brevity, however, it will simply be referred to as "Lambert's," it being understood that it is Lambert's second system that is meant.

used in India, but, apart from this, they are not much favoured in British practice, even in those countries which have a greater east and west than north and south dimension. In such cases, if the use of a single belt would introduce undesirably large scale errors, the general tendency is to divide the country into several belts and to use Transverse Mercator co-ordinates in each. There are two main reasons for this. One is that, when the Y co-ordinate of a point is large, the convergence is also large and this leads to considerable divergence between the grid lines and the true north line for map sheets lying a considerable distance east or west of the central meridian. The second is that conical orthomorphic co-ordinates are not so suitable for point-to-point working as Transverse Mercator co-ordinates are. Consequently, in what follows we shall merely give a general description of the Lambert system and then state the formulæ without further proof or explanation. Those needing further information can consult the United States Coast and Geodetic Survey Special Publication No. 53, which gives the theory, and Special Publication No. 193, which deals with practical computation on both the Transverse Mercator and Lambert systems.

Ordinary conical co-ordinates are co-ordinates developed on the surface of a cone which is tangential to the parallel of latitude through the origin of co-ordinates and whose apex lies on a prolongation of the earth's minor axis. Conical orthomorphic, or Lambert, co-ordinates are ordinary conical co-ordinates modified in such a manner that the system becomes orthomorphic or conformal.

In Fig. 130 (a) OP′ is the tangent to the meridian at the origin O. This tangent intersects the prolongation of the earth's minor axis at P′, which then becomes the apex of the cone on whose surface the rectangular

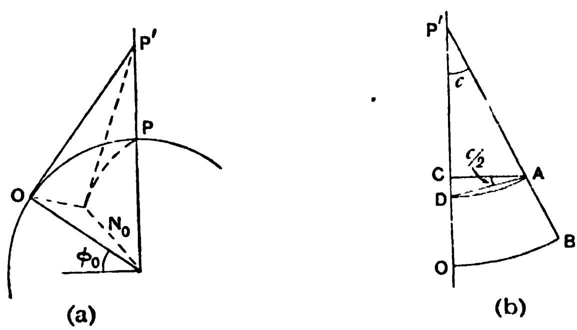


FIG. 130.

co-ordinates are considered to be developed. The cone is tangential to the spheroid along a parallel of latitude through O. From the figure it follows that $OP' = P = N_0 \cot \phi_0$ where N_0 is the normal at, and ϕ_0 the latitude of, the origin O.

Fig. 130 (b) shows the surface of the cone stretched out on the flat. OP′ represents the direction of the meridian through O, P′ the apex of the cone, and the parallel of latitude through O will be a circle with centre at P′ and radius $P'O = P = N_0 \cot \phi$. Let B be any point on this circle

whose difference of longitude from O is dL. Then length $\text{OB}=N_0 \cos \phi_0 dL$ and angle $\text{OP'B} = c = \dfrac{N_0 \cos \phi_0 \, dL}{\text{OP'}} = dL \sin \phi_0$. Let A be any point whose longitude is the same as that of B but let ϕ be the latitude of A. Then, if AD in Fig. 130 (b) represents the parallel of latitude through A, the distance OD can be found from a table of meridian distances. Let m be this meridian distance. Then $\text{P'A}=\text{P'D}=\text{P'O}-\text{OD}=N_0 \cot \phi_0 - m$. From A draw AC perpendicular to OP'. The rectangular co-ordinates of A are then $Y = \text{CA} = \text{P'A} \sin c = (N_0 \cot \phi_0 - m) \sin c$ and $X = \text{OC} = \text{OD} + \text{DC} = m + \text{CA} \tan \dfrac{c}{2} = m + Y \tan \dfrac{c}{2}$.

It can easily be seen that the angle c above is the convergence, γ, at A.

If this system of rectangular co-ordinates is used, it is not orthomorphic. Distances between parallels of latitude and along the parallel of latitude through O are maintained but distances along other parallels of latitude on either side of OP' are not maintained. In order to make the system orthomorphic, the distance OD, instead of being made equal to the true meridian distance m, is made equal to the quantity m' which is given by :—

$$m' = m + \frac{m^3}{6R_0N_0} + \frac{m^4 \tan \phi_0}{24R_0N_0^2} + \frac{m^5(5 + 3\tan^2 \phi_0)}{120\, R_0N_0^3} - \frac{m^4e^2 \sin 2\phi_0}{12(1-e^2)R_0N_0^2} + \frac{m^6 \tan \phi_0(7 + 4\tan^2 \phi_0)}{240R_0N_0^4}.*$$

South of the origin m is negative and terms containing odd powers of m change sign while those containing even powers of m retain their original sign. From this formula a table of modified meridian distances can easily be prepared.

It will be noted that, in this system, the parallel of latitude through the origin plays much the same part as the central meridian in the Transverse Mercator system and that conical orthomorphic co-ordinates are the same as ordinary conical co-ordinates except that meridian distances are increased by the amount given by the above formula.

Formulæ for Conical Orthomorphic Co-ordinates. The following are the formulæ for conical orthomorphic co-ordinates :—

(1) *Conversion of Geographical into Conical Orthomorphic Co-ordinates.*

$$P = N_0 \cot \phi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m' = m + \frac{m^3}{6R_0N_0} + \frac{m^4 \tan \phi_0}{24R_0N_0^2} + \frac{m^5(5 + 3\tan^2 \phi_0)}{120R_0N_0^3} - \frac{m^4e^2 \sin 2\phi_0}{12(1-e^2)R_0N_0^2} + \frac{m^6 \tan \phi_0(7 + 4\tan^2 \phi_0)}{240R_0N_0^2} \quad . \quad . \quad (2)$$

where m is the meridian distance between latitudes ϕ_0 and ϕ. The quantity m' should be tabulated for different values of ϕ. For latitudes south

* For a proof of this formula, up to terms in m^4, see the *Empire Survey Review*, Vol. VI, No. 43, January, 1942.

of ϕ_0 the first, second and fourth terms are negative, the terms in m^4 and m^6 retaining their original signs.

$$\gamma = dL \sin \phi_0 \quad \dots\dots\dots\dots \quad (3)$$

$$Y = (P - m') \sin \gamma \quad \dots\dots\dots \quad (4)$$

$$X = m' + Y \tan \tfrac{1}{2}\gamma \quad \dots\dots\dots \quad (5)$$

In the above, if γ is a small angle, $\sin \gamma$ and $\tan \frac{1}{2}\gamma$ should be found by one of the methods referred to in Vol. I, page 256.

With regard to signs in the above and the following formulæ, dL is positive for points east of the central meridian and negative for points west of it, and ϕ and ϕ_0 are negative for points south of the equator. With these conventions, the formulæ hold for any quadrant and for either hemisphere.

(2) *Conversion of Conical Orthomorphic into Geographical Co-ordinates.* P is known from (1) above.

$$\tan \gamma = \frac{Y}{P - X} \quad \dots\dots\dots\dots \quad (6)$$

$$m' = X - Y \tan \tfrac{1}{2}\gamma \quad \dots\dots\dots \quad (7)$$

$$dL = \gamma \operatorname{cosec} \phi_0 \quad \dots\dots\dots\dots \quad (8)$$

Knowing m', the corresponding latitude can be obtained by interpolation from the table giving m' for different values of ϕ. Alternatively, m can be obtained by successive approximations from (2) and ϕ then found from a table of meridian distances.

(3) *Calculation of Conical Orthomorphic Co-ordinates from Bearing and Distance* :—

$$X' = X + s \cos \alpha - \frac{Xs^2 \sin^2 \alpha}{2r^2} - \frac{s \cos \alpha\, s^2 \sin^2 \alpha}{6r^2} + \frac{(X'^3 - X^3)}{6r^2} \quad \dots\dots\dots\dots \quad (9)$$

$$Y' = Y + s \sin \alpha + \frac{X'^2 s \sin \alpha}{2r^2} - \frac{s^2 \cos^2 \alpha \,.\, s \sin \alpha}{6r^2} \quad \dots\dots \quad (10)$$

$$\alpha' = \alpha \pm 180^\circ + \frac{s \sin \alpha (X + X')}{2r^2 \sin 1''} \quad \dots\dots\dots\dots \quad (11)$$

When X and Y are very large, the smaller terms in (9) and (10) may not be sufficient and it may be necessary to include terms of a higher order. These terms have been worked out by McCaw and are given by Burns in a paper on "Point-to-Point Working for the Conical Orthomorphic Projection" in the *Empire Survey Review*, Vol. II, No. 11, January, 1934. They are as follows :—

To formula (9) add :—

$$\frac{sX^3 \cos \alpha \sec^3 \delta \cos 3\delta \tan \phi_0}{6r^3} + \frac{s^2X^2 \sec^2 \delta \cos 2(\alpha + \delta) \tan \phi_0}{4r^3}$$

$$+ \frac{s^3X \sec \delta \cos (3\alpha + \delta) \tan \phi_0}{6r^3} + \frac{s^4 \cos 4\alpha \tan \phi_0}{24r^3} \quad \dots\dots \quad (12)$$

and to formula (10) add :—

$$\frac{sX^3 \sin\alpha \sec^3\delta \cos 3\delta \tan\phi_{\prime\prime}}{6r^3} + \frac{s^2X^2 \sec^2\delta \sin 2(\alpha+\delta) \tan\phi_{\prime\prime}}{4r^3}$$
$$+ \frac{s^3X \sec\delta \sin(3\alpha+\delta)\tan\phi_{\prime\prime}}{6r^3} + \frac{s^4 \sin 4\alpha \tan\phi_{\prime\prime}}{24r^3} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where δ is an auxiliary angle such that $\tan\delta = \frac{Y}{X}$, $\phi_{\prime\prime}$ is the latitude of the origin and $r^3 = R \,.\, N^2$ for the origin.

Alternatively, when the line is only a few miles in length and not more than about 150 miles from the origin, co-ordinates may be computed from :—

$$S = s\left[1 + \frac{m^2}{2R_{\prime\prime}N_{\prime\prime}} + \frac{m^3\tan\phi_{\prime\prime}}{6R_{\prime\prime}N_{\prime\prime}^2} + \frac{m^4(5 + 3\tan^2\phi_{\prime\prime})}{24R_{\prime\prime}N_{\prime\prime}^3}\right] \quad . \quad . \quad . \quad . \quad (14)$$

or

$$\log S = \log s + M\left[\frac{m^2}{2R_{\prime\prime}N_{\prime\prime}} + \frac{m^3\tan\phi_{\prime\prime}}{6R_{\prime\prime}N_{\prime\prime}^2} + \frac{m^4(5 + 3\tan^2\phi_{\prime\prime})}{24R_{\prime\prime}N_{\prime\prime}^3}\right] \quad . \quad (14\text{A})$$

where m is the true, not the modified, meridian distance of the middle point of the line and $\log M = \bar{1}\cdot637\ 78$.

$$\beta = \alpha + \frac{(Y' - Y)(2X + X')}{6r^2 \sin 1''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\left.\begin{aligned} X' &= X + S\cos\beta \\ Y' &= Y + S\sin\beta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\alpha' = \beta \pm 180^\circ + \frac{(Y' - Y)(2X' + X)}{6r^2 \sin 1''} \quad . \quad . \quad . \quad (17)$$

Here β is the grid bearing but these formulæ are not satisfactory when lines are long or the X's and Y's are very large. For such cases Burns recommends computing the S from formula (14) or (14A) above using the scale factor for the middle point of the line, and the angle θ computed from :—

$$\theta = \alpha + \frac{s\sin\alpha(X + \frac{1}{2}s\cos\alpha)}{2r^2\sin 1''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Then :—

$$\left.\begin{aligned} X' &= X + S\cos\theta \\ Y' &= Y + S\sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and α' is obtained from (11). This method will give results which, for lines up to about 30 miles in length, are accurate to about 1/100,000. For lines much longer than this, or when the highest degree of accuracy is required, no really satisfactory formulæ for point-to-point working directly in terms of conical orthomorhhic co-ordinates have yet been devised, and, in such cases, it is best to compute the geographical co-ordinates by using Clarke's or Puissant's formulæ and then transform these geographical co-ordinates into conical orthomorphic co-ordinates in the usual way.

Distances and Bearings from Co-ordinates. Proceed by approximation

in formulæ (9), (10), (12), (13). The first approximation gives $\tan \alpha = \dfrac{Y' - Y}{X' - X}$ and $s = (X' - X) \sec \alpha = (Y' - Y) \operatorname{cosec} \alpha$. Use these values in the second and succeeding terms and get new values for $s \sin \alpha$ and $s \cos \alpha$, proceeding to a third approximation if necessary.

For short lines not too far from the origin calculate β and S from

$$\tan \beta = \frac{Y' - Y}{X' - X} \quad \ldots \ldots \ldots \quad (20)$$

$$S = (X' - X) \sec \beta = (Y' - Y) \operatorname{cosec} \beta \quad \ldots \quad (21)$$

and s and α from

$$s = S\left[1 - \frac{m^2}{2R_0 N_0} - \frac{m^3 \tan \phi_0}{6R_0 N_0^2}\right] \quad \ldots \ldots \quad (22)$$

or

$$\log s = \log S - \frac{Mm^2}{2R_0 N_0}\left[1 + \frac{m \tan \phi_0}{3N_0}\right] \quad \ldots \quad (22\text{A})$$

and

$$\alpha = \beta - \frac{(Y' - Y)(2X + X')}{6R_0 N_0 \sin 1''} \quad \ldots \ldots \quad (23)$$

Scale and Scale Error. For long lines :—

$$m_2 = m_1 + s \cos \mathrm{A}$$

where m_2 is the approximate meridional distance of the end point of the line, m_1 that of the beginning, and

$$S = s\left[1 + \frac{1}{2R_0 N_0}\left\{\left(\frac{m_1 + m_2}{2}\right)^2 + \frac{1}{3}\left(\frac{m_2 - m_1}{2}\right)^2\right\} + \frac{m_1 m_2 (m_2 + m_1)}{12 R_0 N_0^2} \tan \phi_0 + \frac{s^2 (m_2 + m_1)}{24 R_0 N_0^2} \tan \phi_0\right] \quad \ldots \quad (24)$$

or

$$\log S = \log s + \frac{M}{2R_0 N_0}\left\{\left(\frac{m_1 + m_2}{2}\right)^2 + \frac{1}{3}\left(\frac{m_2 - m_1}{2}\right)^2\right\} + \frac{M m_1 m_2 (m_2 + m_1)}{12 R_0 N_0^2} \tan \phi_0 + \frac{M s^2 (m_2 + m_1)}{24 R_0 N_0^2} \tan \phi_0 \quad \ldots \quad (25)$$

where, as before, $M = \bar{1}{\cdot}638\ 78$, ϕ_0 is the latitude of the origin, ϕ that of the initial point of the line and A the azimuth at that point.

For short lines see formulæ (14), (14A), (22) and (22A).

Reduction of Scale Error. We have seen that, in the case of Transverse Mercator co-ordinates, the maximum scale error can be reduced and even halved by introducing a negative scale error along the central meridian. In an exactly similar manner the maximum scale error in the case of conical orthomorphic co-ordinates can be reduced by introducing a negative scale error along the central parallel of latitude, where it is otherwise zero. In this case, there are two parallels, one north and one south of the central one, along which the scale error is zero. Between these two parallels the scale error is negative and outside them it is positive.

If the scale error is to be reduced in this way by the factor $\dfrac{1}{\mathrm{F}}$, all

measured distances, meridian distances and geodetic functions must be reduced by multiplying them by $\left(1 - \frac{1}{F}\right)$.

COMPUTATION OF PRECISE TRAVERSES

Precise traverses, like triangulation, can be computed either directly in terms of geographical co-ordinates or in terms of some sort of linear co-ordinates, such as Transverse Mercator or Conical Orthomorphic co-ordinates. The method to be adopted will depend very largely on the use to which the traverse is to be put. If it is to be used as a control for geographical mapping on a small scale it will generally be most convenient to compute it in terms of geographical co-ordinates, but, if it is to be used to control large-scale work, it will be more convenient to compute it in terms of linear co-ordinates. Linear co-ordinates can, of course, be converted into geographicals when the geographical co-ordinates of the origin are known, and, similarly, geographicals can easily be converted into linear co-ordinates. When a survey is of sufficient importance for the results to be published, and the work has been computed in terms of linear co-ordinates, the geographical co-ordinates of some, at least, of the more important stations should be computed and tabulated with the other published data.

The main difference between the computation of triangulation and traverses lies in the fact that the lines are much longer in the case of triangulation than they are in the case of traverses. Consequently, certain approximations can often be made in traverse computations which cannot be allowed in triangulation computations.

In what follows, it is assumed that all lengths and angles have been properly corrected and reduced, and that this work has been properly checked, as mistakes in traverse computations are exceedingly easy to make.

Computation of Traverses in Terms of Geographical Co-ordinates. This is the method which is used by the United States Coast and Geodetic Survey for the computation of their first-order traverses. The formulæ used are Puissant's, as modified for use with short lines (page 334). Before the final computations can be made, it is necessary to make a preliminary computation to determine the reverse azimuth of each line, the closing errors between azimuth stations and the closing error in position. After these preliminary calculations have been completed, the traverse is adjusted and corrections to the measured lengths and computed azimuths determined. These corrections can then be used to compute the corrections to the approximate differences in latitude and longitude by means of the following formulæ, which are obtained by direct differentiation of the first terms in Puissant's latitude and longitude formulæ :—

Correction to $d\phi$ in seconds

$$= \frac{ds \cos A}{R \sin 1''} - \frac{s \sin A \, dA}{R}$$

$$= \frac{ds d\phi}{s} - \frac{N' dA \, dL \sin 1''}{R \sec \phi}$$

Correction to dL in seconds

$$= \frac{ds \sin A \sec \phi'}{N' \sin 1''} + \frac{s \cos A \sec \phi' dA}{N'}$$

$$= \frac{ds\, dL}{s} + \frac{R d\phi\, dA \sec \phi' \sin 1''}{N'}$$

where ds is the correction to the length of the leg, dA the correction to the azimuth in seconds of arc and $d\phi$ and dL are the approximate differences in latitude and longitude between the terminal points of the line as obtained from the preliminary computation, both expressed in seconds of arc.

The Canadian Geodetic Survey has evolved and uses a method different from the above. Taking an azimuth station O, Fig. 131, as a pole or origin, the traverse is treated as a traverse on a plane surface and plane rectangular co-ordinates are computed with reference to the origin and the meridian through it. Now imagine lines drawn from the pole or origin to the ends of the different legs. A series of triangles will be obtained, each with the pole as one apex. The spherical excess of each triangle can now be computed from the formula :—

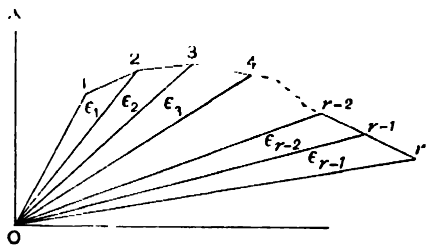


FIG. 131.

$$= \frac{1}{2r^2 \sin 1''} \left[\Delta Y_r . \Delta X_{r-1} - \Delta X_r . \Delta Y_{r-1} \right]$$

where ΔX_r, ΔY_r ; ΔX_{r-1}, ΔY_{r-1}, are the latitudes and departures of the rth and $(r - 1)$th legs respectively. Corrections to the approximate bearings are then worked out from :—

Correction to the bearing of the rth leg :—

$$= \tfrac{2}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3 \ . \ . \ . + \epsilon_{r-1}) - \tfrac{1}{3}\epsilon_{r-1}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \ . \ . \ . \ \epsilon_{r-1}$ are the spherical excesses of the 1st, 2nd, 3rd . . . $(r - 1)$th "polar" triangles. From these corrections to the bearings corrections to the latitudes and departures are computed from :—

$$\text{Correction to latitude of } r\text{th leg} = -\Delta Y_r d\beta_r \sin 1''$$
$$\text{Correction to departure of } r\text{th leg} = +\Delta X_r d\beta_r \sin 1''$$

where $d\beta_r$ is the correction, in seconds of arc, to the bearing of the rth leg. These corrections are then applied to the preliminary latitudes and departures of the different legs and the corrected latitudes and departures obtained. Let X and Y be the corrected co-ordinates of the end of the rth leg. Then :—

$$\tan \beta = \frac{Y}{X}$$

and

$$s = Y \operatorname{cosec} \beta \ ; \ s = X \sec \beta.$$

The value of s obtained in this way is the true length of the line joining the pole to the end of the rth leg, and the azimuth of this line will be :—

$$A = \beta + \tfrac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \ . \ . \ . + \epsilon_{r-1}).$$

Hence, knowing the azimuth and true distance from the origin, whose geographical co-ordinates are assumed to be known, the geographical co-ordinates of any point can be computed by Clarke's or Puissant's formulæ (pages 328 and 332).

Particular care must be taken with the signs of the spherical excesses. These signs are governed by the signs of the latitudes and departures from which the spherical excesses are computed.

The method is more fully described, and examples, showing arrangement of computation, are given in Canadian Survey Publication No. 25, entitled "The Conversion of Latitudes and Departures of a Traverse to Geodetic Differences of Latitude and Longitude," by W. M. Tobey. The chief advantage of this method appears to be that the geographical co-ordinates of any individual point may be obtained without having to work out the geographical co-ordinates of the intermediate points, although corrected latitudes and departures have to be obtained for every leg.

Computation of Traverses in Terms of Transverse Mercator Co-ordinates. As Transverse Mercator co-ordinates are now probably more extensively used than any others in connection with surveys of large areas in which ordinary plane rectangulars are not suitable, and as somewhat similar methods, but with different formulæ, can be employed for the computation of other kinds of linear co-ordinates, we shall describe here those to be adopted when the traverse is to be computed in terms of Transverse Mercator co-ordinates.

The first step is to calculate the convergence at the initial azimuth station and to apply it to the observed azimuth so that the first bearing is a bearing referred to a plane parallel to the central meridian. Using this bearing and the observed angles, the approximate bearings of the other lines are computed as if the traverse were an ordinary traverse on the plane, and these bearings, together with the observed lengths, are used to compute approximate latitudes and departures. The approximate latitudes and departures serve to give approximate co-ordinates from which different corrections, and the convergence at the terminal azimuth station, can be worked out.

From this point two main methods are available. The first is to compute by direct point-to-point working, using the measured lengths and the observed spherical bearings when applying formulæ (10) to (12) of page 359. The second method is to use plane or grid lengths and bearings computed from formulæ (13) to (16) of page 359.

If the traverse is not to be adjusted by least squares, and the closing error in bearing is to be distributed evenly amongst the different traverse legs, the first step in either method is to determine this closing error and to adjust the intermediate bearings.

If the method to be used is that of direct point-to-point working, the approximate preliminary co-ordinates are used to calculate the differences

between forward and back bearings, using formula (12), page 359. These differences are applied to each successive bearing and new bearings obtained. The difference between the final bearing and that obtained by applying the convergence to the observed azimuth at the forward azimuth station is taken as the closing error in bearing and is then distributed among the intermediate bearings.

When grid lengths and bearings are employed, the corrections to lengths and bearings are worked out from the approximate co-ordinates by means of formulæ (13) or (13A) for the lengths and (14) and (16) for the bearings. The grid bearing of the line at the forward azimuth station is compared with the bearing computed directly through the traverse from the observed initial azimuth, and the closing error in bearing obtained and distributed in the usual way.

If the legs of the traverse are short, the grid lengths may be worked out by formulæ (21) or (21A), instead of by (13) or (13A), the terms in Y^4 being neglected in each case as they are inappreciable for short legs.

In the method of direct point-to-point working with true distances and spherical bearings, the original latitudes and departures give the approximate values of the first and largest term in the expansion for $(X' - X)$ and $(Y' - Y)$ and these values can now be corrected by application of the formulæ :—

$$\begin{aligned} \text{Correction to latitude} &= -\Delta Y \,.\, \delta\alpha \,.\, \sin 1'' \\ \text{Correction to departure} &= +\Delta X \,.\, \delta\alpha \,.\, \sin 1'' \end{aligned}$$

where ΔX and ΔY are the latitude and departure of the leg in question and $\delta\alpha$ is the correction, in seconds of arc, to the bearing. After the first terms in the expansions for X and Y have been corrected in this way, the other terms in the expansions in formulæ (10) and (11) can be computed and applied.

When the method of grid lengths and bearings is employed, each of the preliminary latitudes and departures has to be corrected for corrections to both length and bearing. Let the correction to the length be δs and that to bearing $\delta\beta$, where $\delta\beta$ is in seconds of arc. Then :—

$$\text{Correction to latitude} = \delta s \,.\, \frac{\Delta X}{s} - \Delta Y \,.\, \delta\beta \,.\, \sin 1''$$

$$\text{Correction to departure} = \delta s \,.\, \frac{\Delta Y}{s} + \Delta X \,.\, \delta\beta \,.\, \sin 1''.$$

If $\delta\beta$ is the final correction to bearing, including the $(\beta - \alpha)$ and angular closing error corrections, the above two corrections, when applied to the preliminary latitudes and departures, give the final latitudes and departures. Otherwise, or as a check, the latitudes and departures can be re-computed from the beginning using the corrected lengths and bearings.

In all cases where corrections are applied to bearings which have been worked out from observed angles, it is important to remember that a correction applied to one bearing may affect all subsequent bearings. Consequently, care must be taken to see that final corrections to latitudes

and longitudes, or to latitudes and departures, are based on the total correction to each individual bearing (see page 381).

Approximate Methods when Legs are Relatively Short. When the legs of a traverse are relatively short, say under four or five miles, various approximations can be made which are not justifiable in the computations involving longer lines such as are generally met with in triangulation. In many cases, also, and particularly if the legs are very short, say well under a mile in length, many of the stations will probably not be marked by permanent station marks and the co-ordinates of these unmarked stations will not be required.

When the traverse is computed directly in terms of geographical co-ordinates by the use of Puissant's formulæ, the approximate formulæ given on page 334 for lines less than twelve miles may be used, and, even in that case, the term in h^2 in the formula for $d\phi''$ may often be neglected. For very short legs, each section between azimuth stations can be divided into several sub-sections, each of which would normally begin and end at stations that are permanently marked, and ordinary plane rectangular latitudes and departures, referred to the meridian through the initial station of the sub-section, computed. From the local co-ordinates so derived the azimuth and distance from the initial to the terminal station of the sub-section can be calculated and this azimuth and distance used in Puissant's formulæ to compute the geographical co-ordinates and reverse azimuth at the terminal station. In this case greater accuracy will be obtained if the spherical excess of the figure formed by the traverse and the line joining the two stations is calculated, and the azimuth of the line is taken as the azimuth computed from co-ordinates plus one-third of this spherical excess.

In the method of computing in terms of Transverse Mercator co-ordinates by direct point-to-point working a somewhat similar short-cut may be employed. In this case, local co-ordinates, referred to the initial station, can be used as the terms $s \cdot \cos\alpha$ and $s \cdot \sin\alpha$ in formulæ (10) and (11), page 359. If these terms are not very large, those containing s^3 in the formulæ may be neglected.

When the method adopted is that of using grid lengths and bearings, there are several ways by which the labour of computing can be saved if suitable approximations are introduced. In the first place, the full formulæ for scale correction, (13) or (13A), page 359, need seldom be used, and, for most lines that are ordinarily met with in traversing, it is generally legitimate to use formulæ (21) or (21A) for short lines. Here the Y to be taken will be the mean of the Y's of the two ends of the line, though in most cases it will make no appreciable difference if the Y of the initial station is used instead of the mean Y.

For reasonably short lines the formulæ for the correction to directions may also be simplified to

$$\beta - \alpha = -\frac{Y}{2r^2 \sin 1''} \cdot \Delta X$$

where Y is the Y of the middle point of the leg. In this case there is no appreciable difference between the numerical values of the corrections

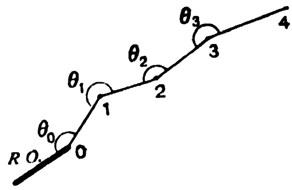


FIG. 132.

at the two ends of the line, as there is for long lines, but, of course, the signs are reversed.

In working out the corrections to bearings which have already been computed from the angles, it must not be forgotten that the correction to one bearing affects those of subsequent bearings. Thus, in Fig. 132 let O be the first azimuth station, R.O. the referring object, and $\theta_0, \theta_1, \theta_2$ . . . the observed angles. Then :—

$$\text{Bearing O to R.O.} = A \pm \gamma = \alpha'_0$$

where A is the observed azimuth O to R.O. and γ is the convergence at O. The preliminary bearings used in working out the preliminary approximate co-ordinates are :—

$$0 \text{ to } 1 = \alpha_1 = \alpha_0' + \theta_0$$
$$1 \text{ to } 2 = \alpha_2 = \alpha_1 \pm 180^\circ + \theta_1$$
$$2 \text{ to } 3 = \alpha_3 = \alpha_2 \pm 180^\circ + \theta_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$r - 1 \text{ to } r = \alpha_r = \alpha_{r-1} \pm 180^\circ + \theta_{r-1}.$$

Let (X_0, Y_0) ; (X_1, Y_1) ; (X_2, Y_2) ; . . . (X_r, Y_r) be the co-ordinates of the points 0, 1, 2, . . . r and let

$$\Delta X_1 = (X_1 - X_0)\ ;\ \Delta X_2 = (X_2 - X_1)\ ;\ \ldots\ \Delta X_r = (X_r - X_{r-1})$$

The plane or grid bearing 0 to 1 is

$$\beta_1 = \alpha_1 - \frac{\Delta X_1}{6r^2 \sin 1''}(2Y_0 + Y_1)$$

Spherical bearing 1 to 0 =

$$\beta_1 \pm 180^\circ - \frac{\Delta X_1(2Y_1 + Y_0)}{6r^2 \sin 1''}.$$

∴ Spherical bearing 1 to 2 =

$$\beta_1 \pm 180^\circ - \frac{\Delta X_1(2Y_1 + Y_0)}{6r^2 \sin 1''} + \theta_1$$

Grid bearing 1 to 2 =

$$\beta_2 = \beta_1 \pm 180^\circ + \theta_1 - \frac{\Delta X_1(2Y_1 + Y_0)}{6r^2 \sin 1''} - \frac{\Delta X_2(2Y_1 + Y_2)}{6r^2 \sin 1''}$$

$$= \alpha_1 \pm 180^\circ + \theta_1 - \frac{\Delta X_1(2Y_0 + Y_1)}{6r^2 \sin 1''} - \frac{\Delta X_1(2Y_1 + Y_0)}{6r^2 \sin 1''}$$
$$- \frac{\Delta X_2(2Y_1 + Y_2)}{6r^2 \sin 1''}$$

$$= \alpha_2 - \frac{3\Delta X_1(Y_0 + Y_1)}{6r^2 \sin 1''} - \frac{\Delta X_2(2Y_1 + Y_2)}{6r^2 \sin 1''}$$

Similarly, grid bearing 2 to 3 =

$$\beta_3 = \beta_2 \pm 180° + \theta_2 - \frac{\Delta X_2(2Y_2 + Y_1)}{6r^2 \sin 1''} - \frac{\Delta X_3(2Y_2 + Y_3)}{6r^2 \sin 1''}$$

$$= \alpha_2 \pm 180° + \theta_2 - \frac{3\Delta X_1(Y_0 + Y_1)}{6r^2 \sin 1''} - \frac{\Delta X_2(2Y_1 + Y_2)}{6r^2 \sin 1''} - \frac{\Delta X_2(2Y_2 + Y_1)}{6r^2 \sin 1''} - \frac{\Delta X_3(2Y_2 + Y_3)}{6r^2 \sin 1''}$$

$$= \alpha_3 - \frac{3\Delta X_1(Y_0 + Y_1)}{6r^2 \sin 1''} - \frac{3\Delta X_2(Y_1 + Y_2)}{6r^2 \sin 1''} - \frac{\Delta X_3(2Y_2 + Y_1)}{6r^2 \sin 1''}.$$

The mode of formation of the different corrections can now be clearly seen, and, for the rth leg, we see that the correction to the preliminary approximate bearing α_r is given by :—

$$\delta\beta_r = -\frac{\Delta X_1(Y_0 + Y_1)}{2r^2 \sin 1''} - \frac{\Delta X_2(Y_1 + Y_2)}{2r^2 \sin 1''} - \frac{\Delta X_3(Y_2 + Y_3)}{2r^2 \sin 1''} - \dots - \frac{\Delta X_{r-1}(Y_{r-2} + Y_{r-1})}{2r^2 \sin 1''} - \frac{\Delta X_r(2Y_{r-1} + Y_r)}{6r^2 \sin 1''}$$

in which the factor $\frac{1}{r^2 \sin 1''}$ is assumed to be the same for all legs in the section.

Note here that this expression is very approximately equal to $\frac{A}{r^2 \sin 1''} = \epsilon''$, where A is the area, and ϵ'' the spherical excess, of the figure contained by the r legs of the traverse, the axis of X and perpendiculars drawn to it from the stations O and r.

If $\delta\beta_r$ and $\delta\beta_{r-1}$ are the $(\beta - \alpha)$ corrections to the approximate bearings of the rth and $(r - 1)$th legs,

$$\delta\beta_r = \delta\beta_{r-1} - \frac{\Delta X_{r-1}(Y_{r-2} + 2Y_{r-1})}{6r^2 \sin 1''} - \frac{\Delta X_r(2Y_{r-1} + Y_r)}{6r^2 \sin 1''}.$$

These corrections are all small quantities, and, in many cases, if the difference between the Y's of the ends of the section is not too large, it will be sufficient to put $Y_1 = Y_2 = Y_3 \dots = Y_m$, where Y_m is the mean Y for the section. In this case the correction to the approximate spherical bearing of the rth leg to reduce it to a plane bearing is given by :—

$$\delta\beta_r = -\frac{Y_m}{2r^2 \sin 1''}\left[2\Delta X_1 + 2\Delta X_2 + 2\Delta X_3 + \dots + 2\Delta X_{r-1} + \Delta X_r\right]$$

$$= -\frac{Y_m}{2r^2 \sin 1''}\left[2(X_{r-1} - X_0) + \Delta X_r\right]$$

and the difference between the corrections to the bearings of the rth and $(r - 1)$th legs by :—

$$\delta\beta_r - \delta\beta_{r-1} = -\frac{Y_m}{2r^2 \sin 1''}\left[\Delta X_{r-1} + \Delta X_r\right].$$

The computation of these corrections is very simple if it is carried out

in a systematic manner, and, for many purposes, it will be quite sufficient if a good slide rule is used for the multiplications. The following is a simple arrangement for a case in which $Y_m = 575{,}000$ ft. and latitude $= 23° 16'$ N., so that $\log \dfrac{1}{2r^2 \sin 1''} = \overline{10}{\cdot}374\,05$ and $\dfrac{Y_m}{2r^2 \sin 1''} = 0{\cdot}000\,1361$.

Leg	ΔX	$2\Sigma_1^{r-1}\Delta X + \Delta X_r$	Correction to Bearing
1	+ 8,600	+ 8,600	− 1″·09
2	+ 6,300	+ 23,500	− 3 ·20
3	+ 4,200	+ 34,000	− 4 ·63
4	− 5,000	+ 33,200	− 4 ·52
5	− 3,800	+ 24,400	− 3 ·32
6	− 9,600	+ 11,000	− 1 ·50
	+ 700		

It will be seen from this example that the entry for any leg in the third column is the sum of the entries in the second and third columns in the line above, plus the entry in the second column in the line itself. Thus, $+33{,}200 = +4{,}200 + 34{,}000 - 5{,}000$, and $-4''{\cdot}52 = 33{,}200 \times 0{\cdot}000\,1361$. The check on the entries in the third column is $2\Sigma_1^5 \Delta X - \Delta X_6 = 2(X_6 - X_0) - \Delta X_6 = 2 \times 700 - (-9{,}600) = +11{,}000$.

Computation of Secondary and Minor Traverses in Terms of Transverse Mercator Co-ordinates. The legs of secondary and minor traverses will generally be much shorter than those of first-order traverses, and, in addition, the work is of a lower order of accuracy so that less accurate methods of computing can therefore be used. Hence, further approximations are permissible.

In the first place, since the latitudes are relatively small, the differences between the spherical and grid bearings will also be small and can usually be neglected.

In the second place, it will hardly be necessary to correct each length to reduce it to grid length, and, instead of this, the traverse may be divided into fairly short sections and the differences in X and Y between the terminal points of sections corrected by means of formulæ (21) or (21A), using the mean Y of the section as the Y to be used in computing the correction. Hence, the procedure is to apply the convergence to the terminal azimuths so as to obtain spherical bearings referred to planes parallel to that of the central meridian, and to use these bearings, and the observed angles, to compute the bearings of the intermediate legs and the closing error in bearing at the end. In order to determine the convergence at the forward azimuth station it may be necessary to compute preliminary co-ordinates but it often happens that the position of the forward azimuth station can be scaled from a map, or from a preliminary plot, or determined with sufficient accuracy by some other means. After

the closing error in bearing has been distributed, the latitudes and departures are computed by the ordinary methods for plane rectangular co-ordinates.

Let (X, Y) be the co-ordinates of the initial point and $(X+\Delta X - Y+\Delta Y)$, the approximate co-ordinates of the end point, ΔX and ΔY being the sums of the latitudes and departures for the section. Then, if Y_m is the mean Y, the corrected sums of the latitudes and departures will be $\Delta X\left(1+\frac{Y_m^2}{2r^2}\right)$ and $\Delta Y\left(1+\frac{Y_m^2}{2r^2}\right)$. Hence the corrected co-ordinates of the terminal point will be :—

$$X' = X + \Delta X\left(1 + \frac{Y_m^2}{2r^2}\right)$$

$$Y' = Y + \Delta Y\left(1 + \frac{Y_m^2}{2r^2}\right).$$

If greater accuracy is desired, or if the section is rather long, the corrected ΔX and ΔY can be computed from formulæ (13) or (13A). Normally, however, the work should be split up into such short sections that the difference between using the exact and the approximate formulæ for scale correction is negligible.

Yates * has shown that the total angular error or twist at the end of a traverse section with short legs, due to neglecting the difference between the spherical and plane angles is given by :—

$$\delta\alpha = -\frac{A}{r^2 \sin 1''} = -\epsilon''$$

where A is the area, and ϵ'' the spherical excess of the figure enclosed by the traverse section, the perpendiculars from the terminal points on to the axis of X, and the X axis itself. This expression could, of course, be used as a correction to be distributed throughout the bearings, but there is no point in doing this since the value will be included and distributed with the apparent closing error in the usual way. It may, however, be useful on occasions, particularly when Y is very large, to compute it and apply it so as to get the real closing error in bearing, thus enabling a better estimate of the quality of the angular work to be determined.

This formula may be compared with the approximate one for $\delta\beta$, derived on page 382. If ΔX_r is very small compared with $(X_r - X_0)$, the expression for $\delta\beta_r$ becomes $-\frac{Y_m(X_r - X_0)}{r^2 \sin 1''}$ which is the spherical excess of the figure formed by the axis of X, the line joining the terminal points of the traverse, and perpendiculars drawn from these points to the axis of X. The expression given on page 382 is based on a mean value for Y, whereas Yates' expression takes the varying values of Y into account, and the difference in the two expressions is the spherical excess of the figure enclosed by the line of the traverse and the straight line joining the terminal points. Yates' formula can, in fact, be derived from the

* *Report of Proceedings, Empire Conference of Survey Officers*, 1928, page 211.

first equation for $\delta\beta_r$ on page 382 by putting $\Delta X_1 = \Delta X_2 = \Delta X_3 = \ldots = \Delta X_r = dX$ and integrating as dX becomes infinitesimally small.

Yates has also shown * that the Y component of the scale correction, when computed from point to point around a closed figure, is zero, but that the X component, when computed from point to point around the same closed figure, is given by the expression $\frac{AY_G}{r^2}$ or $\epsilon'' \cdot Y_G \sin 1''$, where A is the area of the figure, ϵ'' its spherical excess in seconds of arc, and Y_G the distance of its centre of gravity from the X axis. This result can be used either to determine the true scale correction for a traverse with very short legs, taking into account its variation with Y, or else to estimate the magnitude of the error involved by using $\left(1 + \frac{Y_m^2}{2r^2}\right)$ as the scale correction factor for ΔX and ΔY in the formulæ on page 384. Thus, in Fig. 133 the scale correction to the Y's around the closed figure is zero.

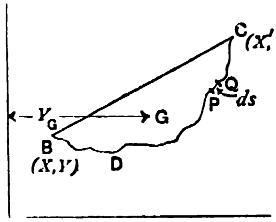


FIG. 133.

Hence, scale correction to the Y's, as computed along the short-legged traverse BDC, is equal to the scale correction computed along the straight line BC which joins the terminal points of the traverse. Consequently, using the formula for long lines, correction to $(Y' - Y)$ is :—

$$\frac{(Y' - Y)}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \frac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right].$$

For the X's, we have scale correction to $(X' - X)$ along the traverse BDC plus scale correction along the straight line CB $= \frac{AY_G}{r^2}$. Hence,

Scale correction to $(X' - X)$ along the traverse

$= \frac{AY_G}{r^2}$ + scale correction to $(X' - X)$ along line BC

$$= \frac{AY_G}{r^2} + \frac{(X' - X)}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \frac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right].$$

As an example, let the co-ordinates of B be $X = 327{,}916{\cdot}08$, $Y = 742{,}466{\cdot}74$ and the approximate co-ordinates of C be $X = 350{,}277{\cdot}24$, $Y = 770{,}808{\cdot}90$, so that $\Delta X = +\ 22{,}361{\cdot}16$, $\Delta Y = +\ 28{,}342{\cdot}16$. Let $\log \frac{1}{2r^2 \sin 1''} = \overline{10}{\cdot}374\ 05$, so that $\log \frac{1}{r^2} = \overline{15}{\cdot}360\ 65$. The traverse having been plotted on a fairly large scale, it was found that $A = 220{,}948{,}000$ square feet approximately and that $Y_G = 752{,}000$ ft. Hence $\frac{AY_G}{r^2} = 0{\cdot}38$ ft. and $\frac{(X' - X)}{2r^2}\left[\left(\frac{Y' + Y}{2}\right)^2 + \frac{1}{3}\left(\frac{Y' - Y}{2}\right)^2\right] = (X' - X) \times 0{\cdot}000\ 65672 = 14{\cdot}68$ ft. The scale correction is therefore $0{\cdot}38 + 14{\cdot}68 = 15{\cdot}06$ ft. and the corrected X' is $X' = 327{,}916{\cdot}08 + 22{,}361{\cdot}16 + 15{\cdot}06 = 350{,}292{\cdot}30$.

* *Report of Proceedings, Empire Conference of Survey Officers*, 1928, page 209.

If the approximate formula $\Delta X\left(1+\frac{Y_m^2}{2r^2}\right)$ had been used we would have obtained :—

$$\begin{aligned} X' &= 327,916\cdot08 + 22,361\cdot16 + 22,360 \times 0\cdot000\ 6566 \\ &= 327,916\cdot08 + 22,375\cdot84 \\ &= 350,291\cdot92. \end{aligned}$$

Hence, the error in computing by the approximate formula is only 0·38 ft. in 22,362 ft. or 1 in 59,000 in the X's.

For the Y's, the true scale correction

$$\begin{aligned} &= \frac{(Y'-Y)}{2r^2}\left[\left(\frac{Y'+Y}{2}\right)^2 + \frac{1}{3}\left(\frac{Y'-Y}{2}\right)^2\right] \\ &= 28,340 \times 0\cdot000\ 6567 = 18\cdot61, \end{aligned}$$

$$\begin{aligned} \text{and true } Y' &= 742,466\cdot74 + 28,342\cdot16 + 18\cdot61 \\ &= 770,827\cdot51 \text{ ft.,} \end{aligned}$$

the same result being obtained from the approximate formula.

In this example, the Y is large, and it can therefore be seen that it is sufficient, in all minor work, to correct the sums of the latitudes and departures between the terminal points of a reasonably short traverse section by means of the approximate formulæ involving the mean Y of the section. In such a case, the correction for height above sea level, when this is appreciable, can also be applied to the total latitude and departure of the whole section, and not to individual legs, the height used in the computation of the correction being the mean height of the section.

The method of determining the scale correction from the area of a figure and the distance of its centre of gravity from the X axis is sometimes useful for computing the total scale correction for a fairly long traverse without having to work through intermediate sections, or as a check against gross error in the computation of the scale errors of the individual sections.. In many cases, the general route of the traverse, if it follows well-defined paths or roads, is shown on a map or else is plotted on a plan, and, in such a case, the area and the Y co-ordinate of the centre of gravity can be obtained with sufficient accuracy by ordinary graphical means.

The derivation of the theorem given above is very simple. Take any small element PQ (Fig. 133) of length ds and let (x, y) be the co-ordinates of the middle point of PQ. Then :—

$$\text{Grid length of } ds = ds\left(1+\frac{y^2}{2r^2}\right)$$

$$\begin{aligned} \therefore \text{ Corrected latitude} & \\ &= ds\left(1+\frac{y^2}{2r^2}\right)\cos\alpha \\ &= dx\left(1+\frac{y^2}{2r^2}\right). \end{aligned}$$

Taking the sum of all the elements around the closed figure BDCB we have :—

$$\text{Total correction to latitudes} = \int\frac{y^2}{2r^2}\,dx,$$

where the integral is the line integral around the closed figure. Also,

$$\int_c \frac{y^2}{2r^2} dx = \int\int_A \frac{y}{r^2} dx dy = \int_A \frac{y}{r^2} dA$$

in which the integral on the right is the surface integral over the figure.

But, by the definition of the centre of gravity,

$$\int_A y dA = A Y_G$$

where A is the area of the figure and Y_G the distance of its centre of gravity from the axis of X.

Similarly the sum of the corrections to the departures around the figure

$$= \int_c \frac{y^2}{2r^2} ds \sin \alpha = \int_c \frac{y^2}{2r^2} dy = \frac{1}{6r^2}(Y^3 - Y^3) = 0.$$

Hence, the theorem follows.

MISCELLANEOUS PROBLEMS

Curves on the Spheroid. Although this point is not one of practical importance in ordinary triangulation with relatively short sides, mention must be made here of three curves on the spheroid which are of interest from a theoretical point of view, while two of them are of some importance in the case of triangles with very long sides. They are separate and distinct curves on the spheroid but they all merge into the one great circle on the sphere.

The first of these curves has already been referred to on page 326 in connection with Clarke's formulæ for long lines. It is the plane curve traced out by the plane containing the line of sight from a point A to a point B and containing the normal at A. This plane, although it contains the normal at A and passes through B, does not contain the normal at B. Similarly, the plane containing the normal at B and passing through A does not contain the normal at A. Each of these planes traces out a separate curve on the surface of the spheroid, and the one containing the normal at A is known as the "plane curve from A to B," or the "curve of normal section at A," and the one containing the normal at B as the "plane curve from B to A," or "curve of normal section at B." For all lines that can be sighted over, the difference in length between the two curves is inappreciable and the angles between them are very small, being given by :—

$$\delta = \frac{s^2}{4N^2} \frac{e^2 \cos^2 \phi \sin 2A}{(1 - e^2) \sin 1''}.$$

When both points lie north or south of the equator, the curves are curves of single curvature, and, if A is the point nearest to the equator, the plane curve from it to the point B lies nearer to the equator than the plane curve from B to A. This can be seen from the fact that the planes containing the curves intersect along the chord joining the two points, and the normal at the point with the smaller latitude cuts the minor axis of the spheroid higher up than does the normal at the point with the greater latitude.

The second curve is the line of shortest distance between the two

points and is called the "geodesic." For points lying on the same side of the equator it is a curve of double curvature, and the relations between the angles contained between it and the plane curves are as shown in Fig. 134. When both A and B lie on a meridian, the geodesic and the plane curves all coincide with, and become part of, the meridian.

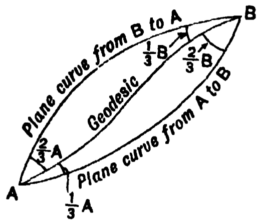


Fig. 134.

The geodesic, at every point along it, satisfies the equation :—

$$R_p \sin A = N \cos \phi \sin A = \text{constant},$$

where R_p is the radius of a parallel of latitude and A is the geodetic azimuth. As a general rule, it lies between the plane curves, but, if the terminal points are in nearly the same latitude, it may cross one of the plane curves.* Another characteristic property of the geodesic is that the principal normal to the curve at any point coincides with the normal to the surface at that point. For other references to this curve see pages 313, 315 and 541.

The third curve is the curve of alignment and it is the curve traced out by the position of a perfectly adjusted theodolite as it moves from point to point and is set at each point so that, when it is properly levelled, the line of sight from it to the point A differs by 180° from the line of sight to the point B. This curve is also one of double curvature and it lies very close to the geodesic, and usually between the two plane curves.

The differences in length between this curve and those of the geodesic and plane curves are inappreciable for all lines that can be sighted over in practice.

Location of Boundaries. Geodetic computations are required in the operation of tracing out long lines, as in marking interstate boundaries situated along meridians, parallels of latitude, or great arcs in any direction.

The best method consists in first running a triangulation chain along the general line and computing the geodetic positions of its stations. The distance and azimuth from those stations to any number of points on the line are then calculated, and the latter are located by traversing or by intersection from the triangulation stations. If the expense of triangulation cannot be incurred, or the distance is comparatively short, the procedure is as follows.

Meridians. To trace a meridian, the terminal stations are approximately located, and their longitudes are determined by means of wireless time signals. The errors in longitude of the assumed positions are eliminated by moving the stations by the linear equivalents of the longitude errors. At the initial station the direction of the meridian is obtained by azimuth observations, and a forward instrument station is established on it. The line is extended by backsighting and foresighting at successive

* The mathematical theory of the geodesic is very fully discussed by McCaw in papers on "Oblique Boundaries" and on "Long Lines on the Earth" in the *Empire Survey Review*, Vol. I, No. 4, April, 1932; Vol. I, No. 6, October, 1932, and Vol. II, No. 9, July, 1933.

instrument stations, and the distances of intermediate points are directly measured. Observations for azimuth are made at intervals. The error discovered on reaching the terminal station is proportioned to the intermediate points.

Arcs of Parallel. In locating a parallel of latitude, the position of the initial point A (Fig. 135) is established by astronomical observations. The direction AP of the meridian is obtained, and a perpendicular AB is set out. The parallel AC is located from this tangent.

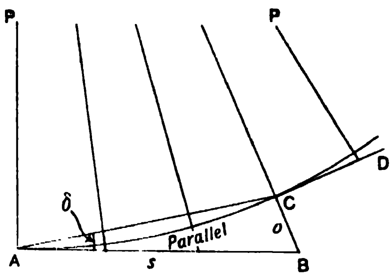


FIG. 135.

Since the azimuth of AB is 90° or 270°, the change of latitude in a distance s along AB is, by Puissant's formula,

$$d\phi = \frac{s^2 \tan \phi}{2NR_M} \qquad \frac{s^2 \tan \phi}{2NR}, \text{ with ample precision.}$$

The offset o is the linear equivalent of $d\phi$, so that

$$o = Rd\phi = \frac{\tan \phi}{2N}.$$

The direction of the offset differs from that of a perpendicular to AB by dA, the convergence of the meridians, and

$$dA = dL \sin \phi, \text{ with sufficient accuracy,}$$

$$\text{where } dL = \frac{s \sec \phi'}{N}.$$

To keep the offsets short, s should not exceed 50 miles, a new tangent CD being then set out. The work is controlled by latitude observations at intervals, and will be found to require some adjustment because of local deflections.

Instead of using offsets, a series of chords to a parallel of latitude may be laid out by means of deflection angles. In Fig. 135 let angle BAC $= \delta$ in seconds of arc and let s be the length of the chord AC. Then :—

$$o = s \tan \delta = s \,.\, \delta \,.\, \sin 1''$$

$$\delta = \frac{}{2N \sin 1'} \,.\, \tan \phi.$$

Hence, if a line of length s is laid out making an angle $90° - \delta$ with the meridian at the initial point, the end of the chord will lie on a parallel of latitude through the first point. When C is reached, the next chord of length s can be set out by turning off an angle of $180° - 2\delta$ from A. The process is therefore very similar to that of setting out a railway curve by means of deflection angles, but, of course, the chords used in setting out a parallel of latitude will be very much longer than those used in setting out railway curves.

A convenient chord length for setting out a parallel of latitude is about six miles, which gives a deflection angle of 2′ 36″·4 in latitude 45°.

At any intermediate point the offset from the chord AC to the parallel of latitude is given by :—

$$x = \frac{s_1(s - s_1) \tan \phi}{2N}$$

where s_1 is the distance of the intermediate point along the chord from A. Hence, intermediate points are easily laid out from the chord.

This method may be preferred to the method of offsets from the tangent, especially if the line has to be cut through bush, as it follows the actual line of the parallel more closely than the tangent does, and the offsets are shorter. Moreover, it produces a cut line free from sharp bends, which is an advantage from a traversing point of view, as short angular sights can thus be more easily avoided.

Oblique Arcs. In locating an oblique arc, the positions of the terminals are first found by astronomical observations. The azimuth between these points is computed, and a line is run on this azimuth from the initial station. Intermediate points are marked at known distances, and the azimuth is checked at intervals, observed azimuths being corrected for convergence. The discrepancy found on reaching the end of the line is distributed to the intermediate points.

EXAMPLES

(1) Assuming that $a = 6,378,300$ m. and $b = 6,356,860$ m., compute the mean radius, $\sqrt{R.N.}$, of the earth at a point in latitude 50° N.

2. A line AB from the above point has a length of 48,532 metres and an azimuth, reckoned clockwise from north, of 344° 17′. Compute the latitude of B.

3. Find the lengths of 1 second of meridian and of parallel at the equator and in latitude 50°.

4. Assuming that $a = 6,378,249$ m. and $b = 6,356,515$ m. (Clarke's 1880 figure), use the formula on page 282 to calculate, with seven-figure logarithms and to the nearest metre, the meridional distances 50° to 51° and 51° to 52°.

5. The following are the observed angles of a triangle :—

$$\begin{aligned} A &= 49^\circ\ 17'\ 23''\cdot 2 \\ B &= 75\ \ 32\ \ 46\ .7 \\ C &= 55\ \ 09\ \ 53\ \cdot 1. \end{aligned}$$

The log length of the side AC in feet is 5·202 3964. Compute the spherical excess of the triangle, adjust the angles and use Legendre's theorem to solve the triangle.

Take $\log \dfrac{1}{2RN \sin 1''} = \overline{10}\cdot 374\ 16.$

6. Assuming that the latitude and longitude of the point A of the triangle in the last example are 21° 40′ 18″·2 South and 15° 18′ 16″·4 East and that the azimuth of the line AC is 221° 14′ 16″·94, reckoned clockwise from north, use Clarke's formulæ for medium lines to compute the latitude and longitude of C and the reverse azimuth of AC. Take

$$\log \frac{1}{2R_M N_M \sin 1''} = \overline{10}\cdot 374\ 16,$$

$$\log \frac{1}{R_M \sin 1''} = \overline{3}\cdot 996\ 0860,$$

$$\log \frac{1}{N_c \sin 1''} = \overline{3}\cdot 993\ 5273.$$

7. Given that the co-ordinates on the Transverse Mercator system of a point A are :—

$$X \text{ (Northing)} = 268{,}462{\cdot}32 \text{ ft.}, \quad Y \text{ (Easting)} = 338{,}692{\cdot}14 \text{ ft.},$$

and that the observed bearing and distance in feet to a point B are :—

$$\alpha = 242^\circ\ 16'\ 08''{\cdot}09\ ; \quad s = 156{,}429{\cdot}2 \text{ ft.},$$

the bearing being reckoned clockwise from north, find the co-ordinates of the point B by using the formulæ for point-to-point working given on page 359 and then check by computing and using the grid bearing and distance.

Take $$\log \frac{1}{2r^2 \sin 1''} = \overline{10}{\cdot}374\ 16.$$

8. It is required to navigate an aeroplane from a point near Freetown, Sierra Leone, in latitude 7° *N*, longitude 13° *W* to a point near Khartoum in latitude 16° *N*, longitude 33½° *E* so as to follow the great circle joining the two points. Assuming the earth to be spherical, use spherical trigonometry to evolve suitable formulæ for calculating the distance between the two points and the azimuths of the great circle at each. Taking the radius of the earth as 3,950 miles, calculate the distance to the nearest tenth of a mile and the azimuths to the nearest decimal of a minute.

9. Use the equation of the ellipse to prove that

$$R = \frac{a(1-e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}$$

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}$$

where $$e^2 = (a^2 - b^2)/a^2.$$

10. Given that the geodetic latitude of a place is 38° 15′ 35″ and that $a = 6{,}378{,}388$ m., $b = 6{,}356{,}909$ m., find the geocentric latitude. Use the equation of the ellipse to prove the formula employed.

In the following examples the meridional distances and geodetic factors used in working out the answers given at the end of the book have been taken from the War Office Publication "Geodetic Tables for the Clarke 1880 *Figure in Feet for Latitudes* 0° *to* 70°." (*H.M.S.O., London. Price* 9*d.*)

11. Calculate the distance and azimuth from a point in latitude 56° 06′ 21″ *N.*, longitude 4° 54′ 52″ *W.* to one in latitude 56° 20′ 17″ *N.*, longitude 4° 58′ 31″ *W.*

12. Given that the origin of co-ordinates is the point 10° North, 8° West, find the co-ordinates in feet on the rectangular spherical system of a point whose geographical co-ordinates are latitude 12° 16′ 12″·98 North, longitude 9° 52′ 31″·64 West. Transform these co-ordinates into others on the Transverse Mercator System and check by computing the Transverse Mercator co-ordinates directly from the geographical co-ordinates.

13. The logarithm of the distance and the observed azimuth of a point B from the point whose co-ordinates are given in the above example are

$$A = 163^\circ\ 16'\ 28''{\cdot}04. \quad \log s = 5{\cdot}104\ 6218.$$

Calculate and apply the convergence at A and find the co-ordinates on the Transverse Mercator system of the point B. The azimuth is, as usual, to be reckoned clockwise from north.

14. Calculate the geographical co-ordinates of the point B in the last example by using Puissant's formula.

15. From the point A in example 12 a traverse, made up of short legs, is carried to a point C. The initial bearing used in the computation of the latitudes and departures was the bearing calculated in example 13. The plane rectangular latitude and departure of C with reference to A were found to be : Latitude = + 82,342·16 ft., departure = − 60,118·90 ft. Find the Transverse Mercator co-ordinates of C.

16. Given that a point A is in latitude 45½° *N.* and longitude 107° *W.*, calculate its conical orthomorphic co-ordinates in feet and the convergence in seconds, the origin being the point 44° *N.*, 105° *W.*

17. A series of chords of 50,000 ft. each were used to lay out a parallel of latitude along the 40th parallel. Calculate the deflection angle to be used at the beginning of each chord and the offset midway along the chord.

If the method of offsets from the tangent were employed, what would be the length of the offset at 50,000 ft. along the tangent ?

Take $$\log \frac{1}{N \sin 1''} = \bar{3}\cdot 993\ 1234.$$

REFERENCES ON GEODETIC SURVEYING

"An Account of the Measurement of a Geodetic Base Line at Lossiemouth, in 1909, together with a Discussion on the Theory of Measurement by Metal Tapes and Wires in Catenary." Ordnance Survey, Prof. Paper No. 1, 1912.

ADAMS, O. S., and CLAIRE, C. N. "Manual of Plane-Co-ordinate Computation." U.S. Coast and Geodetic Survey. Special Publication No. 193.

— "Manual of Traverse Computation on the Transverse Mercator Grid." U.S. Coast and Geodetic Survey. Special Publication No. 195.

ATTWELL, S. W. "Surveying Tapes and Wires." *Dictionary of Applied Physics.* Vol. III, London, 1923.

BAESCHLIN, C. F., *Lehrbuch der Geodäsie*, Zürich, 1948.

BENÔIT, J. R., and GUILLAUME, C. E., *La Mesure Rapide des Bases Géodésiques.* Paris, 1917.

BERGSTRAND, ERIK. "Measurements of Distances by High Frequency Light Signalling." *Bulletin Géodésique*, No. 11, 1949.

— "A Determination of the Velocity of Light." *Arkiv för Fysik*, Vol. II, No. 15. Stockholm, 1950.

BLACK, A. N. "The Method of Systematic Relaxation Applied to Survey Problems." *Empire Survey Review.* Vol. IV, No. 29.

BOMFORD, BRIGADIER G. *Geodesy.* London, 1951.

BRIGGS, H. *The Effects of Errors in Surveying.* London, 1912.

BURNS, E. L. M. "Point-to-Point Working for the Conical Orthomorphic Projection." *Empire Survey Review.* Vol. II, No. 11.

CARD, F/LT. L. C. "Shóran Electronics." *The Canadian Surveyor*, Vol. X, No. 3, 1950.

CARSON, P. A. "Precise Measuring with Invar Wires and the Measurement of the Kootenay Base." Department of the Interior. Ottawa, 1911.

CHURCHILL, J. A., and MORLEY, A. J. "Observing in Haze." *Report of Proceedings. Conference of Empire Survey Officers.* 1935.

CLARKE, A. R. *Ordnance Trigonometrical Survey of Great Britain and Ireland. Account of the Observations and Calculations of the Principal Triangulation.* London, 1858.

— *Geodesy.* Oxford, 1880.

CLARKE, A. R., and HELMERT, F. R "Figure of the Earth." *Encyclopedia Britannica*, 11th Edition, 1911.

CLENDINNING, J. "Experiments to Determine the Incidence and Magnitudes of Errors in Precise Linear Measurements in Surveying Arising from Errors in Temperature Observations." *Report of Proceedings. Conference of Empire Survey Officers.* 1935.

— "Mathematical Theory of the Transit Theodolite." *Empire Survey Review*, Vol. VI, No. 42.

— "The Modified Meridian Distance in the Conical Orthomorphic Projection." *Empire Survey Review*, Vol. VI, No. 43.

CLENDINNING, J., and YATES, F. "Notes on the Application of the Method of Least Squares to the Adjustment of Triangulation and Level and Traverse Networks." Gold Coast Survey Professional Paper No. 1, Accra, 1929.

CLOSE, SIR C. F. "Geodesy of the British Islands." *Report of Proceedings. Conference of Empire Survey Officers*, 1928.

— "Notes on the Geodesy of the British Islands." Ordnance Survey Professional Paper No. 3, 1914.

CLOSE, SIR C. F., and WINTERBOTHAM, H. ST. J. L. *Text Book of Topographical Surveying*, London, 1926.

COLE, J. H. "Computation of Distances of Long Arcs for Radio Purposes." *Empire Survey Review*, Vol. VIII, No. 59.

— "Point to Point Computation of Geographical Co-ordinates of Long Lines." *Empire Survey Review*, Vol. IX, No. 66.

CRANDALL, C. L. *Text Book on Geodesy and Least Squares*. New York, 1914.

DURGIN, C. M., and SUTCLIFFE, W. D. "Manual of First-Order Traverse." U.S. Coast and Geodetic Survey. Special Publication No. 137.

EIMBECK, W. "Formulæ and Tables for the Computation of Geodetic Positions." U.S. Coast and Geodetic Survey. Special Publication No. 8, 1919.

ERRITT, W. A. "The Fixation of Minor Triangulation on the Transverse Mercator Projection." *Empire Survey Review*, Vol. IV, No. 30.

— "Transformation of Rectangular Co-ordinates on the Transverse Mercator Projection." *Empire Survey Review*, Vol. V, No. 32.

GILL, SIR D., etc. Reports on the Geodetic Survey of South Africa. Vols. I–VI. 1896–1932.

GORE, J. H. "A Bibliography of Geodesy." U.S. Coast and Geodetic Survey, Report for 1902, Appendix No. 8.

GUILLAUME, C. F. "Les Aciers au Nickel et leurs Applications à la Géodèsie." International Geodetic Conference Report, 1901. (Translated in Survey of India Prof. Paper No. 7, 1903.)

HART, LIEUT.-COL. C. A. "Mapping by Remote Control with the Aid of Radar." Directorate of Military Survey, Air Survey Research Paper No. 19. London, 1946.

— "Modern Mapping—Including the Employment of Radio Techniques and with Special Reference to Economic Development." *Empire Survey Review*, Vol. IX, No. 65.

— "Some Aspects of the Influence on Geodesy of Accurate Range Measurement by Radio Methods, with Special Reference to Radar Techniques." *Bulletin Géodésique*, No. 10, 1948.

HART, LIEUT.-COL. C. A., and MISKIN, CAPT. E. A. "Range Measurement and Fixation by Radar, and the Effects of Inaccuracies of Range Determination on Fixation." Directorate of Military Survey, Air Survey Research Paper No. 7, 1945.

HENDRIKZ, D. R. "Adjustment of the Secondary Triangulation of South Africa." *Empire Survey Review*, Vol. IV, Nos. 29 and 30.

— "Relaxation and the Co-ordinate Method of Triangulation Adjustment." *Empire Survey Review*, Vol. V, No. 36.

HENRICI, O. and E. O. "The Theory of Measurement by Metal Tapes or Wires in Catenary." Professional Papers of the Ordnance Survey, New Series, No. 2, 1912.

HOSMER, G. L. *Geodesy*. New York, 1930.

HOTINE, MAJOR M. "The East African Arc." *Empire Survey Review*, Vol. II, Nos. 12 and 14, 1934; Vol. III, Nos 16 and 18, 1935.

— "The General Theory of Tape Suspension in Base Measurement." *Empire Survey Review*, Vol. V, No. 21, 1939.

— "The Re-Triangulation of Great Britain." *Empire Survey Review*, Vol. IV, Nos. 25, 26, 1937; Vol. IV, No. 29, 1938; Vol. V, No. 34, 1939.

HUNTER, J. DE G. "The Earth's Axes and Triangulation." Survey of India Prof. Paper No. 16, 1918.

— "Figures of Reference for the Earth." *Empire Survey Review*, Vol. II, No. 8, 1933.

— "Lay-out of Geodetic Triangulation." Geodetic Report of the Survey of India, Vol. VII, 1932.

— "Use and Standardisation of Long Tapes in Catenary." *Proceedings of 1947 Conference of Commonwealth Survey Officers*. London, 1950.

INGRAM, E. L. *Geodetic Surveying and the Adjustment of Observations*. New York, 1911.

JACK, BRIG. E. M. "Report on the Measurement of an Arc of Meridian in Uganda." Vol. I, Colonial Office, London, 1913.

JACKSON, J. E. "The Effect of Altitude on Traverse Errors." *Empire Survey Review*, Vol. II, No. 12, 1934.
— "Notes on Some Experiments with Thermometers and Steel Tapes." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
JAMESON, A. H. *Advanced Surveying*. London, 1949.
JOHNSON, W. H. "Line Standards of Length." *Dictionary of Applied Physics*, Vol. III, London, 1923.
JOLLY, H. L. P. "The Combination of Observations." *Dictionary of Applied Physics*, Vol. III, London, 1923.
— "Traverse Adjustment; Why not Least Squares?" *Empire Survey Review*, Vol. IV, No. 28, 1938.
JORDAN, J. *Handbuch der Vermessungskunde*. Vols. I, II and III, Stuttgart.
LAMBERT, W. D. "The Distance Between Two Widely Separated Points on the Surface of the Earth." *Journal of the Washington Academy of Sciences*, Vol. 32, No. 5.
LEE, L. P. "The Transverse Mercator Projection of the Spheroid." *Empire Survey Review*, Vol. VIII, No. 58.
LEIPOLDT, MAJOR J. G. W. "Observing with the Zeiss and Wild Theodolites." *Empire Survey Review*, Vol. IV, No. 26, 1937.
LOWINGER, V. A. "Standard Traverses." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
MACDONALD, J. A. "Testing Surveyor's Tapes by the Canadian Government." *Eng. News*, Vol. 74, No. 9, 1915.
MACLEOD, MAJOR-GENERAL M. N. "Imperial Geodetic Responsibilities." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
MCCAW, CAPT. G. T. "The Approximately Rigorous Adjustment of Simple Figures." London, 1920.
— "Bowditch Traverse Adjustment and a Modification." *Empire Survey Review*, Vol. IV, No. 29, 1938; Vol. V, No. 31, 1939.
— "Critical Support for Bowditch." *Empire Survey Review*, Vol. V, No. 38, 1940.
— "The Deflection Prism." *Empire Survey Review*, Vol. V, No. 32, 1939.
— "Intervisible Features." *Empire Survey Review*, Vol. III, No. 21, 1936.
— "Long Lines on the Earth." *Empire Survey Review*. Vol. I, No. 6, 1932; Vol. II, No. 9, 1933; Vol. II, Nos. 12 and 14, 1934.
— "Standards of Length in Question." *Empire Survey Review*, Vol. I, No. 6, 1932.
— "The Transverse Mercator Projection." *Empire Survey Review*, Vol. IV, No. 27, 1938.
— "The Trigonometrical Survey of Fiji. Report on." Vol. I (Viti Levu), London, 1916. Vol. II (Vanua Levu and Tavouni), London, 1917.
MERRIMAN, M. *A Text Book on the Method of Least Squares*, New York, 1900.
— *Elements of Precise Surveying and Geodesy*. New York, 1904.
MORLEY, A. J. "The Adjustment of a System of Trigonometrical Chains." *Empire Survey Review*, Vol. V, No. 33, 1939.
— "A Test for the Reliability of First-Order Trigonometrical Observations." *Empire Survey Review*. Vol. VI, No. 41, 1941.
MUNSEY, D. F. "Base Measurement in the Anglo-Egyptian Sudan." *Empire Survey Review*, Vol. X, Nos. 72, 73 and 74, 1949.
NELLES, D. H. "The Projection and Survey of the 141st Meridian Boundary Line." *Trans. Can. C.E.*, Vol. XXVII.
NEY, C. H., and MACTAVISH, W. H. "The Simultaneous Adjustment of Precise Traverses and Triangulation Nets." Canadian Geodetic Survey Publication No. 26, Ottawa.
PERRIER, G. *Une Petite Histoire de la Géodèsie*, Paris, 1939.
POLLARD, A. F. C. "The Application of Kinematic Design to Theodolites and the Determination of their Variance." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
POTTER, A. J. "The Earliest Geodetic Triangulation." *Empire Survey Review*, Vol. I, No. 3, 1932.
RAINSFORD, H. F. "The Clarke Formulæ for Computing Latitudes, Longitudes and Reverse Azimuths for Use with either Logarithms or Machine Calculation." *Empire Survey Review*, Vol. V, No. 33, 1939.

RAINSFORD, H. F. "Long Lines on the Earth: Various Formulæ." *Empire Survey Review*, Vol. X, Nos. 71 and 72.
— "Minor Triangulation on the Transverse Mercator Projection." *Empire Survey Review*, Vol. V, No. 38, 1940.
RANNIE, J. L. "A Field Test of Primary Triangulation Theodolites." *Canadian Journal of Research*, National Research Council of Canada, March, 1934.
RANNIE, J. L., and DENNIS, W. M. "Improving the Performance of Primary Triangulation Theodolites as a Result of Laboratory Tests." *Canadian Journal of Research*, March, 1934.
RANNIE, J. L., and DENNIS, W. M. "Improving the Performance of Wild Precision Theodolites." *Empire Survey Review*, Vol. III, No. 15, 1935.
— "Variable Personal Equation of Bisection in Primary Triangulation." *Canadian Journal of Research*, March, 1934.
REDFEARN, J. C. B. "Transverse Mercator Formulæ." *Empire Survey Review*, Vol. IX, No. 69.
ROSS, J. E. R. "Geodetic Problems in Shoran." Canadian Geodetic Survey Publication No. 76, 1949.
— Shoran: Application to Geodetic Triangulation." *The Canadian Surveyor*, Vol. X, No. 3, 1950.
Royal Geographical Society. *Hints to Travellers*, Vol. I, London, 1935.
— "New Geodetic Tables for Clarke's Figure of 1880 with Transformation to Madrid 1924 and Other Figures." R.G.S. Technical Series No. 4, 1927.
SEARS, J. E., JOHNSON, W. H., and JOLLY, H. L. P. "A New Determination of the Ratio of the Imperial Standard Yard to the International Prototype Metre." *Phil. Trans.*, Series A, Vol. 227, 1928.
SIMONSEN, O. "Preliminary Report on the Triangulation between Denmark and Norway carried out in 1945 by Means of Parachute Flares." *Bulletin Géodésique*, No. 11, 1949.
SPIGL, H. S. "The Survey of the 129th Meridian." *The Australian Surveyor*, Vol. 8.
Survey of India. "Auxiliary Tables. Part IV, Geodetic Tables." Fifth Edition. Dehra Dun. 1931.
— "Auxiliary Tables. Part II, Mathematical Tables." Fifth Edition, Dehra Dun, 1931.
TAYLOR, E. WILFRED. "The Effects of Eccentricity and Misplaced Indices of Divided Circles." *Empire Survey Review*, Vol. V, No. 38, 1940.
— "The Surveyor's Telescope." *Empire Survey Review*, Vol. VI, No. 39, 1941.
— "Testing a Theodolite for Accuracy." *Empire Survey Review*, Vol. VI, No. 40, 1941.
THOMPSON, CAPT. E. H. "A Direct Method of Transforming Rectangular Co-ordinates from the Cassini Projection to the Gauss Conformal Projection." *Empire Survey Review*, Vol. III, Nos. 18 and 20, 1935, 1936.
— "Least Square Solutions with a Calculating Machine." *Empire Survey Review*. Vol. III, No. 20, 1936.
THORNHILL, G. K. "Temperature Correction to Steel Tapes." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
TOBEY, W. M. "The Conversion of Latitude and Departures of a Traverse to Geodetic Differences of Latitude and Longitude." Canadian Geodetic Survey Publication No. 25.
— "Geodesy." Canadian Geodetic Survey Publication No. 11.
UNITED STATES COAST AND GEODETIC SURVEY. "Formulæ and Tables for the Computation of Geodetic Positions." Special Publication No. 8.
— "Manual of Second and Third-Order Triangulation and Traverse." Special Publication No. 145.
— "Uses of Geodetic Control for City Surveys." Special Publication No. 91.
WAR OFFICE. "Geodetic Tables for Clarke 1880 Figure in Feet for Latitudes 0° to 70°." H.M.S.O., London, 1936.
— *Survey Computations*. Second Edition. H.M.S.O., London, 1932.
WARREN, H. A. "Adjustment of a Traverse by Relaxation." *Empire Survey Review*, Vol. V, No. 34, 1939.
WEATHERBURN, C. E. *Differential Geometry of Three Dimensions*. London, 1949.
WHITTAKER, E. T., and ROBINSON, G. *The Calculus of Observations*, London, 1924.

WHITTINGDALE, W. "Map and Survey Grids." *Reports of Proceedings. Conference of Empire Survey Officers*, 1935.
— "The Triangulation and Computation Systems of the Union of South Africa." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
WINTERBOTHAM, BRIG. H. ST. J. L. "An Investigation into the Accuracy of the Principal Triangulation of the United Kingdom." Ordnance Survey Prof. Paper No. 2, 1913.
WOOD, J. C. "Methods of Observing for Primary Triangulation." *Report of Proceedings. Conference of Empire Survey Officers*, 1935.
WOODWARD, R. S. "The Iced Bar and Long Tape Apparatus." U.S. Coast and Geodetic Survey, Report for 1892. Appendix No. 8.
WRIGHT, J. W. "Some Effects of Pulley Eccentricity in Base Measurements." *Empire Survey Review*, Vol. X, No. 74, 1949.
WRIGHT, T. W., and HAYFORD, J. F. *The Adjustment of Observations*, New York.
YATES, F., and WACE, H. "Report on the Survey and Adjustment of a Network of Precise Traverses Measured between 1924 and 1930 in the Southern Portion of the Colony." Gold Coast Survey Records, Vol. III, Accra, 1931.
YOUNG, A. E. "Corrections to be Applied to the Readings of a Spring Balance when used Horizontally or on a Slope." *Min. Proc. Inst. C.E.*, Vol. CLXI.
— "On the Form of a Suspended Wire or Tape, including the Effect of Stiffness." *Philosophical Magazine*, XXIX, January, 1915.

CHAPTER VI

GEODETIC LEVELLING

THE operations of geodetic levelling are directed to the determination of the elevations of a number of vertical control points from which those of any other points throughout the survey may be obtained. The elevations required are absolute elevations from the datum of mean sea level, and, unless the levelling can be connected to that of an adjoining system, it will be necessary to establish the datum by tidal observations.

Methods. The elevations of control points are determined either by precise spirit levelling or trigonometrical levelling, or by both.

Precise spirit levelling represents the highest development of the methods of ordinary levelling, every precaution being taken in the construction of the instruments and in the field work to reduce errors to a minimum. By running lines of precise levels along the most favourable routes through the survey, primary bench marks are established for its ultimate vertical control. From these points subsidiary lines are run and bench marks established for the control of detail, and, in State surveys, for the information of the public. Precise levelling is also required to provide data for geodetic research relating to gravity, secular upheaval or depression, and the comparison of ocean levels. Investigations in connection with certain classes of engineering work, particularly those relating to the flow of water over considerable distances, also demand levelling of a high order of accuracy, and the engineer is sometimes required to produce work comparable to the best precise levelling.

In trigonometrical levelling, differences of elevation are determined from vertical angles and geodetic or horizontal distances The measurement of vertical angles between triangulation stations is usually undertaken at the same time as that of the horizontal angles. The same stations therefore control the survey both horizontally and vertically, and the elevations can be determined economically. In point of accuracy trigonometrical levelling is much inferior to precise spirit levelling, particularly in flat districts. In rugged country the method is of great value, and the results are comparatively more accurate.

Determination of Mean Sea Level. In order to obtain a measurement of the average elevation of the sea, a continuous record of its varying level is required for as long a time as possible. The period of observation should cover an integral number of lunations, and unless it extends to at least a year the effect of the solar components will not be truly represented. The employment of a self-registering tide gauge is almost essential, and its zero must be connected and periodically checked by precise levelling to a permanent bench mark near at hand. Tide gauges are discussed on pages 539–543 of Vol. I, and questions relating to mean sea level as a datum plane for levels on pages 543–546 of the same volume.

PRECISE SPIRIT LEVELLING

There is no difference in principle between precise and ordinary levelling. In the latter, the distances run between checks are relatively short, and, with the usual precautions, the results are sufficiently accurate for everyday purposes. Since very small errors cannot be detected, the relative coarseness of the determinations may, through compensation of errors, give a fictitious idea of accuracy. In precise levelling, on the other hand, the circuits may be of considerable length, and the operations must be conducted so that the uncertainty of each individual determination as well as the actual closing error is reduced to a minimum. This necessitates the employment of high-class instruments of superior sensitiveness and their manipulation in such a manner that instrumental and observational errors are eliminated as far as possible. Success in precise levelling depends upon a due appreciation of the nature and relative importance of those errors and their treatment.

Precise Levels. Precise levels belong in their essentials either to the wye or the dumpy type. Many different designs have been employed, and constructional details vary considerably.

Stability is an important requirement, and is promoted by the provision of a broad levelling base and by having the height above the base and the exposed area as small as practicable. Fittings subject to wear are formed of hard cast steel or iron, and adequate protection must be given to parts exposed to unequal temperature changes.

The telescope is an important feature, and it must afford superior definition, illumination, and flatness of field. The focal length of the objective ranges from 11 in. to over 20 in., and the effective aperture should be at least $1\frac{1}{2}$ in. The eyepiece is generally non-erecting, and the magnifying powers range from 25 to 50 diameters. Stadia hairs are fitted. Focussing is performed either by movement of the eyepiece end or by an internal lens, and the motion must be truly axial.

The level tube is of very uniform curvature, and is furnished with an air chamber (Vol. I, page 34), so that a fairly constant length of bubble can be maintained. The liquid should be sulphuric ether in preference to alcohol on account of its superior mobility. In the more recent instruments of the highest class the angular value of a 2-mm. division of the level tube ranges from about 1·2 sec. to 3 sec. The tube is either attached to the telescope, as in ordinary levelling instruments, or is arranged as a striding level to permit of its reversal without moving the telescope.

The three foot screws are used only in the preliminary approximate levelling of the instrument, for which a circular spirit level or two small bubble tubes at right angles to each other are fitted. An important feature of precise levels is the provision under the telescope of a micrometer levelling screw whereby the telescope can be tilted about a horizontal axis. By rotation of this screw the main bubble is brought to the centre of its run for each observation, the verticality of the vertical axis being neglected. To enable the operator to maintain the bubble central while he is observing, or to inform him of its position at the instant of reading the staff, a reflecting device is essential. This may consist of a mirror or an arrangement of prisms.

The Parallel Plate Micrometer. The parallel plate micrometer is a de ice designed by Wild to reduce errors of estimation of readings on a levelling staff, and it is now generally fitted to most precise levels. It consists of a plate of glass, with opposite faces carefully ground so as to be exactly parallel to one another, mounted on a horizontal axis and fitted in front of the object glass of the level. The level being maintained in a horizontal position, inclination of the glass plate from the vertical displaces the image of the readings on the staff by an amount depending upon the angle of tilt. In taking an observation, the plate is turned by a rod connected with a screw carrying a graduated drum (13 and 14 in Fig. 140) until the image of the nearest graduation is brought into contact with the horizontal hair, and the fractional part of a staff division is read on the drum. By this means readings may be made direct to the nearest thousandth part of a foot and estimated to the nearest ten-thousandth part.

In Fig. 136 ABCD is a plate of glass with carefully ground parallel faces AD and BC. Let this plate be tilted through an angle i with the vertical. Let OM be a horizontal ray of light falling on the face AD at M. Then this ray will be refracted into the glass, following the line MN, and will emerge at N parallel to OM but displaced vertically below it. If PMP′ is the normal to the face AD at M, the angle PMO $= i$ will be the angle of incidence of the ray and it will also be equal to the angle of tilt of the plate from the vertical. Line MN will make an angle $r =$ P′MN with the line MP′ and this angle will be equal to the angle of refraction of the ray into the glass. Hence, by Snell's law of refraction :—

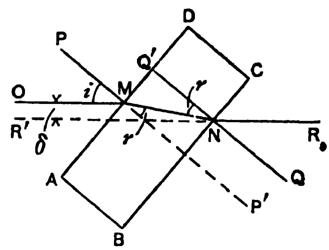


FIG. 136.

$$\frac{\sin i}{\sin r} = \mu \quad .$$

where μ is the index of refraction from air to glass.

Through N draw Q′NQ perpendicular to BC at N and meeting AD at Q′. Then angle MNQ′ $= r$, and, since NR is parallel to OM, angle QNR $= i$.

Let δ be the vertical displacement of NR below OM. Then :—

$$\begin{aligned} \delta &= \text{MN} \sin \text{MNR}' \\ &= \text{MN} \sin (i - r). \end{aligned}$$

But,

$$\begin{aligned} \text{MN} &= \text{Q}'\text{N} \sec r \\ &= t \sec r. \end{aligned}$$

where t is the thickness of the plate. Hence :—

$$\begin{aligned} \delta &= t \sec r \sin (i - r) \\ &= t\left(\sin i - \frac{\cos i \sin r}{\cos r}\right) \\ &= t\left(\sin i - \frac{\cos i \sin i}{\mu \cos r}\right) \end{aligned}$$

$$= t \sin i\left(1 - \frac{\cos i}{\mu \cos r}\right)$$

$$= t \sin i\left(1 - \frac{\sqrt{1 - \sin^2 i}}{\mu\sqrt{1 - \sin^2 r}}\right)$$

$$= t \sin i\left(1 - \frac{\sqrt{1 - \sin^2 i}}{\mu\sqrt{1 - \dfrac{\sin^2 i}{\mu^2}}}\right)$$

$$= t \sin i\left(1 - \frac{\sqrt{1 - \sin^2 i}}{\sqrt{\mu^2 - \sin^2 i}}\right).$$

i is always a small angle and we may therefore put $\sin i = i$ and regard $\sin^2 i$ as negligible. We then have :—

$$\delta = ti\left(1 - \frac{1}{\mu}\right)$$

Consequently, the deflection of the horizontal line of sight is directly proportional to the thickness of the glass plate and to the angle of deflection of the plate from the vertical. If, therefore, when the glass plate is

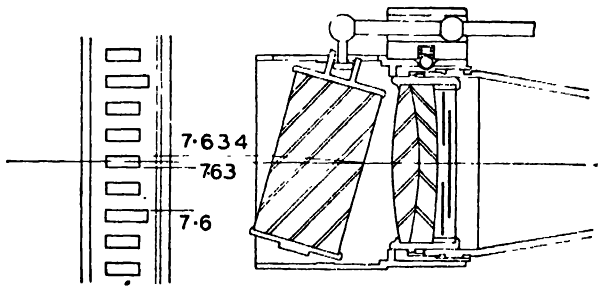


FIG. 137.

(By permission of Messrs. Cooke, Troughton & Simms.)

vertical with the drum adjusted to read zero, the line of sight appears to come between two divisions on the staff, it can easily be brought into coincidence with one of the divisions by rotating the plate, and the amount of the rotation required can be used to measure the difference in elevation between the division and the position of the line of sight. The method of reading will now be clearly understood from Fig. 137, the displacement due to the rotation of the plate being read directly on the graduated drum provided for the purpose.

Use of Colour Filters on Levels and Theodolites. During the last few years the Coast and Geodetic Survey have used a detachable colour filter, placed at the eye end of the telescope of a precise level, to help to eliminate the effects of the scattering of light of short wave length and of "boiling" due to irregular changes in atmospheric refraction caused by heat radiation. The filter found most suitable for general use on a level is a Kodak Wratten No. 26, Stereo Red. In ordinary circumstances, on dull days or even in light rain, the filter is a disadvantage

rather than an advantage, as it tends to absorb too much wanted light; but, in bright sunlight, it has been found to be a great advantage, not only as regards increasing the accuracy attained but also by enabling a longer length of sight to be used, as it reduces "boiling" to a minimum.

The use of colour filters on surveying instruments appears first to have been suggested by J. H. Churchill in an article in the *Empire Survey Review*, Vol. II, No. 7, January, 1933. He suggested that the lengths of sights observed with a theodolite might be increased in hazy weather if a colour filter were used to cut out unwanted scattered light of short wave length. The existing evidence about the advantages of filters on theodolites tends on the whole to be rather contradictory, some surveyors claiming that a filter improves vision in certain circumstances, but others saying that it is no advantage and only reduces the amount of useful light entering the telescope. The general consensus of opinion, however, seems to be that a filter may improve observing in certain atmospheric conditions, and the Coast and Geodetic Survey have now fitted a light amber Wratten Filter No. 8, K-2 Orthochromatic, to a number of their theodolites.

The Binocular Precise Level. The type of level designed by the United States Coast and Geodetic Survey embodies several admirable features, and marked a decided advance in precise levelling instruments when it was introduced in 1900.

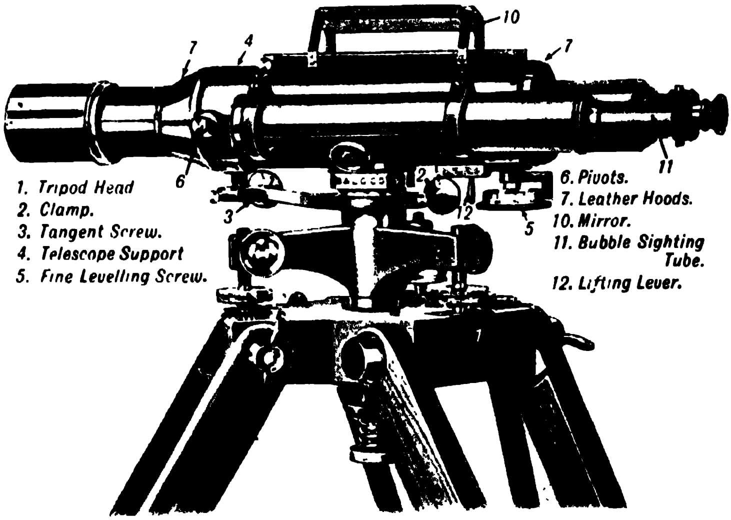


FIG. 138. BINOCULAR PRECISE LEVEL.

The instrument (Figs. 138 and 139) is of the dumpy type with non-reversible bubble tube. As made by various firms in America and England, the telescope objective has a focal length of 16 in. to 18 in., with an aperture of about 1·7 in. Two orthoscopic eyepieces are provided, the powers ranging from 25 to 32 and 40 to 50 respectively. The level tube has a sensitiveness of from 1·2 to 2 sec. per 2-mm. division.

To minimise errors arising from unequal temperature of the instrument, all parts influencing the constancy of the relationship between the line of sight and the level axis are made of nickel iron and nickel

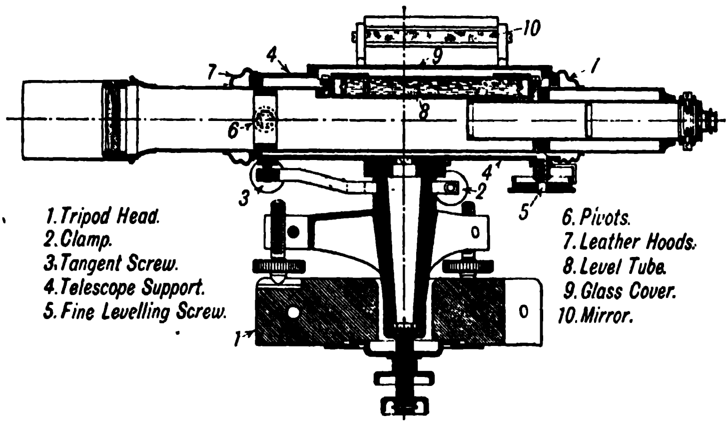


FIG. 139. BINOCULAR PRECISE LEVEL—SECTION.

steel. For the same reason, the bubble tube is placed as near as possible to the line of sight. The vertical axis is long, and carries the cylindrical tube *4*, in which the telescope is supported by the micrometer screw *5* and two nickel steel points *6* diametrically opposite each other. The telescope is pivoted about the latter by rotation of the micrometer screw. The annular spaces between the telescope and the ends of the supporting tube are closed by leather hoods *7*, which shut out air currents without affecting the action of the micrometer. The upper portion of the telescope and encasement is cut away opposite the level, and the outer tube is provided with a glass cover *9* for its protection.

To present to the observer a view of both ends of the bubble with the graduations in their vicinity, the mirror *10* reflects their images into two prisms fitted in a tube *11* alongside the telescope. The distance between these prisms is adjustable to suit the distance between the ends of the bubble. The images are reflected along the tube by the prisms, and the optical arrangement is such that they are presented to the left eye of the observer at the normal distance of distinct vision. A lens suited to the requirements of an observer may be fitted at the eye end of the tube to adapt the normal distance to abnormal vision. The distance between the axes of the telescope and of the bubble-sighting tube is adjustable to suit the pupillary distance between the observer's eyes. A tall tripod is used so that the observer can stand upright, and, while observing, he has merely to examine the staff and bubble alternately without moving any part of the body.

The preliminary levelling is performed with reference to a circular level fitted on the right-hand side of the telescope, and the final levelling by means of the micrometer screw *5*, which has a pitch of 1/100 in. A cam with a lever handle is fitted at *12* for lifting the telescope off the micrometer

in order to avoid possible disturbance of the adjustment during transport of the instrument.

Adjustment of the Binocular Level. The intersection of the vertical hair with the middle horizontal hair having been placed by the maker in the optical axis, the user has to attend to one adjustment only, *viz.* to make the level tube axis parallel to the line of sight. The adjustment is made in the same way as for the engineer's dumpy level (Vol. I, page 124), the level tube having a vertical adjusting screw at the end next the eyepiece.

Exact adjustment is difficult, and, instead of attempting to eliminate very small errors, the error of parallelism may be determined daily and corrections applied to observed differences of elevation. The United States Coast and Geodetic Survey routine in testing the instrument consists in driving two pins about 100 m. apart and setting the instrument about 10 m. beyond one. The average of the readings of the three hairs is taken on a staff held on each point, and the instrument is then moved to about the same distance beyond the second point, and the readings are repeated. A bubble error constant, C, is evaluated from

$$C = \frac{\text{sum of near staff readings} - \text{sum of distant staff readings}}{\text{sum of distant staff intercepts} - \text{sum of near staff intercepts}}.$$

For the best determinations the two distant staff readings are corrected for curvature and refraction. The level is not adjusted if C is less than ·005, and it is not advisable to adjust unless C exceeds ·01, the stadia ratio being 1/333. After an adjustment the value of C is at once re-determined. If C is positive (negative), the line of sight is depressed (elevated) from the horizontal. In deriving one elevation from another, the correction applicable to the observed result is C times the difference between the sum of the backsight staff intercepts and that of the foresights between them. The correction is of the same sign as C if the backsight intercepts are in excess and of opposite sign if the foresights are in excess.

The Zeiss Level. This modern example of a European level (Fig. 140) has been employed by the Ordnance Survey on the Geodetic Levelling of England and Wales. As in the pattern used for ordinary levelling (Vol. I, page 117) the most distinctive features are the methods of mounting the telescope and level tube, the design of the telescope, and the arrangement for showing the observer if the bubble is central while he is sighting.

The level tube *5* is fixed on one side of the telescope. The metal tube carrying it is cut away both on top and bottom, and the whole is protected by being encased in a glass cylinder. The bubble is illuminated by means of the reflector *4*. The telescope with the level tube is capable of rotation through 180° about its own axis, so that the level can lie either on the left- or right-hand side. The level tube is not graduated, but the observer can tell when the bubble is central by means of a combination of prisms contained within the casing *8*, the view presented in the upper prism *11* being as illustrated in Vol. I, Fig. 102.

The instrument is generally used with a parallel plate micrometer of which the operating mechanism is clearly shown in the figure.

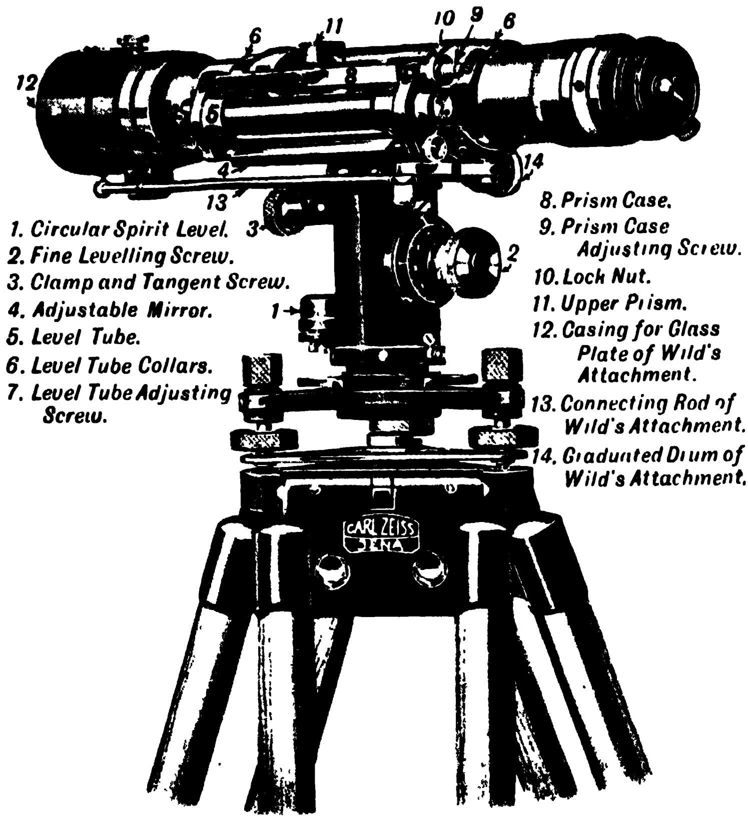


FIG. 140. ZEISS PRECISE LEVEL.

The instrument described above is the one that has been in use for some years in the Ordnance Survey and is still in use in many of the Colonies. A few years ago, however, Messrs. Zeiss brought out an improved model which has an effective objective aperture of $2\frac{11}{16}$ in., with a magnification of 44 diameters. The principal improvement is that the image of the bubble is made to appear in the left-hand side of the field of view of the telescope, in a manner similar to that adopted in the Cooke, Troughton & Simms geodetic level (Fig. 142), and the drum of the parallel plate micrometer can be seen and read through a special fixed magnifier from the observer's position at the eye piece by moving the head a little to one side.

Adjustment of the Zeiss Level. In addition to the means for reversal of the telescope about its own axis, the telescope fittings are such that the ends can be reversed, the eyepiece being inserted in the objective cap. Observations can therefore be taken in four positions, *viz.* :

1, 2. Eyepiece direct, bubble tube on left (right) of telescope ;

3, 4. Eyepiece reversed, bubble tube on left (right) of telescope.

The prism is reversed for positions 3 and 4.

If from one position of the instrument a staff is read in each of these four ways, the mean of the readings is free from instrumental error. The angular error for each position can therefore be determined, as well as the combined error of positions 1 and 2, which may be used in conjunction with each other for precise work. To adjust for any position, it is only necessary to manipulate the fine levelling screw *2* to bring the horizontal hair to the mean of the four readings. The images of the ends of the bubble are then made to coincide by releasing the locknut *10* and displacing the prism case *8* by the screw *9*. To adjust for positions 1 and 2 together, the mean of the readings in those positions must be made to agree with the mean of the four test readings.

The bubble has provision for lateral adjustment to secure parallelism between the vertical planes containing its axis and that of the telescope respectively. The test and adjustment are performed as for the wye level (Vol. I, page 127), and any error should be eliminated before proceeding to the principal adjustment.

Geodetic Level by Cooke, Troughton & Simms. This level, designed for primary work, was adopted before the war for use by the Ordnance Survey for work on the primary levelling of England. Its distinctive features are:—

(1) A large aperture (2 in.) objective which permits of 0·01 ft. being resolved at a distance of 1,350 ft.

(2) Both bubble and staff are seen simultaneously in the eye-piece of the telescope.

(3) The introduction of a scale on the bubble vial which renders precise levelling of the telescope unnecessary, with a consequent saving of time in the field.

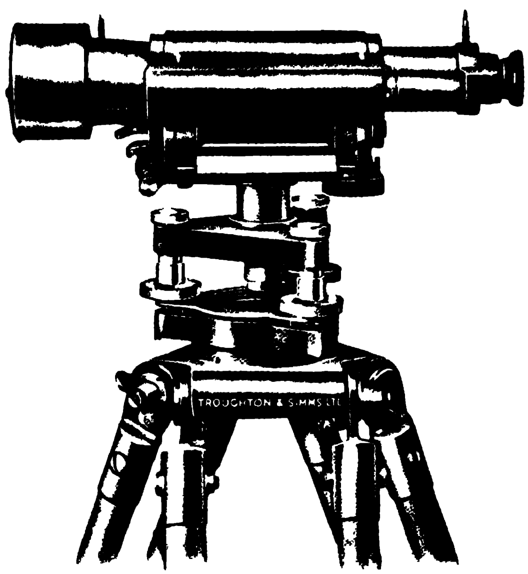


FIG. 141. PRECISE LEVEL BY COOKE, TROUGHTON & SIMMS.

The telescope barrel of this level, which is shown without the parallel plate micrometer in Fig. 141, is made of nickel iron which has approximately the same coefficient of expansion as glass, a point that is of some importance when large lenses are involved, and the whole telescope can be rotated through 180° about the optical axis. The tube of the spirit level attached to the telescope is barrel shaped longitudinally and of circular section at any point throughout its length, so that any longitudinal section is symmetrical about the longitudinal axis. Hence, the instrument can be used first with the spirit level to the left of the observer and then with telescope rotated

through 180° about the optical axis, so that the bubble is on the right of the observer.

As a precaution against temperature changes, the bubble is enclosed in a glass tube, which, in turn, is protected by a metal shield. The vial itself is held in a " geometric mounting " so designed as to prevent the possibility of strain. When the bubble is near the centre of its run images of both ends are brought into the field of view of the telescope as

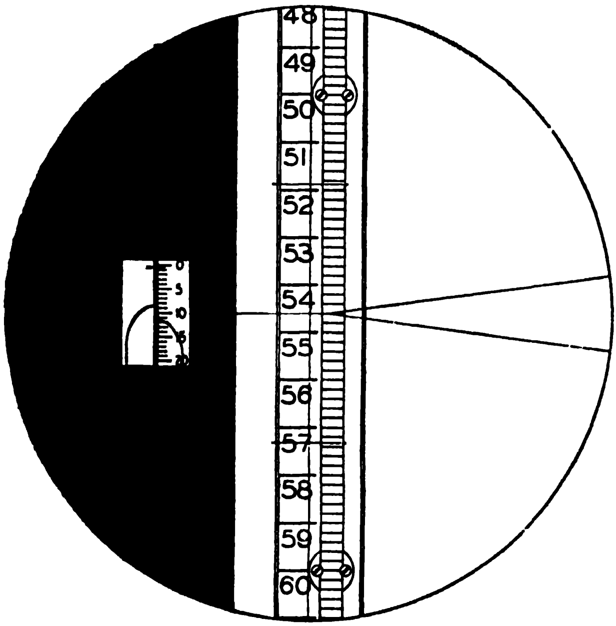


FIG. 142. FIELD OF VIEW OF TELESCOPE EYEPIECE.

The line appearing in both halves of the coincidence system (immediately under the division numbered 0) represents the two index marks on the spirit level.

(*By permission of Messrs. Cooke, Troughton & Simms.*)

shown in Fig. 142. A scale is engraved on the bubble vial, one division of this scale corresponding to approximately 2 seconds of arc, so that accurate measurements of actual dislevelment can be made.

Readings on the staff can be taken in one or other of four ways :—

(1) By levelling the telescope precisely and estimating staff fractions.

(2) As in (1) but using the parallel plate micrometer for determining staff fractions.

(3) By directing the cross hairs on the staff division nearest to true level and recording the position of the bubble, so that the true horizontal reading can be calculated later.

(4) By levelling the instrument approximately and recording the bubble readings and the staff division fraction given by the plate micrometer.

Methods (1) and (2) involve precise levelling of the instrument but give direct results. Methods (3) and (4) avoid exact levelling of the

instrument and estimation of staff division fractions, so saving time in the field, but cause additional computational work. The fourth method is the most accurate and is the one used by the Ordnance Survey. Both it and method (3) also involve taking stadia readings, but in precise levelling it is usual to take these readings in any case in order to check approximate equality of back and fore sights. For the most precise work with this level, readings should in all cases be taken with the bubble in the two positions, once with it on one side of the observer and then with it on the other.

The makers have also placed on the market a cheaper model of the geodetic level in which the bubble is viewed through a separate eyepiece.

Adjustment of C. T. & S. Geodetic Level. The adjustment of this instrument is very simple and consists in placing the longitudinal axis of the bubble parallel to the line of sight. First of all, the instrument is set up and levelled approximately, with the bubble on, say, the right of the observer. The telescope is then pointed at the staff, and, using the fine levelling screw, the images of the bubble are brought into coincidence and a staff reading taken. This procedure is then repeated with the telescope rotated through 180° about its longitudinal axis so that the bubble is on the left of the observer. This will give another staff reading which, in general, will differ from the first. The mean staff reading is then taken and the instrument set so that it reads this amount on the staff. The ends of the bubble will no longer coincide but can be made to do so by turning the milled screw at the rear of the bubble housing. The instrument should then be in adjustment for observation with the bubble on the left-hand side of the observer. This adjustment, however, may cause the calibration marks on the bubble (the line appearing in both halves of the coincidence system immediately under the zero division in Fig. 142) to separate. This does not matter if the instrument is to be levelled exactly for each reading by making both ends of the bubble coincide. If, however the instrument is to be used bubble right and bubble left the calibration marks must coincide and can be made to do so by means of the milled screw at the end of the bubble casing. Any resulting error in collimation will be automatically eliminated by taking the means of the readings with bubble right and bubble left.

Watts Precise Levels. Messrs. E. R. Watts & Sons (now Messrs. Hilger & Watts, Ltd.) make two precise levels that are suitable for geodetic levelling. Each of these instruments is provided with a Watts "constant" bubble (Vol. I, page 38) which is mounted at the side of the telescope in such a way that, as the latter is rotated through 180° about its longitudinal axis, the bubble moves with it and can be read in either position. Hence, as in the Cooke, Troughton & Simms instrument, readings can be taken with bubble left and bubble right, and collimation error can thus be adjusted or its effects eliminated.

The telescope of the smaller of the two levels has an equivalent focus of 15·75 in. and an aperture of 1·6 in., the magnifying power being 35 diameters. The constant bubble, which is accurately reversible to 1·75 secs. and has a sensitivity of 10 to 15 secs. per 2 mm., is read from the eyepiece end of the telescope through a reading lens and a prism.

This instrument is suitable for day-to-day work in precise levelling and 0·01 ft. can be clearly read at 1,200 ft. distance.

The larger level—known as the 21-in. self-adjusting level—is rather heavy for ordinary day-to-day working and is suitable more particularly for levelling across wide gaps, such as wide rivers, or for setting out work where the greatest possible precision is required. One of these instruments, for instance, was used in the laying out and construction of Sydney Harbour Bridge, and the makers claim for this particular model that, to the best of their knowledge and belief, it is the most accurate levelling apparatus of its kind that has yet been made.

The telescope is of 21-in. focal length, with a clear aperture of 2½ in., and is fitted with two eyepieces which give magnifications of 45 and 65 diameters respectively. With this instrument, 0·01 ft. can be read at a distance of 1,600 ft. or more, according to atmospheric conditions.

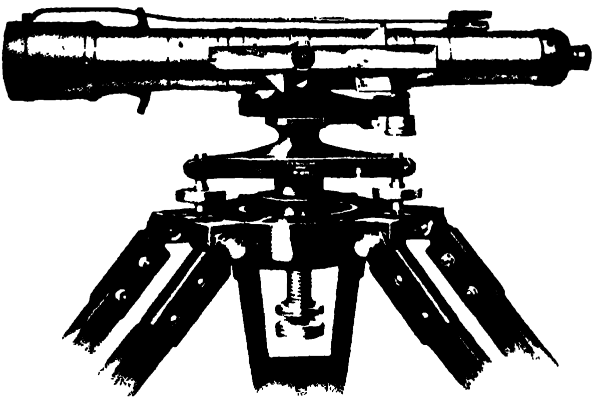


FIG. 143. 21-IN. SELF-ADJUSTING LEVEL BY MESSRS. E. R. WATTS & SON.

The main bubble is accurately reversible to 1·5 secs. of arc, and its sensitiveness is 6 secs. to 7 secs. per 2 mm. run. The final setting of the level before reading is made by means of a patent slow motion differential screw and a special handle is provided for reversing the telescope and bubble about the longitudinal axis of the former.

Both of the Watts precise levels can be supplied with parallel plate micrometers. The readings on these micrometers are not, however, taken on a graduated drum, which is the more usual arrangement, but on a graduated arc fitted at the objective end of the telescope. The ends of this arc can be seen in the illustration (Fig. 143). The graduations are to thousandths of a foot, but the scale is a reasonably open one so that readings can be estimated to ten thousandths of a foot (0·0001 ft.).

Precise Staffs. The various forms of staff employed are of the self-reading type and are usually made in a single length of from 10 to 14 ft. To afford the necessary rigidity the cross section is either rectangular with stiffening side pieces, tee-shaped, cruciform, or a hollow triangle. The staff is usually made of strips of well-seasoned yellow pine, and, before being painted, is subjected to immersion in boiling paraffin or to

other similar treatment in order to minimise the variation in length caused by change of humidity. Small metal plugs are sometimes inserted at intervals for use in determining the actual length. Handles are usually provided, and to ensure steadiness it is desirable to have the staff fitted with two back legs hinged either at its top or at an intermediate point, so that in use the whole forms a rigid tripod. Most types of staff have the base ending in a cylindrical pin of hard metal, the centre of which should lie in the plane of the graduation. Verticality is determined by means of a circular spirit level, or two level tubes, attached at the back. The level adjustment is tested at frequent intervals by a plumb bob, for which a fitting is provided. Temperature is measured on an attached thermometer.

When metric graduation is adopted the division must be sufficiently fine to enable readings to be made to the nearest millimetre, but the pattern must be bold enough to facilitate rapid reading and prevent mistakes. The smallest division ranges from 1 mm. to 1 cm.

When the graduations are in feet and decimals of a foot the smallest graduations are usually in hundredths but sometimes they are in fiftieths (0·02 ft.), the drum of the parallel plate micrometer being graduated accordingly. Readings are taken either to thousandths by estimation or to ten-thousandths by means of the micrometer.

The staff originally adopted by the Ordnance Survey for use on the second geodetic levelling of England

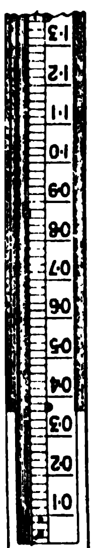


FIG. 144. ORDNANCE SURVEY STAFF.

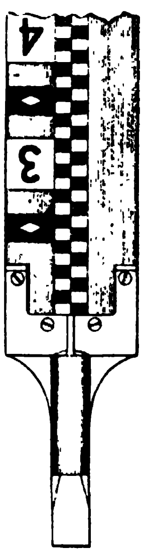


FIG. 145. U.S. COAST AND GEODETIC SURVEY ROD.

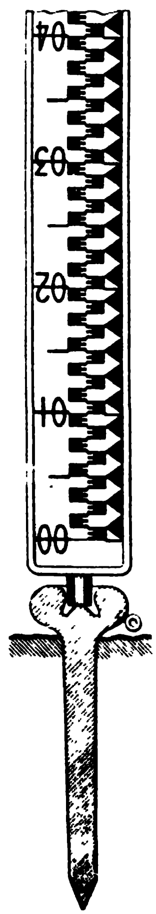


FIG. 146. PROF. MOLITOR'S ROD.

and Wales was made by the Cambridge and Paul Instrument Co. and is illustrated in Fig. 144. The length is 10 ft., and, to eliminate uncertainties arising from temperature and humidity changes, the graduations, to 0·02 ft., are cut on a strip of invar. This is fixed to the foot of the staff, but is otherwise free to slide in a groove in the woodwork without side play.

In more recent years the Ordnance Survey has adopted a folding staff, 10 ft. long, made by Messrs. Cooke, Troughton & Simms. This staff is constructed of two sections of yellow pine into which are let graduated strips of "Nilex" steel, a steel with a low coefficient of thermal expansion. Side pieces of mahogany form a protective edging for the face of the staff. The two sections are hinged so that they can be folded together with the divided strips facing inwards, the hinge being locked in the open position by a clip, which, in the folded position, fits into special recesses in the staff. The steel strips are let into recesses in the face of the staff but are supported clear of the woodwork on brass pads. The lower end of the strip is attached to the staff but otherwise the strip can expand and contract independently of the woodwork. A brass plunger, mounted in a brass box at the back of the staff, causes the strips in the upper and lower sections to maintain contact along a butt joint at the centre of the hinge when the staff is extended. The strips are graduated to 0·02 ft., with black markings on a white background, the wood face being graduated to 0·1 ft.

A plummet, steadying rods, circular spirit levels and holding handles are provided with each staff.

American designs are illustrated in Figs. 145 and 146. The former represents the U.S. Coast and Geodetic Survey rod, which is cruciform in section and carries a strip of nickel steel 10 ft. 2 in. long. The rod is divided to cm., the graduation being marked on the edge of the cross Fig. 146 shows the rod designed by Prof. Molitor. It is tee-shaped in section, has a length of 3·5 m., and is divided to 2 mm.

For the rigid support of a staff at change points either a footplate or pin is used. The former is a cast iron disc of about 6 in. diameter with spikes on the underside. A steel pin driven nearly flush with the ground is found to be more generally satisfactory, and a suitable form is that designed by Molitor and illustrated in Fig. 146.

General Methods of Field Work. For the purpose of providing a check on all bench marks and of ascertaining the quality of every portion of a line of precise levels, the work is invariably duplicated. For the best results, the two lines must be run in opposite directions between the bench marks and preferably at different times of the day or on different days. If the discrepancy between the resulting differences of elevation exceeds a fixed amount, both lines are repeated until the required consistency is obtained. In the best work the allowable discrepancy varies from less than 3 mm.$\sqrt{K}$ to about 4 mm.$\sqrt{K}$, or, say, ·010 ft.$\sqrt{M}$ to ·017 ft.$\sqrt{M}$, where K and M represent the distance between the bench marks in kilometres and miles respectively. (See page 421.)

In place of running two entirely separate lines, simultaneous duplicate lines have been much used. These are less accurate, but can be usefully applied to refined engineering levelling when speed is important. The

usual procedure consists in employing two staffs and having two sets of change points for each position of the instrument. The two lines are affected nearly equally by settlement or heaving of the instrument and by errors due to natural causes, so that the degree of consistency between them is not a good index of their real precision. A simultaneous duplicate line may with advantage be divided into sections run alternately in opposite directions.

Bench Marks. The lines of a large system of precise levelling are of at least two classes—primary and subsidiary—and the bench marks established on them are similarly distinguished. Primary bench marks form the fundamental control points. They are comparatively widely spaced, and are marked in a thoroughly permanent manner. In the revision of the primary levelling of Great Britain, commenced in 1911 by the Ordnance Survey, these fundamental marks are established at intervals of about 25 miles. Their sites are selected with reference to the geological structure so that they may be placed on sound strata clear of mining areas. They are placed at about 3 ft. below the surface, and are embedded in the floor of a covered concrete chamber formed in the solid rock. The new secondary bench marks, about 1 mile apart, are on plates, which are fixed flush on vertical surfaces of walls, etc., with solid foundations. The tertiary marks, averaging about 400 yards apart, consist of copper plugs inserted in horizontal surfaces.

The areas within and between lines of primary levels are dealt with by subsidiary lines closing within shorter distances. The allowable discrepancy between forward and back lines may be greater than in the case of primary lines. The general methods, however, are the same, since the instruments and methods of observation used in primary work are not prejudicial to good progress. The interval between subsidiary bench marks depends upon local circumstances, and should be suited to the needs of the detail surveys. A complete record must be made of the position and description of all bench marks.

Precautions in Running Lines. In running a line of precise levels, the surveyor must consistently adhere to a definite routine of observation. The routine depends upon the instrument used, and should be designed to reduce all possible errors to a minimum.

To limit the effect of instrumental errors, the requirement of equality of backsight and foresight distances is all important. These distances are given by the stadia hair readings, and a limit, of from 1 m. to 10 m., is set to the maximum allowable difference between a backsight and the associated foresight. It is especially necessary that the sum of the backsight distances should balance that of the foresights between benches. The observed staff intercepts should therefore be kept summed by the recorder, and the total difference between the backsight and foresight intercepts should be maintained throughout within narrow limits and be balanced out on reaching a bench mark.

In first-order work the standard graduation error of each staff has to be carefully determined. In the Ordnance Survey levelling this was done in the standard room by means of a standard invar tape for which the errors at every 0·1 ft. were known. Having brought the 0·1-ft graduation on the tape into coincidence with the 0·1-ft. graduation on the invar strip,

the errors e at every 0·1-ft. mark were recorded on a calibration sheet and 100 observation equations of the form

$$x + Ry = e,$$

in which x = index error of the zero mark, y = average graduation error per foot run of staff and the readings R vary from 0·1 ft. to 10·0 ft., were formed and solved by least squares for each staff. In this equation, however, x is the index error of the zero mark on the graduated strip but this does not coincide with the zero error of the base of the staff, which is really the one required. Consequently, the true index error for each *pair* of staffs was found by setting up each staff of the pair alternately on the same mark and taking readings to it from a fixed position of the instrument. After the corrections from the two calibration sheets had been applied, the difference between the results gave the index error for the pair.

For less precise work the graduation error per foot run may be obtained by measuring the distance between the zero and end graduations by means of a standardised invar tape. This should be done at least once a month and also after considerable changes in humidity. The adjustment of the staff level must also be tested regularly.

The staffs should always be used in pairs with the index error of each pair determined as described above. The coefficient of expansion of the invar strip must be known and temperatures should be observed in the field while work is in progress. These temperature observations can be taken at every change point, but, if the coefficient of the invar is small, or if temperature conditions are fairly stable, observations at less frequent intervals will usually be sufficient.

Errors of reading are kept low by limiting the length of sight. The allowable distance depends upon the atmospheric conditions as well as the quality of the telescope and the sensitiveness of the bubble. On the Ordnance Survey the length of sight rarely exceeds 40 yards and never 50. In some surveys it exceeds 100 metres under favourable conditions, but some fairly recent experiments in Ceylon indicate that, for tropical conditions such as are met with there and using Watts and Cooke, Troughton & Simms' precise levels, the ideal length of sight is about 70 ft.* Readings are taken against the three horizontal hairs and averaged. Errors of estimation are thereby reduced, and comparison of the stadia hair readings with that of the centre hair affords a most useful check on the sighting and booking and helps to prevent gross errors.

When a Cooke, Troughton & Simms or other similar type of level is used, one set of readings is taken with bubble left and then a similar set with bubble right.

To minimise errors caused by settlement or heaving of the instrument between observations, as little time as possible should elapse between them, and from every second station the foresight should be read before the backsight. Two staffs should be used to economise time, and therefore the same staff is always observed first after setting up the instrument. The precaution of alternately reading the backsight and foresight first

* See "The Ideal Distance for Precise Levelling" by W. W. Williams in The *Empire Survey Review*, Vol. I, No. 5, 1932.

would eliminate the effect of settlement or heaving if the same change always occurred between observations. Errors caused by change of elevation of a change point, although they are unlikely to be serious when pins are used, are reduced by running the duplicate lines in opposite directions and distributing the discrepancy.

Errors from natural causes may contribute largely to the closing error. The instrument must be shielded from wind, and work suspended when the vibration of the staff interferes with the estimation of readings. Sources of error due to temperature are threefold: (*a*) variable atmospheric refraction; (*b*) irregular refraction near ground level; (*c*) unequal expansion and contraction of the instrument. The effect of change in the coefficient of refraction occurring between backsights and foresights is reduced by taking them in rapid succession. Refraction changes most rapidly in the morning and evening, and is steadiest during the middle of the day (page 426), but, unfortunately, irregular refraction, or shimmer, which causes an apparent trembling of the staff, is then at its worst. This atmospheric disturbance is frequently such as to necessitate a suspension of work during the midday hours, but its effects can be greatly reduced by shortening the length of sight and by avoiding readings near the ground. Unequal heating of the instrument may give rise to serious error, and in sunshine it is essential to shade the instrument during both observation and transport.

The effect of curvature and refraction is generally very small because of the balancing of the backsight and foresight distances. When an unusually long sight is unavoidable, as at a river crossing, reciprocal levelling (Vol. I, page 328) is preferable to equalising the sum of the backsight and foresight distances later, probably under different atmospheric conditions. In the most refined method of reciprocal levelling, possible error due to change of refraction while the instrument is being taken across the river is eliminated by simultaneous observations from both banks. Two instruments are used, and the two staffs are read from each, the long sights being simultaneous. The positions of the instruments are then interchanged, and the observations are repeated. The mean of the four differences of level so obtained is the required difference.

Note-keeping. The columnar arrangement of the field book may take various forms, but at least the following quantities are tabulated in five columns for the backsights and five for the foresights.

1. The readings of the three hairs.
2. Their mean.
3. The two partial staff intercepts and their total.
4. The sum of the whole intercepts up to each observation.
5. The number of the staff observed.

The record also includes, either in tabular form or as remarks, a note of the bench marks between which levelling is being conducted, whether the line is the forward or backward one, the hour of day, the staff temperatures, the state of the weather as regards sunshine, wind, and clearness of atmosphere, and the direction of the line relatively to the sun and wind. The respective sums of the backsight and foresight mean readings between bench marks are brought out, their difference being the observed difference of elevation.

Computation of Levels. Differences of elevation shown in the field book are only approximate since they are subject to various corrections. To facilitate checking, corrections are best set out and applied in tabular form in a book into which are transferred from the field book the quantities required for computation. These include, for both the forward and backward lines. sums of backsights and foresights between bench marks, sums of staff intercepts for backsights and foresights, average staff temperatures, state of weather, and direction of line. The corrections to the approximate difference of elevation are for: (1) instrumental error; (2) staff length and temperature; (3) curvature and refraction; (4) orthometric elevation.

(1) Constant instrumental error, consisting of collimation error and other errors peculiar to particular types of instrument, is corrected by multiplying the difference between the sums of the backsight and foresight staff intercepts, or the corresponding actual distance, by the appropriate constant.

(2) The staff correction to standard and for temperature is proportional to the observed difference of elevation, the temperature correction being derived with sufficient accuracy from the average temperature during the observations between bench marks.

(3) The curvature and refraction correction is applicable only to the difference between the sums of the backsight and foresight distances, and this should be so small that an average value of the refraction coefficient may safely be adopted.

(4) The orthometric correction need not be applied to ordinary levelling but becomes appreciable when precise levelling is involved. It is described in the following paragraphs.

Orthometric and Dynamic Heights and Corrections. The orthometric elevation of a point above mean sea level is the length of the vertical between that surface and the point. Its dynamic elevation is measured by the work required to raise unit mass from mean sea level to the point. A line of constant orthometric elevation is parallel to the mean sea level surface, but, because of the variation of gravity with latitude (page 184), a line of constant dynamic elevation is parallel to mean sea level only when it lies along a parallel of latitude.

The free surface of a still liquid is a dynamically level surface. Elevations obtained by spirit levelling are therefore dependent upon gravity, and observed differences of elevation require corrections to transform them to orthometric differences of elevation. The amount of the correction becomes appreciable for lines at a considerable height above sea level and lying roughly north and south. Let c be the correction to be applied to an observed difference of elevation, as determined from a series of staff readings with a level, to convert it to an orthometric difference of elevation, H the mean elevation of the two points above mean sea level, ϕ their mean latitude and $\delta\phi$ their difference in latitude in seconds of arc. Then, with c and H in the same linear units :—

$$c = \mp 0{\cdot}005302\, H \,.\, \sin 2\phi \,.\, \delta\phi \,.\, \sin 1''.$$

The correction is subtractive (additive) to the observed difference of elevation when the fore-point lies north (south) of the rear-point. As a

general rule it is applied to differences of elevation between successive bench marks, not to differences between individual foresights and back-sights, and in southern latitudes the above rule of signs is reversed. The correction, of course, takes no account of local deviations in the direction of gravity caused by irregularity of the distribution of mass in the earth's crust (page 48).

The difference between orthometric and dynamic heights, and the necessity for the orthometric correction, may, perhaps be better understood from a consideration of Fig. 147. Here A and B are two points at sea level and C and D are two others which are vertically above A and B and lie on a dynamically level or equipotential surface DC. Owing to the variation of gravity with latitude and height of station, the two equipotential surfaces DC and AB tend to converge together towards the pole and to diverge from one another towards the equator. Hence, the height BC is less than the height AD. Dynamically, the height of C is equal to the height of D. so that, if the two points were connected by a pipe line, both would coincide with the water level in the pipe. The orthometric elevations, however, are the heights AD and BC. Hence, although the dynamic heights are equivalent, the orthometric elevations are not equal to one another.

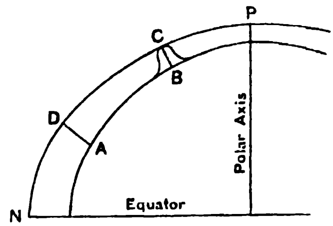


FIG. 147.

In the process of levelling, the axis of the bubble is tangential to the equipotential surface passing through it, so that the actual levelling of the instrument is a "dynamical" operation. On the other hand, the staff readings are the results of a purely "orthometric" operation. Hence ordinary levelling is a mixture of dynamical and orthometric processes in which a series of orthometric measurements are made from lines tangential to different equipotential surfaces. Consequently, in precise work all lines except those running along a parallel of latitude must have a series of small corrections applied at frequent intervals to the observed differences of level—say at the ends of the intervals between successive bench marks—in order to allow for the convergence of the equipotential surfaces at the ends of the lines of sight.

From the point of view of hydraulics, the heights which are of most interest are the dynamic heights. To define the values for these in terms of foot-pounds of work is highly inconvenient in practice. Accordingly, they are often defined as follows: In Fig. 147 let A be any point at sea level nearer than BC to the equator and on some parallel of latitude which is chosen as a standard parallel. Through C draw a level or equipotential surface to intersect the vertical through A at D. Then BC is the orthometric elevation of the hill but AD is taken as its dynamic height. In this way it is possible to compare the dynamic heights of different points since they all become both dynamic and orthometric heights on the standard parallel. This standard parallel is usually chosen so that it is situated about the mean latitude of the area under survey. In England it is the parallel 53° N.

Orthometric heights may easily be converted into dynamic heights by applying the correction

$$h - H = H\left(\frac{1 + 0{\cdot}005302 \sin^2 \phi}{1 + 0{\cdot}005302 \sin^2 \phi_0} - 1\right)$$

where
h = the dynamic height of the point,
H = its orthometric elevation,
ϕ = its approximate latitude,
ϕ_0 = the latitude of the standard parallel.

A table giving this correction for H = 1,000 ft. and for different values of ϕ can easily be compiled and the application of the correction then becomes a very easy matter.

The formulæ given above are easily derived from the expressions for the variation of "g" with latitude and with height above sea level given on page 184.

The orthometric and dynamical corrections are usually very small but still within the limits of observation when precise levelling methods are used. The heights given on the Ordnance Survey maps, when these are based on the new geodetic levelling, are orthometric heights.

For further information on this subject see "The Second Geodetic Levelling of England and Wales 1912–1921" or a paper by Jolly in the *Empire Survey Review*, Vol. II, Nos. 12 and 14, 1934.*

Propagation of Error in Precise Spirit Levelling. (i) *Lallemand's Formulæ.* As a result of a study of the results of the National levelling of France, Mons. C. Lallemand assumed that, apart from gross errors against which every precaution should be taken, precise spirit levelling is subject to errors that may be either accidental or systematic. The accidental errors arise from various causes—errors in estimation of staff reading, errors in sighting, effect of wind on instrument or of settlement of staff, etc.—but the causes which give rise to systematic errors, if these exist, are not so clear. In Great Britain, for instance, the predominant error in the 1912–1921 levelling of England and Wales was such as to make the end point of a single line of levelling too low with respect to the starting point, but this was not the case in Ceylon, where the so-called systematic errors showed considerable variations in sign and direction.

In the computations relating to the second geodetic levelling of England and Wales of 1912–1921, the method adopted for determining systematic error was a graphical one used by M. Lallemand in which differences between the forward and backward levellings, as carried through from the same initial bench mark, are plotted against length of line. If no systematic error existed, these differences would tend to oscillate evenly on either side of the horizontal axis of distances, so that a give-and-take line drawn to even them out would coincide with this axis, and the oscillations on either side of the line would represent errors that are purely accidental. In practice, however, the give-and-take line seldom coincides with the axis of distance but shows a decided tendency to slope in one particular direction, and this slope is taken as an indication, and a measure,

* "Local Attractions and their Influence on the Determination of Height above Mean Sea Level," by H. L. P. Jolly. *Empire Survey Review*, Vol. II, Nos. 12 and 14.

of systematic error. Thus, if S is the intercept in feet on the axis of error for the ends of the give-and-take line, and L is the length of the line of levelling in miles, then σ_K, the probable systematic error of a single line of double levelling in feet per mile, is given by :—

$$\sigma_K = \frac{S}{3L}$$

In many cases a single give-and-take line, which will follow the plotted differences between back and forward levellings reasonably closely, cannot be drawn from one end of the curve to the other, and it may be necessary to divide the curve into several parts and draw separate give-and-take lines for each part. In this case, if S_1, S_2, S_3, . . . are the differences between the end ordinates of the individual give-and-take lines, L_1, L_2, L_3, . . . the corresponding lengths in miles, then S for the whole line may be calculated from :—

$$\frac{S^2}{L} = \frac{S_1^{\,2}}{L_1} + \frac{S_2^{\,2}}{L_2} + \frac{S_3^{\,2}}{L_3} + \ldots$$

where $L = L_1 + L_2 + L_3 + \ldots.$

Fig. 148 shows a plot of forward minus backward levellings in which

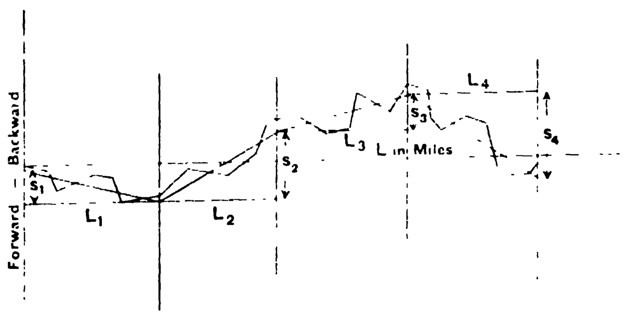


FIG. 148.

four give-and-take lines have been drawn at points where the general direction of the curve changes more or less abruptly.

For a single line of levelling, the probable accidental error derived from the discrepancies between the forward and backward levellings is given by the formula :—

$$\eta_K^{\,2} = \frac{1}{9L}\left\{[\Delta^2] - [r^2]\frac{S^2}{L^2}\right\}$$

where L is the length of the line, $[\Delta^2]$ is the sum of the squares of the differences between forward and backward levellings over the spaces between neighbouring bench marks, and $[r^2]$ is the sum of the squares of the distances between bench marks. This formula, together with the one already given for probable systematic error, is known as Lallamand's formula.

When lines are combined to give general values for η_K and σ_K we get the first two of the formulæ adopted by the International Geodetic Association in 1912, which are as follows :—

$$\eta_r^2 = \frac{1}{9}\left\{\frac{[\Delta^2]}{[L]} - \frac{[r^2]}{[L]^2}\left[\frac{S^2}{L}\right]\right\},$$

$$\sigma_r^2 = \frac{1}{9[L]}\left[\frac{S^2}{L}\right], \text{ for lines not forming a net.}$$

$$\sigma_R^2 = \frac{1}{[L^2]}\left\{\frac{2}{9}[f^2] - \eta_r^2[L]\right\} \text{ for a network containing at least ten polygons.}$$

In these formulæ, [ ] denotes summation over all the lines or network forming the whole survey. In the last formula, f denotes the closing error of a polygon and L refers to the sides of the different polygons. In British practice, S and Δ are usually expressed as decimals of a foot and L in miles ; if metric units are used, S and Δ are usually expressed in millimetres and L in kilometres.

The International Geodetic Association, when adopting these as standard formulæ, also laid it down that "in future there be classified as 'Levelling of High Precision' the levelling on a line, group of lines, or network which has been run in both the forward and backward directions, on different dates if possible, and for which the probable errors, uniformly computed by the given formulæ, do not exceed $\pm$ 1·0 mm. per km.$^{\frac{1}{2}}$ and $\pm$ 0·2 mm. per km." * If probable errors are expressed in decimals of a foot and the lengths in miles these expressions become :—

Probable Accidental Error not to exceed $\pm$ 0·00416 ft. per mile$^{\frac{1}{2}}$.

Probable Systematic Error not to exceed $\pm$ 0·00106 ft. per mile.

On the English geodetic levelling the values actually obtained were :—

$$\eta_r = \pm\ 0{\cdot}00182 \text{ ft. per mile}^{\frac{1}{2}}.$$
$$\sigma_r = \pm\ 0{\cdot}00119 \text{ ft. per mile.}$$
$$\sigma_R = \pm\ 0{\cdot}00063 \text{ ft. per mile.}$$

(ii) *Vignal and Rune's Formulæ.* On very long lines of levelling, or in an extensive network of lines, it has long been noticed that closing errors tend, on the whole, to vary in such a manner that they have most of the characteristics of accidental errors, so that they are proportional to the square root of the length of the line, but that this is not altogether true when lines are short. This fact and other considerations have led M. Jean Vignal, Directeur du Nivellement Général de la France, and other geodesists, to examine and criticise Lallemand's hypothesis of systematic error and to propose other formulæ for calculating the probable errors of precise levelling. These revised formulæ were adopted officially in replacement of those of Lallemand by the International Association of Geodesy in 1936. By that time, however, a good deal of important results had been published in which the precision of the results had been calculated in terms of Lallemand's formulæ.

In the rules proposed by Vignal and adopted by the International Association of Geodesy, two classes of errors are recognised. The first class corresponds to the accidental type in Lallemand's formulæ and consists of "η-errors" which are due to causes that act independently on each successive portion of levelling and which obey the ordinary Gaussian laws of error. "They are characterised by a probable (acci-

* *Comptes Rendus de la Conference Géodésique Internationale,* 1912, II, p. 252.

dental) 'η-error' per kilometre, η, such that the probable error, due to errors of this class, in the difference in elevation between two bench marks D kilometres apart, is equal to $\eta\sqrt{D}$, whatever the distance D may be." The second class of error consists of "ζ-errors" which "are due to causes that act in a similar way on successive or near-by pieces of levelling. They do not obey the Gaussian laws. They act like accidental errors on two bench marks D kilometres apart only if the distance D exceeds a certain limit Z, of the order of magnitude of some tens of kilometres, which is subject to variation with the method of levelling. They are characterised by a probable (accidental) 'ζ-error' per kilometre, ζ, such that if the distance D between two bench marks is greater than the limit Z, the probable error of their difference of elevation due to these sources of error is $\zeta\sqrt{D}$. If the distance is less than the limit Z, the coefficient of proportionality of this probable error affecting the square root of D decreases from the value ζ to zero as D decreases from the limit Z to zero." * These two errors can then be combined to give the "total probable (accidental) error" per kilometre, ϵ, such that, provided D exceeds Z, the total probable error of the difference in elevation of two bench marks D kilometres apart is equal to $\epsilon\sqrt{D}$ and we have

$$\epsilon^2 = \eta^2 + \zeta^2.$$

On the French levelling, it was found that the limiting distance Z was generally of the order of somewhere about 50 to 60 kilometres.

Now, using Vignal's notation in which Σ replaces [ ], let

- r interval between two consecutive bench marks.
- r_m mean value of this interval.
- n_r number of intervals between consecutive bench marks.
- Σr the total length of these intervals, equal to the total length of the net ($\Sigma r = \Sigma L = n_r \times r_m$).
- ρ the discrepancy between the results of the two levelling operations considered (if the levelling considered is the mean of two such operations), for two consecutive bench marks.
- L length of one of the "portions" into which the net may be divided arbitrarily.
- L_m mean length of these portions.
- n_l the number of these lengths.
- ΣL their total length ($\Sigma L = n_l \times L_m$).
- Δ discrepancy between the results of two levellings between the ends of a portion.
- Z limiting distance beyond which the probable error of the difference of elevation between two points is proportional to the square root of the distance between the points.
- P length of the perimeter of a closed surround.
- P_m the mean value of the P's of the different surrounds forming the network,
- n_p the number of the surrounds,
- ΣP the sum of the lengths of their perimeters ($\Sigma P = n_p \times P_m$).

* "International Resolutions Relative to Levelling of High Precision." *Bulletin Géodésique*, No. 61, Janvier-Fevrier-Mars, 1939, page 177.

f = the closing error of a surround after the application of all instrumental and theoretical corrections have been applied, including the orthomorphic and dynamic corrections.

Two sets of formulæ are given, one in which the observations are of equal weight and the other in which they are given weights proportional to r, L and P respectively. The first method gives a better approximation, especially when the number of elements is small; the other involves slightly simpler calculations.

Let

Equal Weights	Weights proportional to r, L, P
$e_r'^2 = \dfrac{1}{9n_r}\Sigma\dfrac{\rho^2}{r}$;	$e_r''^2 = \dfrac{1}{9\Sigma r}\Sigma\rho^2$
$e_L'^2 = \dfrac{1}{9n_l}\Sigma\dfrac{A^2}{L}$;	$e_L''^2 = \dfrac{1}{9\Sigma L}\Sigma A^2$
$e_p'^2 = \dfrac{4}{9n_p}\Sigma\dfrac{f^2}{P}$;	$e_p''^2 = \dfrac{4}{9\Sigma P}\Sigma f^2$

If possible, the value of n in each of these expressions should not be less than 10. Calculate the values of e_L or e_p for different values of L_m or P_m and note when they tend to have a constant value independent of L_m or P_m. The distance where this occurs gives the required approximate value of Z. Let E_l and E_p denote the limits towards which e_l and e_p tend. The probable η- and ζ-errors per kilometre½ and the probable total error per kilometre½, ϵ, are then found from :—

$\eta^2 = e_r'^2 - \zeta^2\dfrac{r_m}{Z}$;	$\eta^2 = e_r''^2 - \zeta^2\dfrac{1}{Z}\dfrac{\Sigma r^2}{\Sigma r}$
$\zeta^2 = E_L^2 - \eta^2$;	$\zeta^2 = E_p^2 - \eta^2$
$\epsilon = E_L$;	$\epsilon = E_p$.

In all cases in these formulæ ρ, A, f, η, ϵ, and ϵ are expressed in millimetres and r, L, P and Z in kilometres.

In the above, when the limit E_p is reached, it is best to take into account the closing error and perimeter of the outer surround in the expressions for $e_p'^2$ and $e_p''^2$. We then write

$$e_p'^2 = \frac{4}{9(n_p+1)}\Sigma\frac{f^2}{P}; \qquad e_p''^2 = \frac{4}{9(n_p+1)}\left[n_p\frac{\Sigma f^2}{\Sigma P} + \frac{f^2}{P}\right],$$

in which the term $\Sigma\dfrac{f^2}{P}$ in the first expression includes the figures for the outer surround, and the first term in the square bracket on the right of the second expression does not include the figures for the outer surround, as the second term, $\dfrac{f^2}{P}$, is intended to embody these figures.

If Lallemand's formulæ have already been used to calculate the "probable systematic error," σ, the results can be used to calculate ζ and ϵ. In this case, for L_m greater than Z,

$$\zeta^2 = E_{LS}^2 - \tfrac{6}{5}\eta^2,$$
$$\epsilon^2 = E_{LS}^2 - \tfrac{1}{5}\eta^2,$$

where E_{LS} is the limiting value of e_{LS} for $L_m > Z$, e_{LS} being calculated from

$$e'^2_{LS} = \frac{1}{9n_L}\Sigma\frac{S^2}{L} \qquad \text{or} \qquad e''^2_{LS} = \frac{1}{9\Sigma L}\Sigma S^2.$$

For any length of L_m the formula for σ_t in Lallemand's formula may be replaced by

$$\sigma^2 = \sigma_S^2 - \tfrac{6}{5}\eta^2 \cdot \frac{1}{L_m} = \frac{e'^2_{LS}}{L_m} - \tfrac{6}{5}\eta^2 \cdot \frac{1}{L_m},$$

where σ is the revised value and σ_S is the value computed by Lallemand's formula.*

Consequent on the adoption of these new formulæ in 1936, the International Association of Geodesy also laid down two new definitions and classifications of precise levelling.

(1) "Levelling of High Precision" in which the total probable error ϵ does not exceed 2 millimetres per kilometre$^{\frac{1}{2}}$.

(2) "Precise Levelling" in which the total probable error ϵ does not exceed 6 millimetres per kilometre$^{\frac{1}{2}}$.

When probable errors are expressed as fractions of a foot and distances in terms of miles, this means that levelling is "levelling of high precision" if ϵ does not exceed 0·0083 ft. per mile$^{\frac{1}{2}}$, and it is "precise levelling" if ϵ does not exceed 0·0250 ft. per mile$^{\frac{1}{2}}$.

Allowable Discrepancies between Forward and Backward Levelling. While the rules just given afford a useful measure of the quality of the completed work as a whole, they do not give the surveyor in the field a convenient working rule by means of which he can know whether his work is acceptable or not. For this purpose, other working rules have been devised which are based on the assumption that the errors of levelling are accidental only, and thus obey the usual square root law. According to the rules adopted by the International Geodetic Association in 1912, the probable accidental error of a single line of levelling should not exceed, in feet, $\pm 0{\cdot}004\sqrt{M}$, where M is the length of the line in miles. This limit should not normally be exceeded if the maximum allowable discrepancy of the forward and backward levellings between a single pair of neighbouring bench marks is taken to be, say, $\pm 0{\cdot}010\sqrt{M}$ where M is the distance between the two bench marks in miles. Usually, the discrepancy actually measured should be less than this, but, if it is more, the line should be relevelled.

In the Ceylon levelling, the rule adopted was that the section was to be remeasured if the difference between forward and backward levellings was more than $\pm 0{\cdot}005\sqrt{F}$, where F is the length of a section between adjacent bench marks in thousands of feet.† This is equivalent to $\pm 0{\cdot}0115\sqrt{M}$, where M is the length of the section in miles. In the

* "Rapport sur L'Évaluation des Erreurs Probable 'ζ' Systématique et Totale par Kilomètre d'une Méthode de Nivellement par le Procédé des Droites Moyennes," by Prof. G. R. Rune and Jean Vignal. *Bulletin Géodésique*, No. 62, Avril-Mai-Juin, 1939, pages 442–453.

† See "Geodetic Levelling of Ceylon," by T. P. Price, *Empire Survey Review*, Vol. I, No. 5, 1932.

Ordnance Survey levelling the rule is that the difference should not exceed $\pm 0{\cdot}0117\sqrt{M}$.

At its 1936 meeting, the International Association of Geodesy gave no easy rules for the allowable tolerances which are so convenient as the earlier rules, but recommended that those parts of a line of levelling which is to be regarded as levelling of high precision should be re-levelled if the discrepancy between forward and backward levellings exceeds the following values in millimetres :—

(1) For the η-discrepancy, $\rho = \pm 6$ to $8\,\eta\sqrt{r}$.

(2) For the ζ-discrepancy. For a length L less than several tens of kilometres, $A = \pm 2$ to $2{\cdot}5\,\epsilon L^{\frac{3}{4}}$ and for a length L greater than several tens of kliometres $A = \pm 6$ to $8\,\epsilon\sqrt{L}$.

In addition, the total probable error ϵ, computed for the whole work, should not exceed the limit of 2 millimetres per kilometre[1] laid down for levelling of high precision.

In all cases, the values of η and ϵ to be used in these rules are those found for "levelling operations performed by the same method as the levelling operations in question."

Adjustment of a Network of Precise Spirit Levels. In theory, the difficulty in the adjustment of a network of precise spirit levels is the assignment of proper weights to the observed differences in elevation between the ends of the different lines. It would, of course, be possible to calculate the probable errors by the rules given above, taking into account the systematic errors by the methods already described, and then to weight the observations inversely as the squares of the probable errors. In any long line, however, the systematic errors vary in such a way over the different parts of the line that it is legitimate to consider them, in the aggregate, as being of the same nature, or rather having much the same characteristics and obeying the same laws, as accidental errors, and this assumption becomes even more justified when a network of different lines is considered as a whole. Thus, in the English levelling, it was found that, for lines exceeding 35 miles in length, the errors were of an accidental nature as regards their effect on the line as a whole. This being so, the probable error tends to be proportional to the square root of the length and hence *the weight to be assigned to the observed difference of elevation of the end points of any particular line should be the reciprocal of the length of the line.* When lines are very short, and polygons are consequently small, this rule may not be entirely valid and it may be advisable to take into account the effects of systematic error. Usually, however, the simplest procedure is to adopt the ordinary rule for a preliminary adjustment, and then, if the corrections seem to be out of proportion, to adopt new weights and make a new adjustment. This should seldom be necessary, and it may be noted that, after the ordinary adjustment with weights inversely proportional to the lengths of the lines, no further adjustment was considered necessary on the English, Indian and American networks, even although the two latter showed much larger systematic errors on long lines than the English levelling did.

After the weights to be used have been decided, the solution is best carried out by least squares, though, for work that is not of the first order, a system of weighted means may be used if necessary.

The whole question of the propagation of error in precise levelling, and of the adjustment of networks of lines of levels, is discussed in considerable detail by H. L. P. Jolly in "The Second Geodetic Levelling of England and Wales, 1912–21," to which the reader is referred for further information.

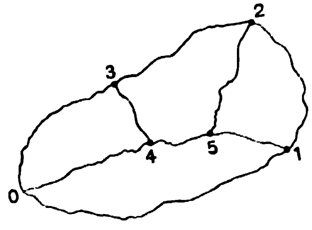


FIG. 149.

As an example of a solution by least squares take the small network shown in Fig. 149 for which the data are set out below.

Elevation of Point 0 = 1,434·9253 ft. above M.S.L.

From	To	Difference of Elevation	Length	Correction.
		Feet	Miles	
0	1	+ 130·0409	151·6	x_1
1	2	311·9118	95·4	x_2
2	3	236·9448	87·2	x_3
3	0	− 205·0201	74·5	x_4
0	4	+ 52·1962	81·4	x_5
4	5	+ 12·9760	32·1	x_6
5	1	+ 64·8295	46·2	x_7
3	4	− 152·7774	36·0	x_8
2	5	− 376·7344	59·1	x_9

All the above differences of elevation, which are purely imaginary, are supposed to have had the orthometric correction already applied to them.

There are 9 observations, so that if observation equations, with assumed approximate heights for the points 1, 2, 3, 4 and 5, were used, there would be 9 observation equations and 5 normal equations. If condition equations are used there are 9 lines, each of which will fix a point, and there are 5 new points to be fixed. Consequently, there will be 9 − 5 = 4 conditions, giving 4 normal equations only. Hence, it will be quicker to use condition equations.*

The first condition equation is for the circuit 01540, and hence the equation is

$$130{\cdot}0409 + x_1 - 64{\cdot}8295 - x_7 - 12{\cdot}9760 - x_6 - 52{\cdot}1962 - x_5 = 0$$

or
$$x_1 - x_5 - x_6 - x_7 + 0{\cdot}0392 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Similarly, for the three other circuits, 1251, 23452 and 3043 :—

$$x_2 + x_7 + x_9 \qquad + 0{\cdot}0069 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

$$x_3 + x_6 + x_8 - x_9 - 0{\cdot}0118 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

$$x_4 + x_5 - x_8 \qquad - 0{\cdot}0465 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

The weights are inversely as the lengths of the lines, so that the reciprocals of the weights will be the lengths of the lines themselves, and, in order to avoid large numbers, it will be convenient to express the lengths in terms of a hundred miles. Hence, we have the following table for forming the correlative equations :—

* The condition that the number of conditional normal equations should be less than the number of observation normal equations is that $l - n$ should be less than n ; that is, $l < 2n$, where l is the number of lines and n the number of *new* points to be fixed.

x	$\frac{1}{w}$	A	B	C	D	Sum.
x_1	1·52	+ 1				+ 1
x_2	0·95		+ 1			+ 1
x_3	0·87			+ 1		+ 1
x_4	0·74				+ 1	+ 1
x_5	0·81	− 1			+ 1	0
x_6	0·32	− 1		+ 1		0
x_7	0·46	− 1	+ 1			0
x_8	0·36			+ 1	− 1	0
x_9	0·59		+ 1	− 1		0

In this table the length of each line has been divided by 100, so that the unit of length is 100 miles.

The correlative normal equations are :—

A	B	C	D	Abs.	Sum.
+ 3·11	− 0·46	− 0·32	− 0·81	+ 0·0392	+ 1·5592
	+ 2·00	− 0·59	0	+ 0·0069	+ 0·9569
		+ 2·14	− 0·36	− 0·0118	+ 0·8582
			+ 1·91	− 0·0465	+ 0·6933

which give the correlatives :—

$A = -0{\cdot}00613$; $C = +0{\cdot}00780$;
$B = -0{\cdot}00256$; $D = +0{\cdot}02321$;

and the corrections :—

$x_1 = -0{\cdot}0093$; $x_4 = +0{\cdot}0172$; $x_7 = +0{\cdot}0016$;
$x_2 = -0{\cdot}0024$; $x_5 = +0{\cdot}0238$; $x_8 = -0{\cdot}0055$;
$x_3 = +0{\cdot}0068$; $x_6 = +0{\cdot}0045$; $x_9 = -0{\cdot}0061$.

Hence, the corrected differences of elevation are :—

0 − 1 = + 130·0316 ; 0 − 4 = + 52·2200 ;
1 − 2 = + 311·9094 ; 4 − 5 = + 12·9805 ;
2 − 3 = − 236·9380 ; 5 − 1 = + 64·8311 ;
3 − 0 = − 205·0029 ; 3 − 4 = − 152·7829 ;
2 − 5 = − 376·7405,

and the corrected elevations are :—

0 = 1434·9253 ; 3 = 1639·9283 ;
1 = 1564·9569 ; 4 = 1847·1453 ;
2 = 1876·8663 ; 5 = 1500·1258.

Since the differences between forward and backward levellings are not given it is impossible to use the International formulæ for the calculation of probable errors, but we can compute a probable error from the formulæ giving the probable errors of conditioned observations. Taking 100 miles as the unit of length so as to keep the weights somewhere in the neighbourhood of unity, those weights are obtained by dividing unity by the quantities tabulated in the second column of the above table. Hence, the weights become 0·66, 1·05, 1·15, etc., and their sum $[w]$ is 15·17. Multiplying the square of each correction by its appropriate weight and taking the sum, we get $[w.v^2] = 0{\cdot}00142434$. Then, using the formula given on page 280, we have :—p.e. of an unadjusted observation of unit weight

$$= \pm\, 0{\cdot}6745 \sqrt{\frac{0{\cdot}00142434}{4}} = \pm\, 0{\cdot}0127 \text{ ft.}$$

and p.e. of an adjusted observation of unit weight

$$\pm\, 0{\cdot}0127 \sqrt{\frac{5}{15{\cdot}17}} = \pm\, 0{\cdot}0073 \text{ ft.}$$

These are the probable errors for a line 100 miles long, and, as probable errors are proportional to the square roots of the lengths, the p.e. for a line M miles long will be given by :—

$$\text{p.e. of an unadjusted observation} = \pm\ 0{\cdot}00127\sqrt{M}$$

$$\text{p.e. of an adjusted observation} = \pm\ 0{\cdot}00073\sqrt{M}.$$

TRIGONOMETRICAL LEVELLING

Although even the best trigonometrical levelling will not give results that are any more accurate than those obtained from ordinary spirit levelling, and, from the point of accuracy, cannot thus be put in the same category as precise spirit levelling, yet trigonometrical levelling forms an important part of geodetic surveying and serves to fix elevations of high points that could not easily be reached by lines of precise spirit levels. Hence, for the sake of convenience it is treated in the present chapter as part of the general subject of precise levelling.

The postulate that precise spirit levelling is not suitable for determining the elevations of high points holds mainly for the common case where these points are on high hills that rise abruptly above the surrounding plain, so that the approach to them is very steep. In such circumstances, the one essential condition for obtaining accurate results by precise levelling methods—the necessity for maintaining equality of foresights and backsights—cannot conveniently be maintained. In any event, extremely accurate determinations of the elevations of high isolated points are seldom, or never, required in practical engineering work, but, with care, trigonometrical levelling will give results which are of sufficient accuracy for all ordinary mapping purposes. Moreover, when extreme accuracy is not essential, elevations determined by trigonometrical means serve to cover quickly a definite belt of country with control heights, whereas a line of levels is confined to a single line only and several lines are required to cover an area of any width. In addition, precise spirit levelling is usually commenced long after the main horizontal framework is complete, or well on the way to completion. Consequently, in nearly all cases, it is economically sound to take the opportunity of observing vertical angles at the same time as the horizontal angles are measured.

Angle Measurement. Refinement in angular observation is necessary in order to obtain satisfactory results with long sights. For the best work the theodolite should have the vertical circle fitted with micrometers reading to single seconds and by estimation to tenths, and the angular value of a division of the bubble tube attached to the micrometer arm should not exceed 2 sec. The instrument known as the vertical circle has been extensively used in trigonometrical levelling. It possesses all the features of the vertical angle measuring part of the theodolite without those for horizontal measurement, and may have the circle of repeating or non-repeating pattern.

The record of an observation must include the height of the instrument axis above the ground point and the height of the observed point above the distant station, in order that, by application of the eye and object correction, the difference of elevation between the station marks may be determined. Each observation is made face right and face left, and is

accompanied by readings of the position of the bubble for application of the level correction. At least two such observations constitute a measurement. The pointings may be made by means of the eyepiece micrometer. If the several angles to be measured at a station are so nearly equal that their differences are within the range of the eyepiece micrometer, these differences may be observed micrometrically with reference to one or more points, the elevation of which is either known or is measured in the usual way. In the case of important determinations, one or more measurements are made each day during the tenancy of the instrument station for triangulation. For points of minor importance, a single observation, face right and face left, is all that is required. Each elevation should be determined by measurements from at least two points, and if observations have to be made from a satellite station, the results must be corrected for eccentricity.*

Large vertical angles can never be measured as accurately as horizontal ones because they cannot be observed on different zeroes, as horizontal ones can. Consequently, it is impossible to eliminate the effects of small errors of graduation of the circle. Small angles, however, may be observed with great accuracy by means of the eyepiece micrometer, as explained above.

Refraction. Owing to the curvature of the line of sight caused by atmospheric refraction, an observed vertical angle differs from the true angle which the straight line joining the instrument to the signal makes with the horizontal through the instrument. The chief source of error in trigonometrical levelling is that arising from uncertainty regarding the amount of refraction. The uncertainty is greater in the case of terrestrial refraction, with which we are here concerned, than for celestial or astronomical refraction, since the sights of trigonometrical levelling are never greatly inclined to the horizontal

The usual way of dealing with the question is by use of the coefficient of refraction. This is defined as the ratio between the refraction angle, or angle at the instrument between the ideal straight line of sight and the actual curved line of sight, and the angle which the two stations subtend at the centre of the earth. The line of sight is assumed to be a circular arc, and the coefficient of refraction to be independent of its elevation or inclination, but these assumptions are not strictly correct. Dr. Hunter, of the Survey of India,† has shown that refraction depends very largely on the rate at which the temperature changes with height and with the change of this rate, and that it also depends on the height above the horizontal plane through the observing station to which the ray extends. Thus, the refraction on a ray of given length differs according as the ray is ascending or descending.

It is well known that refraction varies in amount throughout the day. It is greatest in the morning and evening, and least and steadiest during the middle of the day. Its variation from day to day is found to be less during the period of minimum refraction than at other hours. Vertical

* See Gillespie, *Surveying*, Part II.

† See "Formulæ for Atmospheric Refraction and their Application to Terrestrial Refraction and Geodesy," by J. de Graaf Hunter. Survey of India, Professional Paper, No. 14, 1913.

angle observations should be confined to this period, the duration of which, however, varies in different parts of the earth, and should be investigated in important surveys by preliminary reciprocal observations. Generally speaking, the period from noon to 3 p.m. is the best, but in some cases good observations are possible from about 10 a.m. to 4 p.m. Refraction may thus be least when "shimmer" is at its worst.

The value of the coefficient of refraction to be used in reducing observations is its average minimum value for the district, and should, if possible, be determined by simultaneous observation of reciprocal angles. Otherwise its value must be estimated from the results obtained under similar conditions. For the British Isles, Col. Clarke obtained mean values of 0·0750 and 0·0809 for sights over land and sea respectively.* The average values obtained by the United States Coast and Geodetic Survey range from 0·065 for the interior to 0·078 for lines crossing parts of the sea.† A mean value of 0·065 was found in the measurement of the Uganda arc, and of 0·060 for observations to the highest peaks.‡

Methods of Trigonometrical Levelling. Two general methods are employed to obtain the difference of elevation of two points of known distance apart. In the first, it is determined from vertical angles observed from each station to the other. The object of such reciprocal observations is to remove the effect of uncertainty regarding the value of the coefficient of refraction. The angle of refraction is taken as being the same for both stations, and is eliminated in the reduction. For the best results the reciprocal observations should be taken simultaneously. Owing to the difficulty of arranging for simultaneous observations, the measurements are sometimes made at the time of minimum refraction on different days but with less accurate results.

In the second method, the difference of elevation is computed from the vertical angle measured at one of the stations only, and a knowledge of the value of the refraction coefficient is required. This method is available for determining the elevation of the station occupied by observation to a point of known elevation and it must be used for determining the elevations of inaccessible and intersected points.

Notation. In developing reduction formulæ, it will be supposed that it is required to determine the elevation of a station B from that of a station A, assumed known. The notation adopted is as follows:

h_1 = elevation of A above M.S.L.;
h_2 = ,, B ,, ,, ;
D = geodetic or M.S.L. distance between A and B;
R = radius of the earth at the mid-latitude of A and B;
θ = angle subtended at the centre of the earth by D;
a = observed vertical angle from A to B, corrected in reciprocal observations for the difference in height of the instrument and the signal above the ground;
b = ,, from B to A;

* Ordnance Survey, "Account of the Principal Triangulation," 1858.
† United States Coast and Geodetic Survey Report, 1882, Appendix No. 9.
‡ "Report of the Measurement of an Arc of Meridian in Uganda." Colonial Survey Committee, 1913.

k = coefficient of refraction, $k\theta$ being the angle of refraction;
c = angular eye and object correction for reciprocal observations;
i = height of the instrument above the ground;
s = height of the point sighted above the ground.

Reciprocal Observations. Eye and Object Correction. In trigonometrical levelling generally, it usually happens that the height above the ground, or above the station mark, at which the angle is observed is different from that of the signal. The difference of elevation obtained from an observation at one end of the line is that between the instrument and the point sighted, and is finally corrected for their difference of height above the ground. In the case of reciprocal observations, however, when the point observed to at each station is not that from which the observations are made, the observed angles must be reduced to true reciprocity before they are used in computing the difference of elevation. The amount of this angular correction to the observed angles may be deduced as follows.

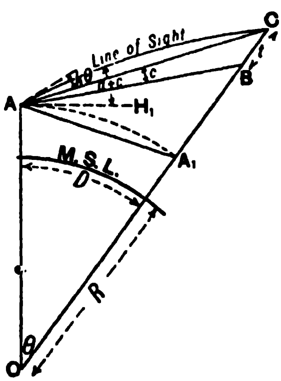


FIG. 150.

In Fig. 150, let A represent the centre of the instrument at one station. Let B at the other station represent a point at the same height above the ground point as A, and let C, at a distance $t = (s - i)$, from B, be the point sighted. It is required to obtain the value of the angle c subtended by t at A. Mark A_1 on OC at the same elevation above mean sea level as A, and join AA_1. AH_1 is the horizontal at A, and $(a + c)$ is the angle of elevation actually observed at A.

In triangle ABC, $\dfrac{\sin c}{\sin ABC} = \dfrac{t}{AC}$,

but $ABC = AOB + OAB = 90° + a + \theta(1 - k)$,

$$\text{and } AC = AA_1 \frac{\sin AA_1C}{\sin ACA_1} = \frac{AA_1 \sin (90° + \frac{1}{2}\theta)}{\sin (90° - a - c - \theta(1 - k))},$$

$$\therefore \sin c = \frac{t \sin (90° + a + \theta(1 - k)) \cdot \sin (90° - a - c - \theta(1 - k))}{AA_1 \sin (90° + \frac{1}{2}\theta)}.$$

$$= \frac{t \cos^2 (a + c + \theta(1 - k))}{AA_1 \cos \frac{1}{2}\theta},$$

where $AA_1 = 2(R + h_1) \sin \frac{1}{2}\theta$.

But, with sufficient accuracy for the purpose, D may be substituted for AA_1, and since θ is small, we have, with a smaller error than that of refraction,

$$\sin c = \frac{t \cos^2 (a + c)}{D},$$

or, since c is a very small angle,

$$c'' = \frac{t \cos^2 (a + c)}{D \sin 1''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the observed angle $(a + c)$ is very small, it is sufficient to take

$$c'' = \frac{t}{D \sin 1''}, \quad \ldots \ldots \ldots \ldots \quad (2)$$

which is the more commonly used formula.

The angle observed at B is similarly corrected for the corresponding value of t, attention being paid to the signs of the corrections.

Difference of Elevation from Reciprocal Observations. Two cases may occur : (1) the difference in elevation of the two points may be sufficiently great that one of the observed angles is an elevation and the other a depression ; (2) the difference in elevation may be small enough in relation to the distance between the points that both angles are depressions.

Case (1). Fig. 151 illustrates the case in which B, the point of unknown elevation, is higher than A. The angles a and b are the observed angles corrected for difference in height of eye and object, and may be treated as the angles which would be observed at and to A and B, the respective ground points.

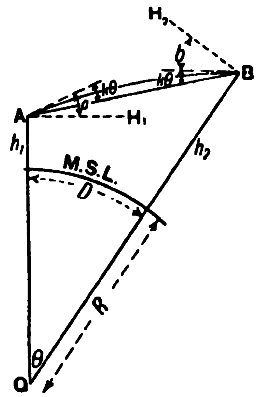


FIG. 151.

The required difference of elevation $(h_2 - h_1)$ is obtained by solving triangle AOB, of which the known quantities are $\mathrm{OA} = (R + h_1)$, $\mathrm{AOB} = \theta = D/R$ in circular measure, and $(\mathrm{OAB} - \mathrm{OBA}) = (90° + a - k\theta) - (90° - b - k\theta) = (a + b)$. Solving, we have

$$\tan\frac{a+b}{2} = \frac{(R+h_2) - (R+h_1)}{(R+h_2) + (R+h_1)} \cot\frac{\theta}{2},$$

$$\text{or } (h_2 - h_1) = \tan\frac{a+b}{2} \cdot \tan\frac{\theta}{2} \cdot (2R + h_1 + h_2).$$

Substituting for $\tan\frac{\theta}{2}$ its expansion, $\frac{D}{2R} + \frac{D^3}{24R^3} + \ldots$,

$$(h_2 - h_1) = D \tan\frac{a+b}{2}\left(1 + \frac{(h_1 + h_2)}{2R} + \frac{D^2}{12R^2}\right) \quad . \quad . \quad (3)$$

In applying this formula, a first approximation is derived from $(h_2 - h_1) = D \tan\frac{a+b}{2}$, and the value of h_2 so obtained is used for a second approximation by the formula.

An alternative method of solution consists in first obtaining the true vertical angles $\mathrm{H_1AB}$ and $\mathrm{H_2BA}$ by eliminating the refraction angles.

$\mathrm{H_1AB} = (a - k\theta)$, and $\mathrm{H_2BA} = (b + k\theta)$, but $\mathrm{H_2BA} - \mathrm{H_1AB} = \theta$,

$$\therefore\ k\theta = \frac{\theta - b + a}{2} \quad \ldots \ldots \ldots \quad (4)$$

Let the values of H_1AB and H_2BA so obtained be denoted by a' and b'. Then, solving triangle AOB, we have

$$\frac{R + h_2}{R + h_1} = \frac{\sin (90° + a')}{\sin (90° - b')},$$

$$\text{whence } (h_2 - h_1) = (R + h_1) \frac{\cos a' - \cos b'}{\cos b'},$$

$$= (R + h_1)\left(\frac{\cos a'}{\cos b'} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

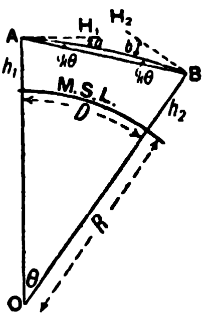


FIG. 152.

Case (2). (Fig. 152). For the first method of solution,

(OAB — OBA) is now $= (b - a)$, and

$$(h_2 - h_1) = D \tan \frac{b - a}{2}\left(1 + \frac{(h_1 + h_2)}{2R} + \frac{D^2}{12R^2}\right) \quad (6)$$

In the second method, $(H_1AB + H_2BA) = \theta$, so that

$$k\theta = \frac{\theta - a - b}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the expression for $(h_2 - h_1)$ is the same as before.

Difference of Elevation from Observations at One Station. In the case where observations are taken at one end of the line only, a value must be assigned to k. A likely value may be assumed, but it is preferable to determine it by means of equation (4) or (7) from simultaneous reciprocal observations taken in the same locality. Otherwise the coefficient may be deduced from an observation to a point of known elevation.

Let A (Fig. 151) be the station occupied, and let a be the uncorrected angle of elevation observed to B. The previous formulæ are applicable if the corresponding angle b at B is deduced. From (4) or (7),

$$b = a + \theta - 2k\theta,$$

a being positive for an elevation and negative for a depression: a negative value of b indicates an angle of elevation at B. The value of b so deduced includes all errors of observation and in the assumed value for k. On substituting this value of b, we have, corresponding to (3) and (6),

$$(h_2 - h_1) = D \tan (a + (\tfrac{1}{2} - k)\theta)\left(1 + \frac{(h_1 + h_2)}{2R} + \frac{D^2}{12R^2}\right), \quad . \quad . \quad . \quad (8)$$

in which a is given a negative sign if a depression. The $\frac{D^2}{12R^2}$ term may safely be omitted, and $\frac{(h_1 + h_2)}{2R}$ may be treated as negligible, in rough determinations.

For the second method of solution we have $b' = (a' + \theta)$, a' being given the negative sign for a depression, and

$$\therefore \ (h_2 - h_1) = (R + h_1)\left(\frac{\cos a'}{\cos (a' + \theta)} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eye and Object Correction. When reciprocal observations are not taken, $(h_2 - h_1)$ represents the difference of elevation between the instrument axis and the point sighted, and must be corrected to give the difference of elevation between the ground points. If i and s denote the respective heights of instrument and signal (if any) above the ground, the required difference of elevation

$$= (h_2 - s) - (h_1 - i) = (h_2 - h_1) + (i - s).$$

Rough Determinations. When, as in plane tabling, the angular observations are not made with a high degree of precision, the reductions are commonly made with the aid of a table of curvature and refraction (page 150). The value of the correction is

$\frac{D^2}{2R}(1 - 2k) = \cdot 574D^2$ ft. for $k = 0{\cdot}07$ and D in miles. A rough value for $(h_2 - h_1)$ in ft. is therefore given by

$$(h_2 - h_1) = 5280\, D \tan a + \cdot 574D^2, \text{ where } D \text{ is in miles ;}$$

or, when a is small, $\tan a = a'' \tan 1'' = \cdot 000004848\, a''$, and

$$(h_2 - h_1) = \cdot 0256\, Da'' + \cdot 574D^2.$$

Errors of Trigonometrical Levelling. The probable error computed from the differences of elevation given by repeated vertical angle observations is not likely to be a trustworthy index of the precision of a result. Even in simultaneous reciprocal observations, constant errors are introduced in the assumptions made regarding the effect of refraction. The quality of a system of trigonometrical levelling is best ascertained by connecting it at intervals to lines of spirit levelling or by reference to its own errors of closure.

If expression (3) on page 429 is differentiated with respect to a and the last two terms in the bracket are neglected, we have :—

$$\delta h = \tfrac{1}{2}D \sec^2 \tfrac{1}{2}(a + b) \,.\, \delta a,$$

in which δh may be considered to be the error in $(h_2 - h_1)$ resulting from an error δa in the measurement of the angle a. Hence, if the angular errors are assumed to be independent of the length of the line, the error in the computed difference of elevation will be directly proportional to the length of the line.

In actual practice, the angle $\frac{1}{2}(a + b)$ is generally small so that $\sec^2 \frac{1}{2}(a + b)$ in the above equation may be put equal to unity. Accordingly, if we assume that all errors are of the accidental type, and if r_h is the probable error of $(h_2 - h_1)$, r_a is the probable error of the angle a and r_b the probable error of the angle b, we can, by the rules for the combination of probable errors given on page 278, write :—

$$r_h = \pm \tfrac{1}{2}D(r_a^2 + r_b^2)^{\frac{1}{2}},$$

and, if r_a and r_b are in seconds of arc, r_h is in feet and D in miles, this becomes :—

$$r_h = \pm 2640\, D \,.\, \sin 1''\, (r_a^2 + r_b^2)^{\frac{1}{2}} = \pm 0{\cdot}0128 \,.\, D(r_a^2 + r_b^2)^{\frac{1}{2}}.$$

This expression gives the probable error of a single line and it varies directly as the length of the line. Now take a long stretch of trigonometrical levelling of length L and made up of a series of N equal lines, each of length D. Further, assume that the probable error of each angle is the same and equal to r_a. Then, combining the different probable errors,

$$\begin{aligned} r_L &= \pm\ 0{\cdot}0128\ r_a[2D^2 + 2D^2 + \ldots \text{ to } N \text{ terms}]^{\frac{1}{2}} \\ &= \pm\ 0{\cdot}0128\ r_a\sqrt{2ND^2} \\ &= \pm\ 0{\cdot}0128\ r_a\sqrt{2LD}. \end{aligned}$$

Hence, in this case r_L is proportional to the square root of the total distance L as well as to the square root of a single line.

In practice, it is usual to express the errors of trigonometrical levelling as being proportional to the square root of the length of the line, although it will be seen from the above that this involves the rather unjustifiable assumption that r_a is proportional to $D^{-\frac{1}{2}}$. Thus, one rule of this kind lays it down that the probable error of reciprocal, but not simultaneous, observations, as computed from closing errors on fixed elevations, should not exceed 0·1 to $0{\cdot}3\sqrt{D}$ ft. for sights of from 5 to 20 miles and 0·2 to $0{\cdot}6\sqrt{D}$ ft. for sights over 20 miles, where D is in miles. Another rule, given in the official publication *Survey Computations*, is that the probable error of a difference in elevation, as carried along a chain of triangulation, which can be expected at the end of the chain is about $0{\cdot}35\sqrt{s}$ ft. for reciprocal (simultaneous) observations and $0{\cdot}5\sqrt{s}$ ft. for single rays with rough signals, s being the total length of the chain in miles reckoned along the shortest route between the terminal points.

As regards errors in refraction, it is obvious from equation (8), page 430, that, provided they obey the ordinary laws of accidental error, these will behave exactly similar to accidental angular errors when the line is observed in one direction only. If reciprocal observations are taken, and the coefficients of refraction at the two stations are not equal, it is easy to show that equation (3), page 429, will contain the extra term

$$\frac{D^2}{2R}(k_2 - k_1)\left[1 + \frac{(h_2 + h_1)}{2R} + \frac{D^2}{12R^2}\right]$$

in which the first term will be much larger than either of the other two, and k_2 and k_1 represent the coefficients of refraction at B and A respectively. Hence, if $(k_2 - k_1) = 2\delta k$ is treated as an error in the refraction, we have

$$\delta h = \frac{D^2}{R}\delta k \text{ approximately,}$$

and, in this case, the error varies as the square of D. On replacing $2\delta k$ by r_k, the probable variation of the coefficient of refraction from its mean value, and combining, as before, the probable errors of N equal lines of length D, we get

$$r_L' = \pm\frac{r_k \,.\, D^2}{R}\sqrt{N} = \pm\frac{r_k \,.\, D}{R} \,.\, \sqrt{LD}$$

where $L = N.D.$ and r_L' is the probable error of the computed difference

in elevation of two points at the ends of the N lines. As r_k will very seldom be known, it is usually impossible to compute this probable error. From this argument, however, we see that r_L' and δh can be proportional to $\sqrt{L}$ only when r_k is proportional to $D^{-\frac{3}{4}}$.

Adjustment of a Network of Trigonometrical Heights. In many cases it is sufficient to adjust a network of trigonometrical heights by weighted means, but, if the quality of the observations is known to be good and it is desired to make the most of them, it will be worth while making a least square adjustment. Either observation or condition equations may be used for the least square adjustment, but the labour will be least by condition equations if the number of lines is less than twice the number of new points to be fixed. Whatever the method used, it is necessary to decide on the weights to be allotted to the different lines. A common rule is to weight each computed difference of level inversely as the length of the line, giving half the normal weight or less to a line observed in one direction only. Another rule is based on the expression $r_h = \pm 0{\cdot}0128 \,.\, D \,.\, (r_{\prime\prime}{}^2 + r_{\prime}{}^2)^{\frac{1}{2}}$ already found. If r_α is the probable error of a single observation of an angle, and there are $n_{\prime\prime}$ observations at A and $n_{\prime}$ at B, then $r_{\prime\prime} = r_\alpha/\sqrt{n_{\prime\prime}}$; $r_{\prime} = r_\alpha/\sqrt{n_{\prime}}$ and $r_h = \pm\, 0{\cdot}0128 D r_\alpha \left(\frac{1}{n_{\prime\prime}} + \frac{1}{n_{\prime}}\right)^{\frac{1}{2}}$. Weights are inversely proportional to the squares of the probable errors. Hence, the weight to be assigned to an observed difference of level will be :—

$$w_h = \frac{1}{r_h{}^2} = \frac{1}{(0{\cdot}0128)^2 D^2 r_\alpha{}^2}\left(\frac{n_{\prime\prime} n_{\prime}}{n_{\prime\prime} + n_{\prime}}\right)$$

$$= \frac{6103}{D^2 r_\alpha{}^2}\left(\frac{n_{\prime\prime} n_{\prime}}{n_{\prime\prime} + n_{\prime}}\right).$$

On the trigonometrical survey of Fiji McCaw used this expression in the form :—

$$w_h = \frac{600}{D^2}\left(\frac{n_{\prime\prime} n_{\prime}}{n_{\prime\prime} + n_{\prime}}\right)$$

In this, however, he took $n_{\prime\prime}$ and $n_{\prime}$ as the number of days on which observations were made at the two stations, and, if more than one observation was taken on one day, two were counted as $1\frac{1}{2}$ and three as two. For a line observed in only one direction he used the formula :—

$$w_h = 90 n_{\prime\prime}/D^2,$$

in which the much lower value of the numerical coefficient makes allowance for the unknown error in the assumed value of the coefficient of refraction.

The above results involve the assumption that the coefficient of refraction is the same for all angles of elevation. It is known that this is not strictly true, and, from a consideration of the likely variation of the coefficient of refraction with angle of elevation caused by variations of the atmospheric pressure and temperature with height above sea level or height of line above ground level, Morley has suggested that lines should be weighted (1) inversely as D, (2) inversely as $(h_2 - h_1)$, (3) directly as $\frac{1}{2}(h_2 + h_1)$, where h_1 and h_2 are station heights above ordinary ground level. (See *Empire Survey Review*, Vol. IV, No. 23.)

REFERENCES ON GEODETIC LEVELLING

AVERS, H. G., and COWIE, G. D. "Precise Levelling from Reno to Las Vegas, Nev., and from Tonopah Junction, Nev., to Laws, Cal." U.S. Coast and Geodetic Survey, Special Publication No. 39, 1916.

BAGNALL, G. "Long Distance Levelling." *Min. Proc. Inst. C.E.*, Vol. CXXI.

BOOCK, C. F. V. "The Main Levelling of Copenhagen, 1907-8." *The Surveyor and Municipal and County Engineer*, Vol. XLI, 1912.

CHURCHILL, J. H. "The Application of an Infra-Red Filter to Surveying Instruments." *Empire Survey Review*, Vol. II, No. 7, 1933.

COLE, J. H. "Systematic Error in Spirit Levelling." Survey of Egypt, Departmental Paper No. 35, 1919.

CROSTHWAIT, H. L. "Levelling Across the Ganges at Damukdia." Survey of India, Prof. Paper No. 7, 1903.

ECCLES, J. "Experiment to test the Changes in the Length of a Levelling Staff due to Moisture and Temperature." Survey of India, Prof. Paper No. 7, 1903.

HARDESTY, W. P. "Precise-level Survey of the City of Portland, Ore." *Eng. News*, Vol. 76, No. 2, 1916.

HENRICI, E. O. "Precise Levelling." *Min. Proc. Inst. C.E.*, Vol. CCIX.

HUNTER, J. DE G. "Formulæ for Atmospheric Refraction and their Application to Terrestrial Refraction and Geodesy." Survey of India, Prof. Paper No. 14, 1913.

— "Trigonometrical Heights and Terrestrial Atmospheric Refraction." *Dictionary of Applied Physics*, Vol. III.

INTERNATIONAL UNION OF GEODESY AND GEOPHYSICS. "International Resolutions Relative to Levelling of High Precision." *Bulletin Géodésique*, No. 61, 1939.

JOLLY, H. L. P. "Local Attractions and their Influence on the Determination of Heights above Mean Sea Level. *Empire Survey Review*, Vol. II, Nos. 12 and 14, 1934.

KOOP, F. W. "Precise Levelling in New York City." *Eng. News*, Vol. 70, No. 10, 1913.

LALLEMAND, CH. and PRÉVOT, E. "La Nivellement Général de la France de 1878 á 1926. Exposé Preliminaire." Paris, 1927.

McCAW, G. T. "Distances and Refraction on or near the Apparent Horizon." *Empire Survey Review*, Vol. IV, No. 24, 1937.

— "The Trigonometrical Survey of Fiji. Report on." Vol. I (Viti Levu). London, 1916. Vol. II (Vanua Levu and Taveuni). London, 1917.

MOLITOR, D. A. "The Theory and Practice of Precise Spirit Levelling." *Trans. Am. Soc. C.E.*, Vol. XLV.

MORLEY, A. J. "Evaluation of the Coefficient of Terrestrial Refraction." *Empire Survey Review*, Vol. IV, No. 25, 1937; and Vol. VI, No. 40, 1941.

— "The Adjustment of Trigonometrical Levels and Evaluation of Coefficient of Refraction." *Empire Survey Review*, Vol. IV, No. 23, 1937.

Ordnance Survey, Reports of the Progress of (Annual), from 1912.

— "The Second Geodetic Levelling of England and Wales, 1912-21." London, 1922.

PRICE, T. Y. "The Geodetic Levelling of Ceylon (1926-1929)." *Empire Survey Review*, Vol. I, No. 5, 1932.

"Report of the Measurement of an Arc of Meridian in Uganda." Vol. I. Colonial Office, 1913.

RAPPLEYE, H. S. "Manual of Geodetic Leveling." U.S. Coast and Geodetic Survey, Special Publication No. 239. Washington, 1948.

— "Manual of Leveling Computation and Adjustment." U.S. Coast and Geodetic Survey, Special Publication No. 240. Washington, 1948.

RUNE, PROF. G. A., and VIGNAL, JEAN. "Rapport sur l'Evaluation des Erreurs Probables 'ξ' Systematique et Totale par Kilomètre d'une Methode de Nivellement par le Procédé des Droits Moyennes." *Bulletin Géodésique*, No. 62, 1939.

SHAW, H. G. "Observation of Atmospheric Refraction, 1905-09." Survey of India, Prof. Paper No. 11, 1911.

VIGNAL, JEAN. "Évaluation de la Précision d'une Méthode de Nivellement," *Bulletin Géodésique*, No. 49, 1936.

WARWICK, H. C. "Use of Color Filters on Geodetic Instruments." *Field Engineer's Bulletin, U.S. Coast and Geodetic Survey*, No. 10, December, 1936.

WILLIAMS, W. W. "The Ideal Distance for Precise Levelling." *Empire Survey Review*, Vol. I, No. 5, 1932.

CHAPTER VII

TOPOGRAPHICAL AND RECONNAISSANCE SURVEYING

Topographical Surveys. Topographical surveys have for their object the preparation of maps as complete in detail as the scale will allow. They are distinct from cadastral surveys, the primary object of which is the location and representation of boundaries and the measurement of areas of land. National topographical surveys are undertaken for the production of maps which, while on smaller scales than those of cadastral surveys, will fulfil the various purposes, civil and military, for which reference to general maps is required.

The representation of relief is an essential feature of a topographical map, and to the engineer such maps are of the greatest utility as an aid to the location of railways, roads, canals, reservoirs, pipe lines, etc. In the absence of existing maps he must employ the methods of topographical surveying in location surveys. Methods applicable to the production of large scale contoured plans of small areas have been dealt with in Vol. I, Chapter VIII. In the following pages are considered the field operations required for the preparation of maps of large areas on smaller scales.

Topographical surveying is a subject in itself and it is only possible in these pages to give a general sketch of the methods used. Specimens of topographical maps of various countries will be found in Close and Winterbotham's *Text Book of Topographical Surveying*, a book which should be consulted by anyone who is called upon to undertake topographical work in the field.

General Features of Topographical Surveys. To prevent excessive accumulation of error, a topographical survey must be based upon a system of horizontal control points located with such precision that their errors of position cannot be detected on the scale of the map. Between these points detail surveys are conducted by methods suited to the character of the country and with a standard of accuracy so related to the distance between their control points that the errors in location of detail may also be negligible. The most rigid framework is available if the country has been covered by triangulation of geodetic standard. Triangulation affords much the best control, but the principal system need not necessarily attain first-order precision, and may be of a much lower standard when the mapping is to be executed only on a small scale. In flat and densely wooded country, for which triangulation is unsuitable, a framework of primary traverse must take its place.

In a similar manner, the survey of relief should be based upon a framework of accurate, though not necessarily precise, spirit levelling. The distances between the bench marks established must be such that the less refined dependent work may not develop errors sufficient to affect appreciably the location of the contours. The methods to be adopted both in the spirit and the subsidiary levelling depend not only upon the scale but also upon the character of the country. In regions characterised by flat slopes a given error of elevation will produce a more serious displace-

ment of the contours than in rugged country, and a higher grade of work is required.

The scale adopted for a general topographical map depends not only upon the physical character of the country, but in greater degree upon its state of development and wealth. The purposes for which the map will be used are, however, so varied that it is impossible to select a scale to suit them all, and a wide range—between 1/25,000 and about 1/125,000—has been adopted for what may properly be classed as topographical maps. The more important nations possess topographical maps on more than one scale; those of the Ordnance Survey are 1/63,360 and 1/126,720, or one inch and half-inch to a mile. For topographical surveys in connection with engineering schemes the requirements as to scale depend upon the nature of the work. The scale adopted is commonly larger than the above owing to the necessity for the inclusion of what would in general mapping be regarded as minor topographic features.

Geographical Surveys. Where the expense of a deliberate topographical survey is not warranted, the survey may be executed with a view to reproduction on a smaller scale. Surveys on scales of less than about half-inch to one mile are sometimes distinguished as geographical, but no hard and fast distinction can be drawn between topographical and geographical surveys, since both aim at the production of maps as accurate as the scale will allow.

Exploratory and Reconnaissance Surveys. As the name implies, an exploratory survey is one made to record the geography of the country passed through by an exploring party. The same methods are applicable to rapid reconnaissance executed as an aid to the planning of deliberate surveys, as well as to military reconnaissance conducted under active service conditions. They are employed by the civil engineer for preliminary location surveys and in connection with reports on proposed schemes. A high grade of work is not expected, since only the most rapid methods are available and the most portable instruments are used. The scale is usually small, and may be as low as 1/1,000,000.

For locating a system of points to control the sketching, triangulation is to be preferred, even although it is of low grade, and time-saving methods are freely used. If the survey can proceed from or terminate on stations of an existing survey, difficulties of base measurement are removed, but, even so, triangulation may be impracticable on account of the limited time available or the nature of the ground. In this case, a system of points may be located by astronomical observations, or a rough traverse is conducted along the route followed. The disadvantages and inaccuracies of a primary control consisting solely of astronomical observations have been discussed on page 31, but the inaccuracies to be expected are of no great importance if the survey is to be drawn on a very small scale or is of an exploratory nature only. The method of route traversing has been greatly favoured by explorers because of its rapidity. This method, however, is of low precision, and errors must be kept in check and adjusted at intervals by means of observations for latitude and azimuth, and, if the route lies east and west, for longitude difference. A sufficient number of elevations to control the sketching of relief are obtained by trigonometrical levelling if a theodolite is carried.

Otherwise, the barometer or the boiling-point thermometer, or both, are employed.

Methods. In the absence of an existing framework for the ultimate control of a topographical or geographical survey, the principal factors which influence the planning of the work are the nature of the country and the accuracy required. Reconnaissance will reveal whether triangulation can be economically conducted over the whole area or whether primary traverse may have to be substituted in parts. The substitution of traversing for triangulation as primary control is justifiable only in densely wooded or flat country, where the latter would involve unreasonable delay and expense. Unless great care is taken with it, particularly with regard to the question of the elimination of gross error in the field work or of serious mistakes in the computations, traversing is inferior to good triangulation. The question of accuracy is, however, largely dependent upon the scale of the mapping. No greater refinement is warranted than is necessitated by the requirement that the errors will not show in scaling.

There is a considerable choice of methods for the survey of detail, and experience is required to enable the topographer to select that best suited to a particular case. The plane table is by far the most useful instrument when the country is fairly open and a close network of control points has been established, as the advantage of being able to sketch the topography with the ground in view is universally recognised. Whatever system is adopted for the survey of relief, the sketching of topographic form is a highly important feature of the work, and the topographer must be qualified to produce a good representation of the characteristic physical features of the country and to eliminate the unessential. To this end some slight geological knowledge, particularly that relating to the causes and origin of common land forms, is a great asset to the surveyor. In exploratory surveys only the most prominent features can be recorded, and the sketching is controlled by fewer points.

Like almost every other kind of survey, topographical work must proceed from the whole to the part, which means a reasonably accurate horizontal and vertical framework followed by less accurate methods of detail survey. The methods in use for the survey of the framework and detail may be summarised as follows :—

Class of Survey	Topographical and Geographical	Exploratory and Reconnaissance
Horizontal Control	Triangulation Primary, Secondary, Tertiary, or Topographical Triangulation of Different Grades Primary Theodolite Traverse	Rapid Theodolite Triangulation Plane Table Triangulation Latitude and Azimuth Traverse Astronomical Positions
Horizontal Detail	Plane Tabling Resection, Intersection, Traversing	Plane Tabling Resection Route Traverse Sketching

Class of Survey	Topographical and Geographical	Exploratory and Reconnaissance
Horizontal Detail	Secondary Theodolite Traverse Tacheometry Compass Traverse Photographic Surveying Sketching	
Vertical Control	Spirit Levelling Trigonometrical Levelling	Trigonometrical Levelling Barometric Levelling
Vertical Detail	Spirit Levelling Trigonometrical Levelling by Theodolite, Tacheometer, or Plane Table Clinometer Barometric Levelling Photographic Surveying Sketching	Barometric Levelling Sketching

HORIZONTAL CONTROL

Topographical and Geographical Triangulation. Although the refinements of first-order triangulation are not required when the results are to be applied only to mapping, the methods employed for the principal control of deliberate topographical surveys of extended areas are very similar, particularly if it is necessary to produce maps of certain districts on larger scales than the general scale adopted. On the other hand, considerable deviation from the rigorous methods of primary triangulation is allowable in work to be mapped on a geographical scale only. For any survey, standards of accuracy must be set for the various operations, and rules are formulated to ensure consistency in the quality of the work of different field parties.

In the highest class of topographical survey, the conduct of the preliminary operations of reconnaissance, signal building, and station marking is the same as that described under geodetic surveying. The erection of high towers is, however, avoided, even at the expense of introducing figures not as well-conditioned as would be required in first-order triangulation. If the work is not connected to an existing triangulation system of suitable precision, a base, at least a mile long, must be measured with an accuracy represented by a probable error of 1/100,000 to 1/250,000, the latter for the most extended work. This degree of precision may be attained with less labour by the use of invar tapes or wires than by steel tapes, but either are used. If the catenary apparatus (page 173) is not employed, the tape is suspended in catenary between mark posts with copper or zinc strips to receive the scratches, a constant tension being applied by a tape stretcher through a spring balance. The base must be measured at least once in each direction. The methods of horizontal angle measurement must be such as will yield a triangular error not exceeding 3″ to 5″ in the principal system, 10″ to 15″ in the second, and 20″ to 30″ in the third.

In the majority of topographical surveys and in geographical surveys,

the standard of accuracy may be considerably lower, and progress is more rapid. Although the time available for the field work may be limited, a preliminary reconnaissance by plane table or compass can never be dispensed with, unless existing maps sufficient for the purpose are available. During reconnaissance as many stations as possible are beaconed by cairns of stones or other simple means, but artificial signals are frequently dispensed with in the triangulation proper if well-defined hilltops, isolated trees, rocks, etc. can be used as station points. Bases necessary should be at least half a mile long, special attention being paid to securing a well-conditioned expansion. The precision of the base measurement is of the order 1/10,000 to 1/50,000. Invar or steel tapes are used suspended between mark posts, or, alternatively, the tape may be laid on the ground, which is first graded between the posts. The allowable triangular errors of the angular measurements may be increased to twice those cited above.

Computation of Topographical and Geographical Triangulation. In the highest class of extended topographical triangulation the methods of adjusting the angles and calculating the sides and station positions follow those given in Chaps. IV, V, the approximate methods and formulæ being sufficient. In topographical surveying generally and in geographical surveying no attempt at elaborate adjustment is required. Spherical excess can generally be neglected, and the angles of each triangle are adjusted for its total error without reference to side conditions but the check provided by computing each side in two ways should not be omitted.

Theodolite Traverse. The methods of first-order traversing described in Chap. III are more particularly applicable to geodetic work, to the control of a survey which is to be plotted on a large scale, or to other work in which great precision is required. For purely topographical or geographical work, a very high degree of accuracy is not necessary, and the methods already described may therefore be modified accordingly. If the area to be covered is very large, an accuracy of about 1/30,000 should provide a good primary control for topographical work and the methods used can be based on those described on pages 246 to 248 for the survey of second-order traverses. For medium or smaller areas—say less than about 2,500 square miles—ordinary good theodolite traversing, carried out by the methods described in Vol. I, can safely be used as a primary topographical control, provided angles are measured with a theodolite reading direct by micrometer to 10″ of arc and by estimation to 1″ or 2″ and distances are measured with carefully standardised steel tapes, used in catenary as described on pages 155 to 158 of Vol. I. For this class of work, angles may be observed on two zeros, with change of face between each zero. This last class of traverse may also be used to break down large blocks formed by a network of traverses surveyed by second-order methods. If the traverse is a long one, azimuth observations should be taken at intervals not exceeding twenty miles, or fifty instrument stations, whichever is the shorter.

Minor Theodolite Traverse Work. The main control traverses for a topographical survey should divide the country into blocks in which these main traverses are not more than thirty or forty miles apart, and

those blocks should, where possible, be sub-divided into several smaller blocks by means of minor theodolite traverses. Short minor traverses can be surveyed, if necessary, by vernier instruments, with angles observed on two zeros and change of face between each zero, distances being measured with a steel tape used either in catenary or stretched along the ground. Slopes on these minor traverses, unless needed for working out heights, need not be observed in flat country, but should be read, or distances very carefully stepped (Vol. I, page 142), in broken or hilly country, and, unless they are likely to be exceedingly variable during working hours, temperatures need not be observed.

Computation of Theodolite Traverses. Topographical surveys may either be plotted in terms of some system of rectangular co-ordinates or by means of geographical co-ordinates, but, even if geographicals are used for plotting, it is generally convenient to compute a theodolite traverse in terms of some system of rectangular co-ordinates in which the rectangular values of some, or all, of the fixed permanent points can easily be converted into geographicals later. If a special system of co-ordinates, such as spherical rectangular or Transverse Mercator, is already in use in the area under survey, the primary traverses should be computed in terms of it, using, if necessary, any approximations, such as those described in pages 380 to 386, which are justified when the work is not of first-order accuracy. Indeed, if such a system of co-ordinates is not already in use, it is advisable to select the one most suitable for the area under survey, and to work in terms of it, because triangulation and traverses executed as a primary control for purely topographical work may, in a later state of development, often be incorporated as second- or third-order work in a framework intended for more general purposes. Both the rectangular spherical and Transverse Mercator systems are more suitable for traverse work than the conical orthomorphic one, and should be used in preference where the choice exists.

Approximate methods of computing traverses in terms of the Transverse Mercator system are explained in pages 383 to 386, and, when legs are short and azimuths are properly corrected for convergence, they merely involve application of scale correction to the plane latitudes and departures between the permanently marked stations or between azimuth stations. If spherical rectangular co-ordinates are used, convergence is applied in the usual way to the observed azimuths to give bearings referred to the central meridian, and, after the intermediate bearings have been adjusted for closure, plane rectangular co-ordinates are computed from one azimuth station to another. As the spherical rectangular co-ordinates of the starting point are known, the latitude and departure between the azimuth stations can be used as the $s \,.\, \sin \alpha$ and $s \,.\, \cos \alpha$ terms in formulæ (7) and (8) on page 348 for point-to-point working. After corrected rectangular co-ordinates have been obtained, the geographical co-ordinates can easily be computed.

An alternative method is to use plane rectangular co-ordinates which are local for each traverse section and are based on the true meridian at the initial station of the section as axis of X. Application of local convergence to the observed azimuth at the second station will give a bearing referred to the meridian through the first station which can be

used for the adjustment of the intermediate bearings. Plane rectangular co-ordinates will then give the total latitude and departure of the second station with reference to the first, and these become the quantities $s \cdot \cos A$ and $s \cdot \sin A$ to be used in Puissant's formulæ (page 333) for computing differences of latitude and longitude from azimuth and distance.

When the geographical co-ordinates of the end azimuth station of the section have been obtained, those of intermediate points can be computed by assuming them to be proportional to their latitude and departure, as compared with the total latitude and departure of the end point with respect to the initial point.

In the case of minor theodolite traverses a still simpler method of obtaining geographical co-ordinates can be employed. The initial bearing to be used in this case is the true azimuth, and intermediate bearings are adjusted to the true azimuth at the terminal point of the section, and not the azimuth corrected for convergence, so that all intermediate bearings are very approximately true azimuths. Latitudes and departures are then computed in the usual way and it is now assumed that the resulting latitudes are meridional distances and the departures approximate arcs of parallels of latitude. Hence, if the traverse latitude is divided by the number of feet in one second of arc measured along the meridian, the result will be the difference in geodetic latitude, in seconds of arc, between the two points concerned. Similarly, if the departure is divided by the number of feet in one second of longitude, taken out for the mid-latitude of the line, the result will be the difference in longitude in seconds of arc.

In general, the geographical co-ordinates of fixed permanent points or of the terminal points of the section only need be computed in this manner; if those of intermediate points are needed they can be obtained from simple proportion from the rectangular co-ordinates, although the labour of computing them in the manner above described is not very great. This method is described more fully in Close and Winterbotham's *Text-book of Topographical Surveying*, which also gives an example and contains the necessary tables (Tables III and IV). Only short minor traverses, which are not intended to reach an accuracy greater than about 1/3,000 to 1/5,000, should be computed in this way as it is only approximate. Major compass traverses, in which magnetic bearings are corrected for declination, so as to reduce them to true azimuths, can also be computed in this manner.

In the case of all theodolite traverses, the calculation of the latitudes and departures should be thoroughly checked by an independent computation involving the use of auxiliary bearings (Vol. I, page 262).

Rapid Triangulation. In applying theodolite triangulation to the control of exploratory and reconnaissance surveys, the required rate of progress can be maintained only by considerable deviation from the methods of rigid triangulation. Time is saved not only by omitting refinements of measurement, but in greater degree by the adoption of devices for continuing the chain of triangles which would not be allowable in deliberate work. Less attention is paid to the proportions of the triangles, and greater dependence is placed upon stations fixed by intersection, or astronomical observations.

Methods comprised under rapid triangulation vary greatly according to the speed required, the strength of the party, and the equipment carried. Under favourable circumstances they merge into those of geographical surveying, and the results are then much superior to those which can be expected from an explorer using the most portable of instruments under conditions demanding the utmost speed.

Triangulation may proceed simultaneously with the sketching of topography. Otherwise the triangulation is executed by a party working in advance of the detail surveyors. In the latter case, the geographical co-ordinates of the stations fixed are worked out as soon as possible, and this information, along with a description of the sites, is sent back at frequent intervals for insertion on the plane table sheets. Preliminary reconnaissance and beaconing of stations before occupation are, as a rule, impracticable, except perhaps for the base net. The surveyor will have only a general idea of the direction in which the triangulation will develop, so that success depends greatly upon his skill in selecting and using elevated features which come into view as the party advances.

Rapid Base Measurement. When the survey is isolated from existing systems, a base, at least half a mile long, is measured, but it may be a difficult matter to find a suitable site unless time is spent in reconnoitring. Since a refined measurement is not attempted, a very level site is not required, and the inexactness of the measurement makes it more than ever necessary to have a good extension net. If the base is well-conditioned, the error of a rough base affects only the scale of the survey, and can be corrected by the measurement of a check base when a suitable site is reached.

Whenever possible, the base should be measured by tape. Usually the best that can be done is a double measurement by steel tape with the slopes measured or the tape held horizontally. When a suitable site for a taped base cannot be obtained, the measurement may be made tacheometrically with an improvised subtense bar. Otherwise, an astronomical base, lying roughly north and south, may be computed in the manner of a latitude and azimuth traverse (page 445) from the observed latitudes of its end stations and the observed or computed azimuth of the line joining them. Because of the uncertainties arising from local deviation, the greater the distance between astronomical base stations the better, and this method should be regarded only as a temporary expedient until a base can be measured.

Angle Measurement of Rapid Triangulation. In some classes of rapid triangulation a 5-in. micrometer theodolite is used, but when weight must be reduced to a minimum a light reconnaissance or mountain transit is all that can be carried. Remarkably good work can be done with modern 3-in. and 4-in. theodolites. These may be obtained with vernier reading to 1′ for the 3-in. instrument and to 1′, 30″, or 20″ in the case of the 4-in., the latter having a weight as low as 14 lbs., including tripod, cases, and accessories. A compass, preferably of trough form, is essential.

The azimuth of the base and the latitude of one end must be observed. If possible, a telegraphic longitude or one based on wireless time signals will be obtained: otherwise, a value is assumed for the longitude. The error of the assumed value affects all the positions equally, and may be

corrected out when the longitude of any station can be obtained telegraphically or by wireless time signals.

In measuring the angles of the base net, only very definite marks must be used for observing upon because of the comparative shortness of the sights. Whenever possible, base extension stations should be beaconed. It is then justifiable to observe four rounds of angles, with change of face and zero between each round, but otherwise it is sufficient to take face right and face left readings, with change of zero between each face.

From the stations of the base net the surveyor must observe to all prominent natural features which may prove suitable for theodolite stations. Directions are also measured for the future fixing by intersection of such additional points as are likely to serve either as plane table stations or as guides to the sketching. Face right and face left rounds should be taken more to guard against mistakes of reading and booking than to eliminate instrumental error. In judging as to which of the points before him will serve as future theodolite stations, the surveyor is usually faced with the difficulty that he will be unable to visit points unless near the route to be followed. The future course of the journey is probably not definitely known, and must be estimated from information obtained as to easy routes of travel and positions of camping grounds and water. It is therefore advisable to observe as many prominent objects as possible in the time available, as it frequently becomes necessary to occupy a station which was intended only as an intersected point.

Having completed the observations at a station and descended to the route of the march, the surveyor will find that many of the points to which sights have been taken are no longer visible, and their changing aspect cannot be watched as the journey proceeds. When the next station is occupied, it will be found no easy matter to recognise them from the new point of view.

Identification of Points. To avoid a breakdown in the continuity of the triangulation through failure to identify stations, the triangle sides must be kept shorter than would be necessary if the stations were regularly beaconed. The lines of the chain should seldom exceed twenty miles in length, although longer sights are allowable to intersect points outside it. In observing, a good general rule is to sight only to the highest points of hills, but if the top is rather flat and presents no definite mark for sighting, the observation is made on an isolated tree or prominent rock near the summit. Such marks, however, are apt to prove especially troublesome to recognise from different standpoints.

Additional precautions affecting the routine of observation must be regularly adopted. At each station the theodolite should be oriented by compass, so that the observed directions are magnetic bearings. On proceeding to occupy a point, reference to the back bearing of a preceding station affords a valuable check against gross mistakes of position. The record at each station should include notes and sketches of the appearance of the marks bisected and of the outline of the surrounding ground as seen through the telescope.

As an aid to the identification of points, a plane table proves of great service, and, if at all possible, the triangulator should carry one of light

pattern, even although he may not be required to take topography. By its means a small scale diagram of the triangulation is kept up by drawing a ray to every point observed with the theodolite. The diagram, besides affording a ready means of identifying points, also exhibits the state of the work at any time, and enables the surveyor to judge the quality of intersections. Points which have been observed for the first time are approximately located on the sheet by the intersection of rays drawn from the theodolite station and from a subsidiary plane table station fixed by resection two or three miles distant in a direction transverse to the triangulation chain. The forward points should be easily recognised, as their appearance will not have changed greatly in the limited distance between the two positions of the plane table. During the journey, when opportunities occur of obtaining good views of the forward points, the plane table is set up, and its position is plotted by resecting on back stations or intersected points. Then, by pointing the alidade through the plotted points representing either intended theodolite stations or points to be further intersected, their changed appearance may be examined as an aid to their future identification. At the same time, points to which only one ray has previously been drawn are approximately located, and a new set of points may be plotted during the journey for use in resecting the surveyor's position when finding his way towards a station. Plane table resections provide a check against setting up the theodolite on the wrong peak, but the identification of the actual spot to be occupied must be based upon the sketches of the site made at back stations.

Use of Intersected and Resected Stations in Rapid Triangulation. When it is impracticable to observe all three angles of each triangle, the system can be carried forward by the aid of intersected points. In Fig. 153, A, B, C, D, and E are stations lying along the route followed, and only these are occupied. Points a, b, c, d, and e are among those intersected, and, because of their favourable situation and natural marking, are selected for use in continuing the chain. The length of side AB having been computed through the preceding figures, triangle ABa is solved for Ba, and the result is used in solving BCa for BC. Distance CD may now be obtained by solving BCb for Cb, and CDb for CD, but it may also be derived from the results of the observations to c by solving BCc for Cc, and CDc for CD. The mean of the two values for CD is to be adopted for continuing the computation. Whenever possible, a pair of intersected points should be thus used: these may lie on the same side of the line of occupied stations or on opposite sides, as at d and e.

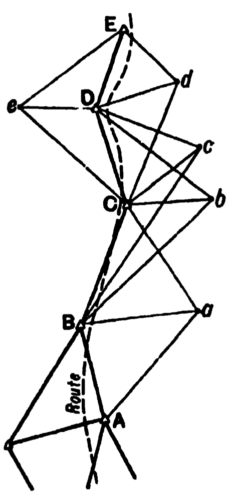


FIG. 153.

It is sometimes necessary to occupy a point to which no forward sights have been taken. Observations must then be made upon previously located points to enable the position of the instrument station to be computed by the method of trigonometrical

interpolation or resection (Vol. I, page 564). A sufficiently good result may be secured if attention is paid to the quality of the fix, and this can first be tested on the plane table. The accuracy of the method is much improved by observation of an astronomical azimuth to one of the back stations.

It may be found impracticable to maintain a continuous system of triangulation, particularly when the route passes through belts of wooded country. Triangulation is then confined to the more favourable ground, each section having its own base line determined by latitude and azimuth or measured roughly along the ground. The intervening gaps may be bridged by latitude and azimuth traverse, or, if the route lies nearly east and west, longitude may be determined by wireless, by transport of chronometer watches, or by an absolute method.

Graphic Triangulation. Triangulation by means of a small plane table is sometimes employed for the main control of reconnaissance surveys. The general features of the field work are similar to those of rapid triangulation by theodolite. The method is best adapted for running subsidiary chains in topographical and geographical surveys, the sketching of detail proceeding simultaneously. The conduct of the work and its adjustment between known positions are described in Vol. I, Chap. VII.

Latitude and Azimuth Traverse. If the route lies approximately north and south, the method of latitude and azimuth traverse is available for the rapid location of a system of rather widely separated but intervisible points. If A and B are two such points, a latitude determination is made at each, and the azimuth of the line between them is observed at A or B or from both stations. From these data the length of AB and the longitude difference are computed.

The accuracy of the method is controlled not only by that with which the latitude and azimuth determinations are made, but also by the value of the azimuth. Errors of observation have least influence when the course lies north and south, and their effect increases as the direction deviates from the meridian. The method is impossible of application in the case of a course lying east and west, and should not be used for azimuths within 45° of that direction. Since errors of astronomical observation and the unknown effects of local deviation are independent of the distance between the stations, the latter should be as great as possible compatible with good visibility.

Although not capable of a high grade of accuracy, the method is useful for rapid work in rough country. The circumstance that a forward sight is not essential proves an advantage when it is found necessary to depart from the intended route of march. When, after conducting a latitude and azimuth traverse so far, the general direction of the route turns towards the east or west so that the azimuths cannot be kept within 45° of the meridian, triangulation or ordinary traverse must be substituted until the method can be resumed.

The method of calculating distances and differences of longitude is described on pages 339 and 340.

Control by Astronomical Positions. In this method the latitudes and longitudes of rather widely spaced points are obtained, the subsidiary work consisting of rough traverse. The method is suitable for the rapid

survey of flat country, in which latitude and azimuth traversing cannot be employed. The application of wireless telegraphy to longitude determination has removed the difficulty of obtaining satisfactory longitudes, and with its aid the method has been found particularly useful in regions barren of detail. In such circumstances the route traverse need not be continuous.

VERTICAL CONTROL

The methods of precise and trigometrical levelling have been described in Chap. VI and those of ordinary spirit levelling in Vol. I, Chap. VI. Where possible, precise levelling should be employed as the basis for the primary vertical control of important or extended topographical surveys, but ordinary spirit levelling may be used for less important or smaller surveys. It may, however, be impracticable to run more than a few lines of precise or ordinary spirit levels, from which trigonometrical levelling may proceed, and very often the latter is exclusively used. Barometric levelling, although much less suitable for control than for detail work, is frequently applied to the entire levelling of reconnaissance surveys.

Barometric Levelling. In barometric levelling the relative altitudes of points are determined by ascertaining the difference of their depths below the upper surface of the atmosphere, an operation analogous to sounding in hydrographical work. The measurement is performed by observing the atmospheric pressure at the several points and deducing the corresponding relative elevations. The pressures are measured by (1) mercurial barometer, (2) aneroid barometer, (3) boiling-point thermometer. The mercurial barometer measures the height of a column of mercury which balances the column of air above the instrument. The aneroid records the pressure exerted against a hermetically sealed elastic box. The temperature of boiling water as ascertained by the boiling-point thermometer affords an indirect method of barometry, since water boils when the elastic force of the vapour equals the external pressure on the water surface.

Scope. The great merit of barometric levelling lies in the rapidity with which measurement may be made of the difference of elevation of points at considerable horizontal or vertical distances apart. On this account it is greatly used in preliminary reconnaissance and exploratory surveys, but the degree of accuracy to be expected is much inferior to that of spirit levelling.

Mercurial Barometers. The mercurial barometer consists essentially of a vertical glass tube about 33 in. long, closed at the top and having the bottom placed in a bath of mercury. The tube being exhausted of air, and the bath exposed to atmospheric pressure, mercury will stand in the tube at a height sufficient to balance the pressure of the atmosphere on the free surface of the mercury in the reservoir.

The arrangement of a vertical tube standing in a bath of mercury is that adopted in the cistern barometer. In the syphon barometer the tube is bent up at the lower end to form a short vertical branch, the mercury in which is exposed to atmospheric pressure by the admission of air through a small opening near the top. The cistern barometer is the better type of mercurial instrument.

The Cistern Barometer. Figs. 154 and 155 show the arrangement of a cistern barometer on the Fortin system, the particular feature of which consists in having the bottom of the cistern formed of fine-grained leather. The leather is permeable to air, and the mercury is subject to the action of the atmosphere without danger of leakage. The leather bag is attached to the bottom of the cistern, and is supported by the screw *5*, by means of which the level of the free surface of the mercury can be adjusted. At the top of the cistern there is fitted a glass sighting cylinder held against packings at top and bottom by four screws. The glass barometer tube enters the cistern through a collar where a secure joint is formed by leather or silk and glue. This collar carries the ivory gauge peg *7*.

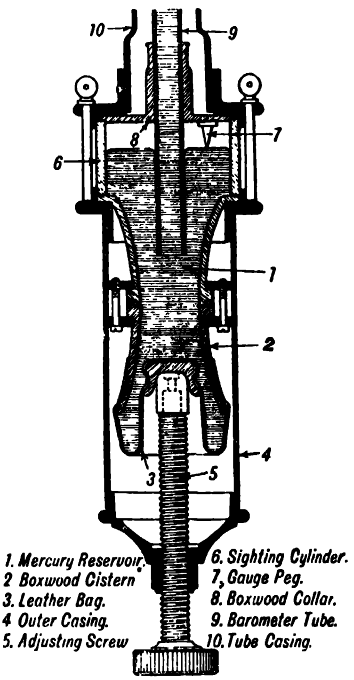


FIG. 154. FORTIN BAROMETER.

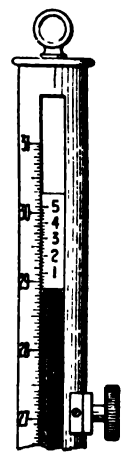


FIG. 155. READING ARRANGEMENT.

The barometer tube is enclosed in a brass casing, the upper part of which is slotted on both sides to expose the mercurial column (Fig. 155). Graduations to ·05 in. are marked on one side. A tube, fitting the inside of the external casing, carries a vernier index, and, by means of a rack actuated by a pinion through the milled head shown, the lower edge of the vernier is brought to coincidence with the top of the mercury, and readings can be carried to ·002 in. About midway up the barometer tube a thermometer is attached inside the external casing and close to the glass tube. For levelling work the instrument is mounted on a tripod, the head of which is arranged to ensure that the instrument maintains a vertical position.

To read the barometer, the surface of the mercury in the cistern is viewed through the glass *6*, and is adjusted by the screw *5* until it just touches the point of the gauge peg. To effect this with the necessary precision, the eye should be placed level with the surface of the mercury

and the cistern tapped gently to overcome adhesion between the mercury surface and the glass. The vernier index must then be set to the top of the mercurial column. The glass tube is first tapped near this point, and the index is racked until its front and back lower edges appear tangential to the convex surface of the mercury. The heading can then be observed. At each reading of the barometer it is necessary to record the temperature registered by the attached thermometer, and this reading should be taken first.

Other forms * of mercurial barometer are also used in levelling. Some of these are designed to afford maximum portability with minimum risk of breakage.

The great disadvantage of the mercurial barometer is the ease with which it is broken and its consequent lack of portability. When a barometer of the Fortin type is being transported from one place to another, the adjusting screw *5* should be screwed up until the mercury completely fills the barometer tube, and the instrument should always be carried in a vertical position with the cistern down.

The Aneroid Barometer. The features essential to all aneroids are (1) a thin metal box exhausted of air and hermetically sealed, (2) some mechanical or optical means whereby the small displacements of the surface of the box due to changes of atmospheric pressure may be magnified and read.

The form almost exclusively used for levelling purposes is of the Vidi type, in which the movement is multiplied by delicate levers and is shown by a pointer travelling over the graduations on a dial of from 2 in. to 6 in. diameter (Fig. 156). The interior arrangement is shown diagrammatically in Fig. 157. The vacuum box is made in the form of a flat cylinder, and

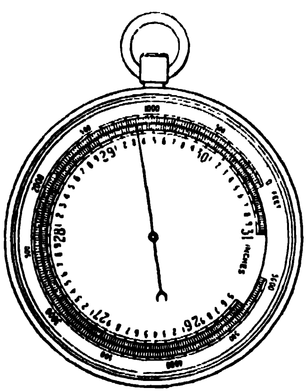


FIG. 156.
ANEROID BAROMETER.

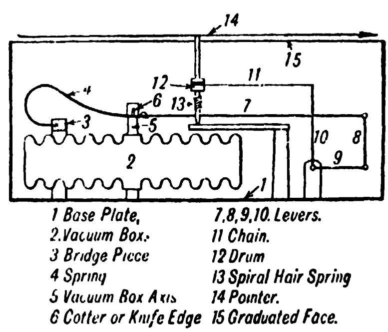


FIG. 157. DIAGRAMMATIC SECTION OF ANEROID BAROMETER.

is corrugated at top and bottom to increase its sensitiveness. The bottom of the box is fixed at the centre to the base plate, and the post, or axis, *5* is attached to the top and communicates the tension of the spring *4*, which is as broad as the casing. Increase of atmospheric pressure causes

* See *Hints to Travellers*, Vols. I and II.

the knife-edge *6* to descend and *vice versa*, and this motion is transferred by the levers *7*, *8*, *9*, and *10* through the watch chain *11* to the pointer. The hair spring *13* keeps the chain taut by pulling against it with a force just sufficient to overcome axial friction.

The graduations on the dial represent inches of mercury, the spacing being obtained by subjecting the aneroid to known pressures under an air pump or by comparison with a standard mercurial barometer. By means of an adjusting screw at the back of the instrument the pointer may be moved over the scale, and the instrument thereby set to agree at any time with a standard barometer under the pressure then prevailing. A scale of altitudes is usually provided, the divisions being spaced by conversion from the scale of inches by the use of a formula, but the indications of the altitude scale cannot be accepted for the best work unless a correction for the temperature of the air is applied. The altitude scale is commonly engraved upon a movable ring so that the graduation representing the known elevation of a station occupied can be set under the pointer. The absolute elevations of other points to which the instrument is carried can then be read directly. This is a convenient arrangement for rough observations, but introduces an additional source of error, since rate of increase of altitude is not equal to rate of decrease of pressure.

In reading the aneroid, it should always be held in the same position, vertically or horizontally, and the case tapped lightly to overcome any friction in the recording mechanism. In estimating the position of the pointer, care must be exercised to avoid error due to parallax. In order to enable this to be done some aneroids are provided with a thin reflecting circular strip placed near, and concentric with, the graduated scale. Readings on the scale are then taken when the pointer and its reflected image appear to coincide.

On account of its portability and the greater speed with which observations may be made, the aneroid is preferred to the mercurial barometer in taking topography. With the necessary precautions, results may be obtained with sufficient accuracy for the plotting of 25 ft. contours on small scales, but in point of precision the ordinary aneroid ranks considerably below the mercurial instrument (see page 446), especially when absolute heights, as opposed to differences of elevation, are required.

One trouble with the ordinary aneroid barometer is a certain amount of friction in the working parts, and of sluggishness or "hysteresis" in responding to rapid changes in pressure. To overcome this, the Paulin Co. of Stockholm, for whom Messrs. E. R. Watts & Sons are the English agents, have produced an improved type ot aneroid for which it is claimed that the effects of friction and hysteresis are virtually eliminated. In this instrument, the movement of the diaphragm of the vacuum box is balanced by the tension of a precise spring, this tension being controlled by a knurled knob to which a pointer is attached. When a reading is to be taken, the knob is turned until the pointer coincides with a fixed index line. Turning the knob also rotates a cam, and a second pointer, working against this cam, then indicates the pressure or height on a graduated scale. Since the whole apparatus works on the principle of a

balance, there are practically no moving parts, such as there are in an ordinary aneroid. The latest model of this instrument, the "surveying micro-altimeter," reads to single feet and has a range of 6,000 ft. (— 1,000 ft. to 0 to + 5,000 ft.). Other models, known as "surveying altimeters," are graduated at intervals of 2 ft., 5 ft. and 10 ft., with ranges of — 760 ft. to 0 to + 3,600 ft., — 900 ft. to 0 to + 9,700 ft. and — 500 ft. to 0 to + 14,500 ft. respectively. Much more accurate results may be obtained with these instruments than with an ordinary aneroid, especially if the results are carefully corrected for air temperature and humidity.

Observations. The barometric determination of elevations would be comparatively simple and accurate if the atmospheric pressures were always equal at points of equal elevation. In reality, since the air is continually in motion, this is far from being the case, and, although the assumption of static equilibrium of the atmosphere forms the basis of the barometric levelling formulæ in practical use, the observations should be made in a manner designed to eliminate as far as possible errors due to variation in pressure throughout level layers of the atmosphere.

Changes of atmospheric conditions may be classified as those due to (1) gradient, (2) temperature, (3) humidity.

Gradient. If the atmosphere were in static equilibrium, surfaces passing through all points at which the pressure is the same would be level surfaces. Actually, surfaces of equal pressure are irregular, and are subject to continual change of form. Under normal conditions such a surface may within a limited area be regarded as a plane, the inclination of which to the horizontal is termed the barometric gradient. The direction and amount of the barometric gradient are subject to continual changes, which may be analysed as periodic, due to diurnal * and annual fluctuations in temperature, and non-periodic, arising from local conditions as to weather and topography.

Temperature. The weight of the column of air of depth equal to the difference of level between two stations being compared depends upon its temperature. Observations for temperature should be taken at both stations, but the mean of these may be considerably different from the mean temperature of the air column, since the thermometer is read in the stratum of air at the surface of the earth, which is warmer than the air column throughout the day and cooler at night. The consequent uncertainty with regard to the mean temperature of the air column forms an important source of error. The lapse-rate formulæ given on pages 456 to 457 are designed to reduce this source of error as much as possible.

Humidity. The proportion of water vapour present in the atmosphere is continually changing, and, as its density is much less than that of dry air, change of humidity is a second factor in varying the weight of the air column. In the best work, observations for humidity are taken by means of the wet bulb hygrometer, but as these are necessarily made near the ground, where the proportion of vapour is greatest and most variable, estimation of the mean humidity of the air column is as uncertain as that of its temperature.

* See "Diurnal Atmospheric Variation in the Tropics, and Surveying with the Aneroid," by T. G. Gribble. *Min. Proc. Inst. C.E.*, Vol. CLXXI.

Field Work. A good determination of the difference of level between distant points can be obtained barometrically only from an extended series of observations at both stations by means of mercurial barometers. The greater the distance between the stations, the greater is the number of observations required for averaging. In the ordinary operation of taking topography by aneroid the usual problem, however, is to find the elevations of points at moderate distances from a station, the elevation of which has been otherwise ascertained. The determinations are then based upon either (1) single observations, (2) simultaneous observations.

Single Observations. This is the commoner but less accurate method. The instrument is read at the reference station, and is then carried from point to point, and a reading is taken at each. When speed is of greater importance than accuracy, use is made of the altitude scale, which is first set at the base station to the known elevation of that point, and the elevations of the other points are read directly. The results of single observations may be very considerably in error if no attempt is made to eliminate errors arising from variations in atmospheric conditions.

The accuracy of the method may, however, be improved without much extra trouble. Change of gradient may be roughly ascertained by arranging the route so that a return to the reference station is made at intervals. By observing the aneroid on such occasions, the rate of change of pressure is ascertained, and intermediate readings can be roughly corrected by time intervals on the supposition that the variation at the base represents that throughout the surrounding area. Otherwise, the rate of change may be roughly obtained during a circuit by remaining at one point for half an hour or so and noting any variation which may occur. If the route extends between two stations of known elevation, any change of pressure which has occurred during the journey will be evidenced at the second station by a discrepancy in the difference of level obtained, and can be roughly distributed to the intermediate points. The atmospheric temperatures are taken at the several points for correction of the barometer readings.

Simultaneous Observations. In this method two instruments, which have previously been compared, are employed. One is kept at the base station, and is read at regular intervals throughout the day : the second is carried to the various points to be levelled, and each reading is taken simultaneously with an observation at the base. Otherwise, the transported instrument is read at irregular intervals as required, and the simultaneous pressure at the base is deduced by interpolation from the values observed there.

The method is designed to reduce the effects of atmospheric changes to a greater degree than is possible by single observations, and is to be preferred, particularly where reference stations are widely separated.

The barometer used at the base station may be a good self-recording micro-barograph, with a suitable range of reading, as this will obviate the necessity for a special observer at the base. The scale on the chart of an ordinary barograph is not large enough for this class of work and hence a micro-barograph must be used instead.

Aneroid Observations in the Tropics. It has been found that in some parts of the tropics great variations in the readings of the barometer do not occur at any particular place, and that the main variation is one which occurs during the course of the day and repeats itself with great regularity from day to day, with slight variations from season to season. Readings taken at one station throughout the day tend to show maxima about 3 to 4 p.m. and minima about 8 to 9 a.m., but the variations from the mean curve are least between about 7 a.m. and 2 p.m., although these times may vary slightly in different localities. This fact is sometimes taken advantage of to introduce corrections to aneroid readings, even when systematic observations are not taken at regular intervals during the day at a central station.

The procedure in this case is to take half-hourly readings on each instrument at a fixed station for several successive days, the time of each reading being carefully noted. Each day's readings for one instrument are plotted against time on a single piece of squared paper and curves are drawn through each set of readings. A mean curve is then drawn to represent the average of all the curves, and this mean curve is used as a basis for drawing up a table of corrections to be applied to readings at any times of the day to reduce them to the equivalent readings at a certain time chosen as standard. In the field, therefore, the work consists of observing the time at which a particular reading is taken and applying the correction necessary to reduce it to the equivalent height at the standard time.

Correction curves should be obtained at fairly frequent intervals in case there should be any very appreciable change in the form of the diurnal wave. Time will be saved if each surveyor of a group is supplied with several aneroids, so that some can be used in the field while others are sent to have their individual waves determined at a central camp. Alternatively, a battery of several aneroids may be kept at the central camp and corrections, based on the readings of all the aneroids in the battery, sent out for field use.

This system is only used when a good framework of control heights, established by spirit levelling, already exists, and each line of aneroid heights begins and ends on a bench mark established by spirit levelling. Such lines are not generally allowed to exceed 15 miles in length, and the closing error on the end bench mark is distributed between the intermediate readings proportionately to the distance from the initial bench mark. The method is used extensively in West Africa, where it forms the basis of the detail levelling for contouring for the standard official maps on the 1/62,500 and 1/125,000 scales with contours at 50-ft. or 100-ft. vertical intervals. It is only applicable in countries where the diurnal pressure wave is known to be regular and steady.

It is obvious that this method demands a reasonably accurate knowledge of time. If an ordinary wireless set is not available, or for some reason cannot be used, local time can be obtained very simply by observing the sun, either with an Abney level or with a simple clinometer, when it is at a standard elevation—say 15°—above the horizon. For this purpose, a table giving these times to the nearest minute at half monthly intervals

can easily be computed for any particular latitude and longitude. Exact times are not needed, as errors of several minutes in the time will make no practical difference to reduced heights, and, once computed, a table of the kind described will be valid for several decades. The formulæ for computing it are :—

$$\tan^2 \frac{t}{2} = \frac{\sin (s - p) \sin (s - \lambda)}{\sin s \sin (s - z)},$$

where t = sun's hour angle in angular measure,

p = sun's polar distance = 90° — sun's declination,

z = sun's zenith distance = 90° — sun's altitude,

λ = co-latitude of place = 90° — latitude,

$s = \frac{1}{2}(p + z + \lambda)$.

Then, mean time of observation when sun's zenith distance is $z = 12^h \pm t$ (converted into time) $\pm$ equation of time. If the sun is observed at an altitude of 15°, z may be taken as 75° 03½′, as the refraction for an altitude of 15° is about 3½′.

For a change in latitude of $d\phi$, the corresponding change in t, in time, is given by :—

$$dt = d\phi \operatorname{cosec} \lambda \cot A/15$$

in which A, the sun's azimuth, is computed from :—

$$\tan^2 \frac{A}{2} = \frac{\sin (s - z) \sin (s - \lambda)}{\sin . s . \sin (s - p)}.$$

An auxiliary table, giving dt in minutes of time for $d\phi$ = 1° or 60′, can easily be computed for the middle day of each month, and this table used to obtain the correction for a *small* difference (up to 2° or 3°) from the latitude for which the main table has been calculated.

Reduction of Observations. From the nature of the case, a simple formula for difference of elevation is not available. The surveyor uses tabular values prepared from one or other of the various formulæ which have been proposed.

Theoretical investigation of the subject was first made by Laplace, who showed that

Difference of elevation = $C (\log H_1 - \log H') \times a \times b \times c$ . . (1)

where H_1, H' = barometer readings in inches at lower and upper stations respectively, both reduced to 32° F.,

C = a constant, variously estimated as 60,159 to 60,384 for ft. units,

a = a factor allowing for the mean temperature of the air column and an average amount of moisture = $\left(1 + \frac{t_1 + t' - 64}{900}\right)$, where t_1, t' = air temperatures in degrees Fahr. at lower and upper stations respectively,

b = a factor allowing for the variation of gravity with latitude = $(1 + \cdot0026 \cos 2\phi)$, where ϕ = mid-latitude of the stations,

c = a factor allowing for the diminution of gravity with altitude
$= \left(1 + \dfrac{X + 52,252 + 2h_1}{R}\right)$, where X = difference of elevation as obtained without this factor, h_1 = elevation of lower station, and R = radius of earth in ft. = $20 \cdot 89 \times 10^6$.

The reduction of barometric readings to the temperature 32° F. requires a knowledge of the temperature, T, of the instrument as given by the attached thermometer. The reduction formula is

Reading reduced to 32° F. = Actual Reading $(1 - \alpha(T - 32))$, . . (2)

where α = differential coefficient of expansion of mercury and the metal (brass) of the scale = say $\cdot 00009$.

Readings of compensated aneroids are not subject to this reduction, and the term 52,252 in the c factor is omitted in their case.

The Laplace formula has been tabulated in various ways. Loomis' Tables are frequently used, and are published in the Indian Survey Auxiliary Tables, the Smithsonian Miscellaneous Collections, *Hints to Travellers*, etc.

Several modifications of the formula have been made. That by Baily is

Difference of elevation in ft. $= 60,346 (\log H_1 - \log H') \times d \times a \times b$. . (3)

where H_1, H' = unreduced barometer readings at lower and upper stations respectively,

d = temperature correction to barometer $= \dfrac{1}{1 + \cdot 0001(T_1 - T')}$, where T_1, T' = readings of the attached thermometers at lower and upper stations respectively,

a = as before,

b = latitude factor $= (1 + \cdot 002695 \cos 2\phi)$, where ϕ = mid-latitude.

Baily's Tables are given in the Indian Auxiliary Tables, the Smithsonian Collection, Close and Winterbotham's *Text Book of Topographical and Geographical Surveying*, etc.

The formula now adopted as a standard for international use takes into account the actual humidity of the air at the times of observation, and although not so convenient as those given above, may be employed for the most precise work. With heights in feet, pressures in inches of mercury and temperatures in degrees Fahrenheit, it is :—

Difference of elevation in feet =

$$60370 (\log H_1 - \log H') \times \left(1 + 0 \cdot 00264 \cos 2\phi + \frac{h_1 + h^1}{R} + \frac{3P}{8H_m}\right) \times (1 + 0 \cdot 002036(t_m - 32)), \quad . . . . (4)$$

where H_m is the mean barometric pressure in inches, t_m the mean temperature at the two stations and P is the mean water vapour pressure.

P is calculated from :—

$$P = P_w - 0 \cdot 00045 H_m (t_d - t_w), \quad . . . . (5)$$

where t_d and t_w are readings on the dry and wet bulbs of the hygrometer, in degrees Fahrenheit, and P_w, the saturation pressure of water vapour, must be taken from tables such as those given in *Hints to Travellers*, Vol. II.

For logarithmic work this formula takes the form :—

$$\log (h^1 - h_1) = 4{\cdot}780\ 82 + 0{\cdot}00264\, M \cos 2\phi + \frac{M(h_1 + h^1)}{R}$$
$$+ \frac{3MP}{8H_m} + \log (1 + 0{\cdot}002036\, (t_m - 32)\,) + \log \left[\log \frac{H_1}{H^1}\right] \quad . \quad . \quad (5)$$

where M is the modulus of the common logarithms ($\log M = \bar{1}{\cdot}637\ 78$).

The classical Laplace formula for difference of barometric pressure with difference of height is easily derived as follows: the change of weight per unit area (*i.e.*, the change of pressure) dp for a change of elevation dh is given by $dp = -g\rho dh$, where ρ is the density of air at pressure p and *absolute* temperature T and g is the acceleration due to gravity. But, by the Boyle-Gay-Lussac law for a perfect gas, $p \,.\, v = G \,.\, T.$, where G is the gas constant, v being the volume of the gas. Let v_s, ρ_s and p_s be the volume, density and pressure of the gas at a standard absolute temperature T_s. Then

$$\frac{pv}{T} = \frac{p_s v_s}{T_s}.$$

$$\therefore\ v = \frac{p_s v_s}{p} \,.\, \frac{T}{T_s}.$$

For the same mass of gas, $v\rho = v_s\rho_s$.

$$\therefore\quad \rho = \frac{v_s\rho_s}{v} = \frac{p}{p_s} \,.\, \rho_s \frac{T_s}{T}.$$

$$\therefore\ dp = -gp \,.\, \frac{\rho_s}{p_s} \,.\, \frac{T_s}{T} dh.$$

$$\therefore\ dh = -\frac{p_s}{g\rho_s} \,.\, \frac{T}{T_s} \,.\, \frac{dp}{p}.$$

Assuming the same absolute temperature T for both upper and lower stations, and integrating between the limits h_1 and h^1 and p_1 and p^1, we get

$$h^1 - h_1 = \frac{p_s}{g\rho_s} \,.\, \frac{T}{T_s} [\log p_1 - \log p^1].$$

Putting $p_1 = k \,.\, H_1$ and $p^1 = k \,.\, H^1$ and

$$\frac{1}{g} = \frac{1}{g_o}(1 + 0{\cdot}002695 \cos 2\phi)\left(1 + \frac{X + 52{,}252 + 2h_1}{R}\right),$$

where g_o is the value of gravity at the equator, we have

$$h^1 - h_1 = \frac{p_s}{g_o\rho_s} \,.\, \frac{T}{T_s} (\log H_1 - \log H^1)\,(1 + 0{\cdot}002695 \cos 2\phi)\left(1 + \frac{X + 52{,}252 + 2h_1}{R}\right).$$

If p_s and ρ_s are taken to be the pressure and density of unit volume of air at the freezing point of water, then $\frac{p_s}{g_o\rho_s}$ is a constant which we can put equal to C. If we also take T as the means of the absolute temperatures at the two stations, and T_s as the absolute temperature of the freezing point of water, we have $T = \frac{1}{2}(t_1 - 32^\circ + T_s + t^1 - 32^\circ + T_s) = T_s + \frac{1}{2}(t_1 + t^1 - 64^\circ)$. Hence the expression becomes: —

$$h^1 - h_1 = MC(\log H_1 - \log H^1) \times (1 + 0{\cdot}002695 \cos 2\phi) \times \left(1 + \frac{t_1 + t^1 - 64}{2T_s}\right)$$
$$\times \left(1 + \frac{X + 52{,}252 + 2h_1}{R}\right)$$

in which the modulus M of the common logarithms has been introduced to reduce Naperian to common logarithms.

But freezing point on the absolute scale corresponds to 491 Fahrenheit degrees, so that $2T_s = 982^\circ$. In order, however, to allow for average humidity, the figure 900 has been taken instead of 982 in the formula for a in (1) above.

Lapse-rate Formulæ for Aneroid Height Scales. The formula given above is derived on the assumption that the temperature of the air column is uniform at all heights. This, however, is contrary to normal experience as it is known that, up to a certain limiting height, the temperature of the air decreases with increase in elevation. The fall in temperature does not obey an absolutely linear law but normally it does not depart greatly from one. The whole question of the variation of barometric pressure with elevation, with which is connected the altitude scale on aneroids, has assumed new importance in recent years owing to the needs of aviation, and this has led to a certain amount of research work on the subject. Accordingly, in 1930, an inter-departmental committee, consisting of representatives from the Admiralty, War Office, Air Ministry and the National Physical Laboratory, to which representatives of instrument makers were added later, was formed to investigate the question of the most suitable formula on which to base the height scale for aneroids. This Committee has now recommended certain formulæ and has calculated tables * which make allowance for the variation of temperature with height. Both formulæ and tables are based on the assumption of a constant "lapse-rate"—that is, the rate of decrease of temperature per unit difference of height—amounting to 3·566° F. or 1·981° C. per 1,000 ft.

For a dry-air lapse-rate k it can be shown that, with an absolute temperature T_e at sea level and $T_e - kh$ at height h above sea level,

$$\frac{p}{p_e} = \left(\frac{T_e - kh}{T_e}\right)^{\frac{g}{k}\cdot\frac{\rho_e T_e}{p_e}},$$

where p_e is the atmospheric pressure at sea level, p is the corresponding pressure at height h above sea level, ρ_e is the density of the air at sea level and g is the acceleration of gravity.

If p_s and ρ_s are the pressure and density of the air at some other temperature T_s, which is chosen as a standard temperature at sea level, we have from the laws relating to a perfect gas

$$\frac{\rho_e T_e}{p_e} = \frac{\rho_s T_s}{p_s} = \frac{1}{c},$$

where c is a constant for the gas. Accordingly, we can write

$$\frac{p}{p_e} = \left(\frac{T_e - kh}{T_e}\right)^{\frac{g}{kc}}$$

where p_e and T_e can be chosen at convenience.

Assuming that $g = 980·62$ c.g.s. units (the accepted value at latitude 45°), that at mean sea level the temperature is 15° C. or 59° F. and the barometric height, reduced to 0° C, is 760 mm. of mercury, that in these conditions 1 cc. of air weighs 1·2257 milligrammes, and that the lapse-rate is 1·981° C. per 1,000 ft., this gives :—

$$\frac{p}{p_e} = \left[(288 - 0·00198h)/288\right]^{5·256},$$

where h is in feet.

* "Aneroid Tables Based on a Standard Atmosphere and a Standard Lapse-Rate." H.M.S.O., London, 1935.

This formula, which is based on one due to L. Touissant, is the one recommended by the Committee as a basis for the graduation of aneroid barometers and the figures in the second column of the following table have been computed from it on the assumption that $p_e = 29{\cdot}921$ inches and $T_e = 59°$ F.

Height. Feet above Mean Sea Level.	Pressure. Inches of mercury at 32° F. with g = 980·62 dynes.	Diff.	Temperature. Fahrenheit.
− 1,000	31·018		62°·6
		− 1·097	
0	29·921		59 ·0
		− 1·065	
+ 1,000	28·856		55 ·4
		− 1·035	
+ 2,000	27·821		51 ·9
		− 1·004	
+ 3,000	26·817		48 ·3
		− 0·975	
+ 4,000	25·842		44 ·7
		− 0·946	
+ 5,000	24·896		41 ·2
		− 4·368	
+ 10,000	20·528		23 ·3

Temperature correction may be obtained from :—

$$h_c - h = 0{\cdot}001929h(t - t_{\prime\prime})$$

in which h_c is the corrected height, h the height at standard temperature, both in feet, t the observed temperature in degrees Fahrenheit and $t_{\prime\prime}$ the standard temperature in degrees Fahrenheit for height h — that is, the temperature given in the last column against height h or computed from $t_{\prime\prime} = (59° - 0{\cdot}003566 \,.\, h)$.

In the tables published by the Committee, the correction for temperature is combined with one for humidity and it is advisable to use the tables when it is desired to apply these corrections, especially as the humidity correction is not one that can be computed very easily. The explanation at the beginning of the tables gives the procedure to be followed when selecting a scale to be adopted for an aneroid that is to be used throughout at an average temperature differing greatly from the standard temperature of 15° C. or 59° F.

The corrections for variation of gravity with height and latitude are both small, and, if required, can be computed in the ordinary way, but it is much easier to take them from the official tables where they are combined together in Table A.

The formula given above for the relation between the pressures at sea level and at height h above it may easily be derived in a manner similar to that used in deriving the Laplace formula. Taking T_s as the absolute temperature at sea level, the temperature at height h above sea level will be $T_s - k \,.\, h$. Hence, if p_s, ρ_s and v_s are the pressure, density and volume of the gas at sea level, we have as before :—

$$\frac{pv}{T_s - k \,.\, h} = \frac{p_s v_s}{T_s},$$

and this leads to

$$dp = -g p \frac{\rho_s}{p_s}\left(\frac{T_s}{T_s - kh}\right)dh.$$

Integrating, we get

$$\log p = \frac{g\rho_s}{kp_s} T_s \log (T_s - kh) + \text{constant}.$$

But $p = p_e$ when $h = 0$, and $p = p$ when $h = h$. Hence,

$$\frac{p}{p_e} = \left(\frac{T_e - kh}{T_e}\right)^{\frac{gp_eT_e}{kp_e}} = \left(\frac{T_e - kh}{T_e}\right)^{\frac{gp_sT_s}{kp_s}} = \left(\frac{T_e - kh}{T_e}\right)^{\frac{g}{kc}}.$$

As regards the temperature correction, let p' be the observed pressure. Corresponding to this pressure, the height in the standard atmosphere (that is the height registered on the aneroid or given in the table) will be h and the pressure at sea level in this atmosphere will be p_{se}, the temperature there being the standard temperature T_{se}. Assuming the same sea level pressure at the time of observation but a new sea level temperature T_e,* the observed temperature at height h_e will be $(T_e - k \,.\, h_e)$. Accordingly, we may write

$$\frac{p'}{p_{se}} = \left(\frac{T_{se} - kh}{T_{se}}\right)^{\frac{g}{kc}} = \left(\frac{T_e - kh_e}{T_e}\right)^{\frac{g}{kc}},$$

which reduces to

$$h_e - h = \frac{h(T_e - T_{se})}{T_{se}}.$$

so, assuming that $T_e - T_{se}$ is the same as $t - t_u$, and, as $T_s = 518°$, this gives :—

$$h_e - h = \frac{h(t - t_u)}{518} = 0{\cdot}001929h(t - t_u).$$

Sources of Error. These may be classed as :

(1) Errors due to Natural Causes ;
(2) Instrumental Errors ;
(3) Errors of Observation ;
(4) Errors of Reduction.

(1) Errors due to atmospheric conditions are the most important. The most difficult of elimination are those arising from the existence of permanent gradient and the impossibility of accurately estimating the temperature and humidity of the air column. Observations must, of course, be suspended during storms, and should not be made in situations where wind eddies will cause abnormal readings, as may occur on the leeward side of obstructions.

(2) Instrumental errors are of minor importance in the case of a good mercurial barometer if the necessary precautions are taken in using and transporting it. The instrument with its attached thermometer must have been compared with a standard, and a table of corrections obtained.

The ordinary aneroid, on the other hand, cannot be regarded as a precise instrument. Accurate graduation is a matter of considerable difficulty, and the readings should be compared under various temperatures with those of a standardised mercurial instrument, and the corrections noted. These corrections do not, however, remain constant with the lapse of time. The variations arise from changing elasticity of the vacuum box and the mainspring, wear, and temperature effects (in

* The assumption that the pressure at mean sea level is the same at the time of observation as it is in the standard atmosphere leads to the omission of a term which, for absolute determinations at a single station, may not be negligible, but which is eliminated when the difference of height between two stations is being determined. Assuming that there is no change of pressure at sea level between the observations at the two stations, the value of this term is $\frac{c}{g} \,.\, T \,.\, \frac{\delta p}{p_{se}}$, where $\delta p = (p_e - p_{se})$. See paper by G. T. McCaw on " New Aneroid Tables " in the *Empire Survey Review*, Vol. III, No. 16, 1935.

"compensated" as well as uncompensated instruments). Refined reading is impracticable. The fineness of reading possible depends upon the range of the instrument, as the graduation is, of course, more open in an instrument designed for reading elevations from 0 to 1,000 ft. than in one ranging from 0 to 10,000 ft. For work covering large variations in altitude, aneroids of different ranges are carried, *e.g.* 0 to 5,000 ft., 4,000 to 10,000 ft., etc., overlaps being necessary since the indications are uncertain towards the extremes of the scale.

(3) In observing, precautions are necessary to avoid anomalous indications arising from unrepresentative temperature conditions and sluggishness or drag in the instrument. Barometers should therefore be carefully shielded from the rays of the sun. The existence of drag is evident after a sudden change of elevation, as the instrument does not respond immediately. After ascending or descending a steep slope, a stoppage should be made for a few minutes, and the barometer then gently tapped and read.

Errors of parallax in reading the instruments are almost always present. In the mercurial instrument they occur in inaccuracy of contact between the gauge peg and the surface of the mercury, in setting the vernier index, and in reading the thermometer. Parallax error in reading the aneroid is considerably greater, and depends upon the distance between the pointer and the dial. This error, however, can be greatly reduced or practically eliminated if the instrument is provided with a "parallax mirror" (page 449) set concentrically with, and close to, the scale. Errors of reading are reduced to negligible limits by repetition of observations.

(4) The formulæ by which differences of barometer readings are reduced to differences of elevation are not of an exact nature, but it may be taken that errors thus introduced are negligible in comparison with those due to the foregoing causes.

Limits of Error. Looking to the uncertainties inherent in barometric levelling as performed in taking topography, it is very difficult to set a value on the degree of accuracy which may be expected. Some remarkably close agreements with the results of spirit levelling have been obtained by prolonged observations with mercurial barometers, but such methods are usually impracticable in ordinary topographical surveying. The errors arising from the use of an aneroid are, of course, greater than those arising from the use of mercurial barometers, and, with a single aneroid in the field controlled by simultaneous observations at the base station, the error of any single observation may be anywhere between 10 and 100 ft., but this error may be reduced to from 5 to 20 ft. if batteries of not less than 3 aneroids are used at a time and atmospheric conditions are relatively stable. In some parts of the tropics, duplicate runs, or even single runs over lines already levelled by spirit levelling, with corrections applied from the diurnal wave, have shown comparatively small discrepancies of from 2 to 20 ft., but it is known that single errors much larger than this will sometimes occur. Adjustment between fixed points of known height will, of course, do something to reduce error taken on the whole but it will not appreciably reduce errors of large amount. These remarks apply to work with ordinary aneroids, but much greater accuracy may be obtained if instruments of the Paulin type are

used. The whole question of accurate height determinations has assumed new importance in connection with air survey, and research is now going on, or is likely to be commenced, with a view to improving the accuracy of aneroid and altimeter determinations. Already the Paulin instruments have set new standards for ground survey work, and excellent results have also been obtained with the surveying altimeter manufactured for the U.S. War Department by the Wallace and Tiernan Products Inc., of Belleville, New Jersey, U.S.A., an instrument which, like the Paulin, is remarkably free from hysteresis and drag.

Errors in levelling will, in general, tend to be greatest at high altitudes and at latitudes near the poles and equator.

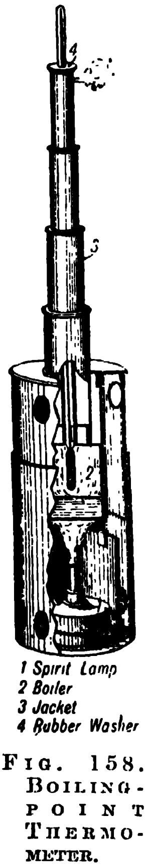


FIG. 158. BOILING-POINT THERMOMETER.

The Boiling-point Thermometer. Fig. 158 illustrates the usual portable form of boiling-point thermometer or hypsometer. Sensitiveness is an essential requirement in the thermometer, which is graduated from about 180° to 215° F. and divided to 0·2°. The thermometer is held by the rubber washer *4*, so that the bulb is immersed in the steam in the boiler, and the stem is subjected to a current of steam in the telescopic jacket *3*.

In using the instrument, the boiler is about one-third filled with rain-water. The thermometer is introduced through the tube leading from the boiler, and is adjusted so that the bulb is just clear of the water. The spirit lamp is lit, and the thermometer is read when the mercury becomes stationary. When not in use, the thermometer is carried in a brass tube with a rubber lining.

Observations and Reduction. The general procedure in observing is the same as that with the barometer. Simultaneous observations give the best results, but are often impracticable. In the method of single observations, if the surveyor is remote from the reference station, he must rely upon estimated values for the atmospheric pressure there at the times of his various observations. Mean pressures at different seasons are given in published charts.

Before reducing, the thermometer readings are corrected for index error, which should be ascertained periodically. Difference of elevation may be obtained from barometric tables if the barometer readings equivalent to the boiling-points are computed. Several tables of elevations corresponding to boiling-points are, however, available. Examples will be found in *Hints to Travellers* and Close and Winterbotham's *Text Book of Topographical and Geographical Surveying.*

In the absence of a table, the following empirical formula may be used :

$$E = t\,(521 + \cdot 75\,t) \times a,$$

where E = elevation in ft. above that at which water boils at 212°,

t = degrees Fahr. of boiling-point below 212°,

a = air temperature factor of the barometric formulæ.

Only rough results are possible since the thermometer can be read only to the nearest ·01 in., which corresponds to over 50 ft. of elevation.

THE SURVEY OF DETAIL

With the exception of photographic surveying, the methods for the survey of detail tabulated on page 437 have already been dealt with in this volume and in Vol. I, Chap. III, IV, VI, VII, VIII and XI. Features of the work special to small scale mapping are given below, but the subjects of theodolite traverse and tacheometry are not treated further. Although excellent for close contouring, tacheometry, on other than the subtense bar system, is of minor utility for rapid work on small scales.

Plane Tabling. In view of the great importance of plotting detail in the field, plane tabling is the most extensively used method for the survey of topography. It is particularly useful in open ground, but for surveying through tracts of bush the plane table is employed only to a limited extent in small scale mapping, and compass traversing is preferred. The methods of plane table surveying have been treated in Vol. I, Chap. VII, and it remains to outline here the routine followed in the mapping of detail on topographical and geographical scales.

Field Sheets. The sheets on which the field mapping is performed are called field sheets. The scale adopted is commonly that on which the final map is to be made, but if it is intended to reproduce the survey on more than one scale, the largest must be used for plotting the field sheets. In country favourable for plane tabling the field sheets are completed on the plane tables. If, however, the survey includes compass traversing in parts, such work is usually better plotted on a larger scale in the first instance. These subsidiary surveys are laid down on auxiliary sheets, and are subsequently reduced and incorporated in the field sheets in camp.

Each field sheet is to contain a definite part of the survey, bounded by meridians and parallels, and commonly embraces a quarter of the area included in the final map sheets. A degree of latitude and of longitude can be taken on the 1/250,000 scale, and proportionately less on larger scales. This leaves sufficient margin on a medium-sized board.

Preparation of Field Sheets. Before proceeding to take topography, the surveyor must prepare his sheet, after it has been properly seasoned, by drawing the graticule and plotting the positions of the trigonometrical points by which his mapping will be controlled. The graticule, or network of meridians and parallels, for the area embraced by the field sheet is constructed as described on page 498. The graticule interval must be a simple fraction of a degree, the smaller the greater the scale. It is generally made 15′ for mapping on geographical or topographical scales, and may be subdivided to 5′. The graticule must extend beyond the limits of the area to be surveyed to give a surrounding margin of sufficient breadth to contain trigonometrical points which, although outside the section, will be used to control detail near the edge.

As soon as the graticule is inked in, the positions of triangulation and traverse stations and of intersected points are plotted thereon as described on page 500. In addition to checking the plotting by scaling between

the points, their plotted positions are verified in the field. The table is set up at one of them which occupies a commanding position, and is oriented by the ray to any other. The plotting of all stations within view is then checked by sighting them successively.

Field Work. Plane table stations are almost exclusively located by resection from trigonometrical points. The stations are situated as far as possible on elevated ground, but occasional fixings are required on the lower ground for the survey of detail which cannot be obtained from the hill stations. It is economical to work from high ground to low. The distance between plane table stations naturally depends upon the openness of the country, and is likely to be greater with an experienced than with an inexperienced topographer. On the easiest ground four or five fixings per square mile are recognised as sufficient for work on the 1/62,500 scale. For smaller scales the number of points occupied may be considerably reduced.

Detail is surveyed by sketching between a few intersected points. Elevations are taken by vertical angles with the telescopic alidade or the Indian clinometer (Vol. I, page 351), by aneroid, or by a combination of vertical angles and aneroid readings. The elevations of plane table stations are obtained by observing the vertical angles and scaling the distances to two trigonometrical stations, one result serving to check the other. The elevations of intersected points are obtained from those of the plane table stations by similar measurements, and the contours or form lines are then sketched in by estimation. Barometric levelling is a valuable adjunct for determining spot heights around a station or on the route between stations. As the scale decreases, fewer points are fixed for the control of sketching: the requirements of geographical mapping are met by sketching topography entirely from the plane table positions and elevations.

The art of sketching topographic form is one which demands considerable judgment. The inexperienced surveyor usually errs by including unimportant detail too small to show clearly upon the finished map. The topographer should possess a sound knowledge of land forms in order that he may select for representation the significant features which express the character of the region. The ability to sketch these with the desired accuracy is acquired by experience. The natural tendency to exaggerate the roughness of difficult country and the smoothness of flat ground should be recognised and avoided.

The area mapped in each field sheet should extend beyond the strict boundary of the section by the inclusion of an additional strip, about half a mile to a mile wide, round it. This overlap between adjoining sheets is necessary to ensure a good connection between them by the adjustment of small discrepancies.

Compass Traversing. Recourse is had to compass traversing (Vol. I, page 245) between theodolite stations for the survey of detail in thickly wooded regions. The lengths of the course are usually strictly limited by the nature of the country, and this adds to the other inaccuracies of the method. When a wide region has to be surveyed by compass, the area is divided up by systems of deliberate compass traverse forming a framework from which the bulk of the detail is surveyed by rapid traverses

of short length. For economy, the framework itself usually comprises traverses of different grades of precision, and in extended work may consist of primary, secondary, and tertiary systems.

Primary Compass Traverse. Routes for primary traverse are selected over ground favourable for linear measurement, and so that as long courses as possible may be obtained with a minimum of clearing. The instrument is either a circumferentor or a large size prismatic mounted on a tripod. Bearings are read to the nearest 5′, both forward and back bearings being observed at each station. The steel tape is stretched under a constant tension, and the slopes are measured by clinometer. A double measurement is made as a check. The chainages at which roads, streams, etc. are crossed are entered in the field book, and important features of detail on either side are surveyed by intersection, bearing and distance, or rough offsets. Clinometric or aneroid heights are observed along the traverse.

When there is a considerable distance between control points, primary traverses should be arranged to intersect each other and form closed figures capable of adjustment. Traverses must also fit between the control points on the theodolite framework. A graphical adjustment is usually sufficient in small scale work.

Secondary and Tertiary Compass Traverse. Secondary traverses are run between stations of the main framework or of the primary compass traverses. Because of the smaller distance between checks, rougher work is allowable, and greater speed is attained. A 4-in. prismatic compass on a tripod or staff is suitable, and forward and back bearings are read to the nearest 10′ or 15′. Ordinary steel taping is sufficient, the tape being held horizontally or the inclinations measured. Distances should be checked for mistakes by pacing.

In tertiary traverse the forward and back bearings need only be read to the nearest degree with the prismatic held in the hand. Steel taping is commonly employed, but the use of a light steel wire rope 200 ft. or 300 ft. long is sometimes preferred on account of its superior strength. A chain is objectionable in dense undergrowth.

It is usually sufficient to plot secondary and tertiary traverses by protractor and to make a graphical adjustment between control points.

Rope and Sound Traverse. This method of rough traversing is much used in certain heavily forested parts of the tropics for supplying relatively unimportant detail, such as semi-permanent forest paths, minor streams and the positions of spot heights. Distances are measured with light hempen ropes 310 ft. long and magnetic bearings are observed on small hand compasses—2-in. liquid compasses are generally used—to a sound made by whistle or shouting at a point three rope lengths ahead. The extra 10-ft. length on the rope is supposed to make allowance for average twist of path so that each rope length counts as 300 ft.

It has been found by experience that a fairly experienced observer can estimate the direction of the sound signal to within about 2° or 3°. In systematic mapping on the 1/62,500 scale, rope and sound traverses are usually not allowed to exceed about 6 to 8 miles in length, and they begin and end on points established by a more accurate order of survey except when several traverses, which all begin at fixed points, end at a

common central point. Aneroid readings are commonly taken at the same time as the bearing is read—that is, at the end of every third rope length—but, if the ground is very broken, or any sharp breaks in slope occur, intermediate readings may be taken.

Traverses of this kind are plotted on the auxiliary sheets in the field as the work progresses. The aneroid readings are also corrected for the diurnal wave and the form lines sketched in at the same time.

Measurement of Distance with Range Finder. The ordinary range finder, such as is used for military purposes, enables the lengths of lines of medium length to be measured with sufficient accuracy for much topographical work, and is specially useful when a compass or minor theodolite traverse is being carried along a wide river, or in other similar cases

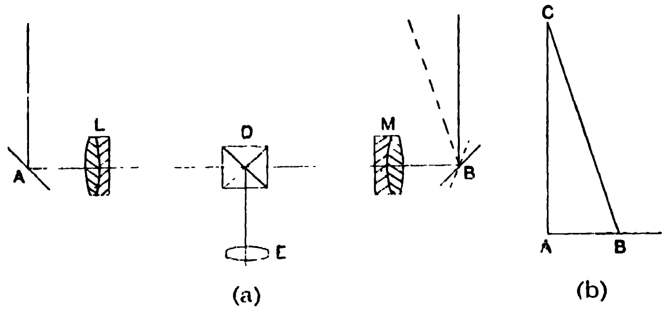


FIG. 159.

where distances have to be measured over gaps that are inaccessible for ordinary chaining and are too long for observation by tacheometer.

The principle of the range finder is very simple and is illustrated in Fig. 159. Here A and B are two mirrors which, in an instrument intended for surveying, are set at the ends of a base about 1 metre long. Mirror B is movable but A is fixed at an angle of 45° with the base AB. Rays falling on A and B pass through the objectives L and M and are reflected by the prisms at D into the eyepiece E. When a relatively near object is sighted, two separated images are seen in the eyepiece, but these can be brought into coincidence by rotating the mirror B. The amount of rotation required to bring the two images into coincidence is a function of the distance of the object, and this distance can be read directly on a suitably graduated scale. What the instrument does, in fact, is to solve the triangle ABC, Fig. 159 (*b*), for the side AC, the base AB being the distance between the centres of the mirrors and the angle ACB twice the angle through which the mirror at B is rotated.

The accuracy of the range finder falls off very quickly as the distance is increased; in fact, the theoretical error varies as the square of the distance. Thus, R.Ll. Brown gives the following figures for the approximate uncertainty of observations under favourable conditions with instruments having bases 80 cm. and 1 m. long respectively: *

* See "Optical Distance Measurement," by Major R.Ll. Brown, R.E., *Empire Survey Review*, Vol. I, No. 4, April, 1932.

Distance. Yards.	Uncertainty of Observation.	
	80 cm. Base.	1 m. Base.
	Yards.	Yards.
500	1·2 = 1/416	1·0 = 1/500
1,000	4·7 = 1/213	3·8 = 1/263
2,000	19 = 1/105	15 = 1/133
4,000	76 = 1/53	61 = 1/66
8,000	. . . . .	244 = 1/33

The chief disadvantages of the range finder are its weight and expense.

Route Traverse. Under this heading may be classed all methods of traverse of lower grade than tertiary. They are extensively used in reconnaissance and exploratory surveys, when speed is of greater importance than precision. Such work is sometimes controlled only by astronomical determinations, and the resulting map of the route followed is necessarily of inferior accuracy. On the other hand, rough traversing between frequent control points is a useful method of surveying thickly wooded belts in deliberate mapping. Detail on either side of the route is surveyed by intersection of compass bearings, bearing and distance, and by sketching.

Bearings are observed as in tertiary traverse, but in thick bush it is sometimes impracticable to obtain sights of more than a few yards, and bearings are taken towards the sound of a whistle at the invisible forward station as in a rope and sound traverse. The linear measurements of the traverse legs and for the inclusion of adjacent detail are made by rough methods, such as by wheel, pacing, time, and sound.

Measurement by Wheel. On level ground remarkably good results are obtained by running a wheel along the line and determining the distance as the product of the number of revolutions by the circumference of the wheel. The reliability of the method is reduced on rough ground owing both to the presence of slopes and the effects of jolts. The revolutions may be counted by watching a piece of bunting attached to one of the spokes, or the count is performed automatically by odometer or by perambulator. The odometer is a device which is fixed to the wheel of a vehicle and exhibits on a dial the number of revolutions made or the distance run. The perambulator resembles the front wheel of a bicycle with fork and handle-bars : the wheel is usually of wood, and the tyre of hard brass. The recording mechanism moves a pointer over a small dial graduated to read to yards.

Measurement by Pacing. The ability to maintain a constant length of pace is acquired by practice. For the same person the length of the pace increases with increase of speed, and decreases with increase of slope of the ground, whether going uphill or downhill. The surveyor should ascertain the length of his step by walking at average speed over a known distance on level ground. In measuring by pacing, he should keep to his natural step instead of trying to pace yards, but the stride should be lengthened on going up or down slopes as an attempt to cover a constant

horizontal distance at each pace. It is usual to count only every second step, *i.e.* those of either the right or the left foot.

The trouble of counting is eliminated by carrying a pedometer. Externally this instrument resembles a watch, and it is carried upright in the pocket. The jolt given at each step actuates the mechanism, and the distance walked is recorded on the face, which is graduated to $\frac{1}{4}$ mile and is read by estimation to $\frac{1}{10}$ mile. The instrument is adjustable to suit the length of pace of the user. In a very similar instrument, called the passometer, the readings represent the number of paces only, so that no adjustment is required for its use by different observers.

If the surveyor is mounted, distances may be obtained by counting the paces of the animal, but the accuracy is rather less than that of human pacing because of the effect of slopes and varying speed. The value of a pace should be ascertained by riding a measured distance both at a walking gait and a trot, the number of strides of one of the forelegs being counted. A pedometer may be carried and calibrated in this manner for use on horseback.

Measurement by Time. In this method distances are estimated from the time taken in travelling. Note is made of the times at which the beginning and end of each traverse course is reached as well as of the estimated rate of march between. The method is useful for work on horseback, since the speed of a particular animal is fairly constant when either walking or trotting. It is frequently applied in running traverses on rivers. Experienced oarsmen can maintain a nearly uniform stroke, and the speed of the boat can be determined over a known distance when rowing with and against the stream. If the traverse is made with the boat drifting downstream, the speed of the current should be ascertained at intervals by timing the boat over a measured distance. When a power launch is used, its speed should be measured for various rates of running the engine.

Measurement by Sound. This consists in firing a gun at one end of the line and noting at the other end the time which elapses between seeing the discharge and hearing the report. Two or three repetitions are made, and the results are averaged. It may be taken that sound travels at the rate of 1,090 ft. per second in still air at a temperature of 32° F. and that the speed increases by 1·1 ft. per second for every degree rise in temperature. The accelerating or retarding effect of wind is eliminated by firing and observing at each end of the line, the mean result giving the required measurement.

When the stations are not intervisible, two observers, A and B, take up positions at either end of the distance to be measured : each is provided with a revolver, and A has a stop-watch. A fires, and observes the time. Immediately on hearing the report, B fires, and A notes the time at which the sound reaches him. The interval between the two times observed by A is found to be greater than that required for sound to travel from A to B and back to A, chiefly on account of an inevitable delay by B between hearing A's shot and replying. Huddart * finds that the subtractive correction to the time interval is practically independ-

* "Sketch Mapping with Special Reference to Southern Nigeria." *Min. Proc. Inst. C. E.*, Vol. CLXIX.

ent of the distance, and with experienced observers is fairly constant at 0·7 to 0·8 sec.

Route Sketching. It is frequently desirable to make a scale sketch of the topography while running a route traverse. To facilitate sketching, various types of sketch board have been designed in the form of a miniature plane table with an attached compass, whereby bearings to side objects, as well as those of the traverse, may be observed. Fig. 160 shows the Verner Cavalry Sketch Board. The instrument is strapped to the left forearm for use on horseback, but can also be mounted on a Jacob staff when required. The board measures about 9 in. by 7 in., and carries two rollers over which a continuous roll of paper is passed. Orientation is performed by reference to the small compass fitted on one side. The functions of an alidade are performed by a scale which is held in any required position by two rubber bands as shown.

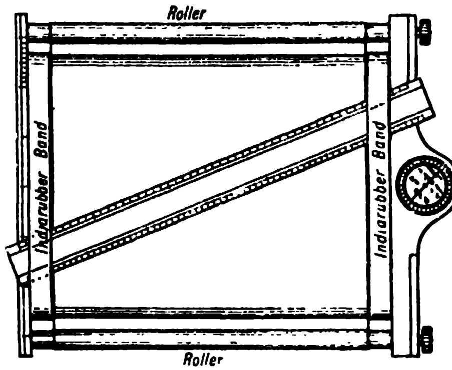


FIG. 160. CAVALRY SKETCH BOARD.

PHOTOGRAPHIC SURVEYING

Photographic surveying, or phototopography, is a method of surveying in which the detail is plotted entirely from photographs taken at suitable camera stations. The method has been extensively employed throughout the Continent and in Canada. It is adapted for the small scale mapping of open mountainous regions, but, unless the photographs are taken from the air, it is useless in flat or wooded country.

In its general features, photographic surveying resembles plane tabling by intersection and resection, but, as the plotting is performed in the office, the accuracy obtained in the representation of topographic detail is necessarily inferior to that of field sketching. The principal merit of the system is the rapidity of the field work: the time spent in the field may be put at about one-third that of plane tabling. This proves a valuable feature under adverse climatic conditions. In mountainous country there may occur only brief intervals during which the peaks are clear of mist, and the operations at a camera station can be completed with much less delay than would occur with any other system. Moreover, the method can be used to survey very precipitous and mountainous country which it would be quite impossible to survey by ordinary ground methods, and it is mainly for this reason, and because of the necessity for devising some means of dealing with surveys in the Rocky Mountains, that the practical development of the method is very largely due to the late Dr. E. Deville, who was Surveyor-General of Canada from 1885 until 1924. The original inventor of photographic surveying, however, was Laussidat in France, who commenced experimenting in 1851 and in 1861 completed a survey of a village near Versailles.

General Principles. In plotting the map, the distances and elevations required must be obtained from the perspective dimensions on the photographs. The process—termed iconometry—is therefore the reverse of perspective drawing.

In Fig. 161, O represents the optical centre of the camera lens. OP, its principal axis, intersects the vertical negative at P, the principal point of the picture, of which HH′ is the horizon line and VV′ the principal line. The image thrown on the plate is an inverted perspective, but an upright perspective of the same dimensions can be imagined received on a glass plate parallel to the negative and placed, as shown by dotted lines, in front of O with its principal point on the axis of the lens and at a distance from it equal to OP. Since the lens is focussed for distant objects, OP equals f, the focal length of the lens. This perspective corresponds to the photographic print, which is therefore a perspective with view point O.

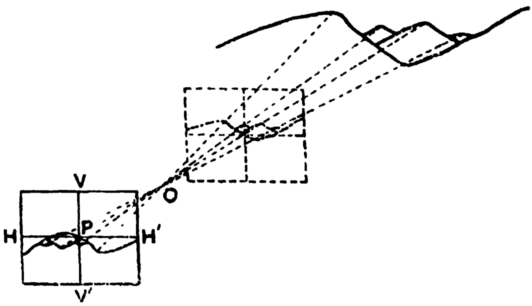


FIG. 161.

A single photograph does not furnish the complete data required for plotting the features it includes. If, however, a second photograph of the same area is taken from another known station, and the orientation and elevation of the camera at the two stations are known, the points appearing in both photographs can be located both horizontally and vertically. Thus, in Fig. 162, let O and O′ be the plotted positions of the stations, then the data regarding the orientation of the camera will enable XX and YY, the picture traces, to be correctly placed on the plan at f in. in front of O and O′ respectively. The abscissæ P_1a_1, P_1b_1, P_2a_2, P_2b_2, etc. are transferred by dividers from the prints to the corresponding picture traces, and rays from O through the points a_1, b_1, etc. thus obtained will yield with corresponding rays from O′ the required intersections A, B, etc.

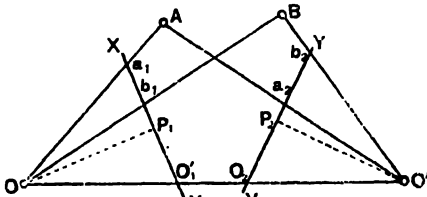


FIG. 162.

The elevations of intersected points can be ascertained from that of either camera station. The effects of curvature and refraction may be neglected, so that the horizon line on a print intersects points whose elevation is that of the centre of the camera. To obtain the elevation of any other

point, such as A, the ordinate Aa_1 is measured on the print. By similar triangles, this represents an actual distance of

$$h = \frac{Aa_1}{Oa_1} . OA,$$

and the elevation of the distant point = the elevation of the camera station + the height of the camera above the ground $\pm h$.

The Surveying Camera. An ordinary camera may be utilised for surveying if it is fitted with spirit levels so that the photographic plate may be set accurately vertical. The instrument should have a ground glass screen through which the view embraced by each photograph may be examined to ensure that desired features near the sides are included. The lens should have a large field free from distortion. A knowledge of the distance from the optical centre of the lens to the sensitised surface of the plate is required in plotting, and it is therefore a great advantage to have a camera of the fixed focus type.

The labour of plotting is much reduced, and the camera is adapted for survey work, if means are provided for exhibiting on the negatives the trace of the horizontal plane passing through the centre of the lens, as well as the position of the principal point, in which the optical axis meets the plate. Their positions may be marked by four needles or notches in the frame against which the plate rests, and these are photographed upon the negative at each exposure. Two of the marks define the horizon line and the other two the principal line, and the intersection of lines drawn through them on the print or negative gives the principal point. Alternatively, these lines are reproduced on the negative by having the frame fitted with hairs which are stretched just in front of the plate.

If the camera is provided with no other attachment, it must be used in conjunction with a theodolite, horizontal angles being taken to fix the positions of camera stations and to orient the photographs, and vertical angles to control elevations. To economise weight, the same tripod and base plate may be used both for the theodolite and the camera, as in Canadian practice. In Europe the surveying camera has been combined with the theodolite to form the photo-theodolite.

The Photo-Theodolite. Several combinations of theodolite and camera have been devised and used.* The instrument designed by Mr. Bridges-Lee, and made by Messrs. C. F. Casella and Co., embodies all the essential features of a photo-theodolite in a simple form, and is illustrated in Fig. 163.

The camera is of the fixed focus type for 5 in. by 4 in. plates, and is mounted on an axis in the same manner as the upper plate of a theodolite. It carries a vernier by which the horizontal circle is read to single minutes. Upper and lower clamps and tangent screws are fitted as in the theodolite. A telescope with a vertical arc is mounted on the top of the camera box, and cannot be moved in azimuth relatively to the camera, the line of sight being in the same vertical plane as the optical axis of the camera lens.

Inside the camera box is a vertical frame 1, carrying a vertical and a horizontal hair KK', situated in the same vertical and horizontal

* See Flemer. *Phototopographic Methods and Instruments.*

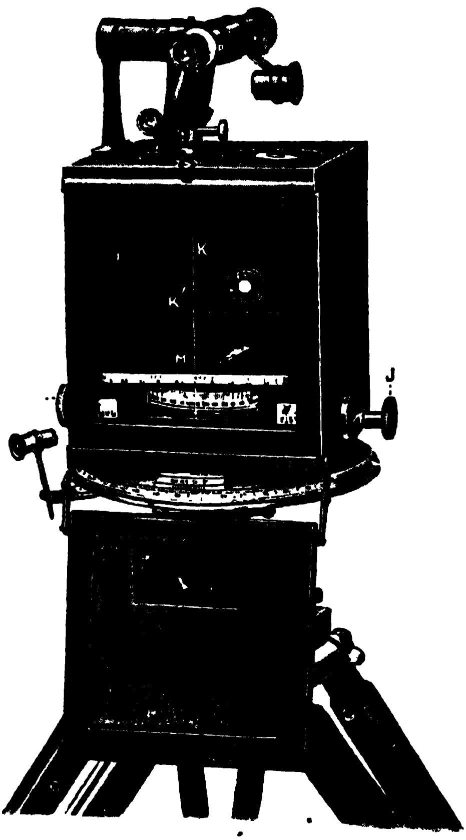


FIG. 163. BRIDGES-LEE PHOTO-THEODOLITE.

planes respectively as the optical axis. Attached to the same frame there is a horizontal transparent scale of angular distances, the graduations of which serve to show to the nearest 5′ the angular distances from the vertical line of points on the picture. The frame I is rigidly connected to a base plate which supports a circular magnetic compass M, the needle of which carries a vertical cylindrical transparent scale divided to half-degrees.

When the photographic plate is in position and the shutter of the dark slide is drawn out, the internal frame can be racked back by the screws J until the cross-hairs just touch the plate. The scale of angular distances is at the same time brought close up to the plate, and both it and the hairs are reproduced on the negative. In addition, the action of racking back places the compass needle on its pivot, and brings the compass scale sufficiently close to the surface of the plate that the graduations in the neighbourhood of the vertical hair are also distinctly shadow-graphed. The reading of the compass scale at which it is intersected by the vertical line on the photograph represents the magnetic bearing of the pointing, so that the orientation of each view is automatically recorded. The bearing to any other point is obtained by applying to that magnetic bearing the angular distance of the point from the principal plane as shown on the scale of angles.

Determination of Focal Length of Camera Lens. If the focal length f of the lens is unknown, it should be ascertained at the outset. The method given by Deville is as follows.

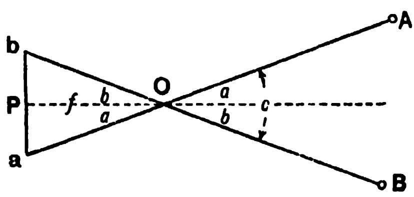


FIG. 164.

The horizontal angle c (Fig. 164) subtended at the camera station O by two distant points A and B is measured by theodolite. A photograph is taken to include these points, and their distances

Pa and Pb from the principal line are measured on the negative. Then

$$\frac{\text{Pa}}{f} = \tan a, \text{ and } \frac{\text{Pb}}{f} = \tan b,$$

$$\text{but } a + b = c,$$

$$\text{whence } \tan c = \frac{\dfrac{\text{Pa} + \text{Pb}}{f}}{1 - \dfrac{\text{Pa} \cdot \text{Pb}}{f^2}}, \text{ a quadratic for } f.$$

Determination of Principal and Horizon Lines. The positions of these lines must be obtained and marked in the case of a surveying camera out of adjustment or in using an ordinary camera for surveying. A photograph of a suspended plumb line gives the directions of the lines. By including in the photograph three points subtending two known angles at the instrument, the position of the principal line is obtained as follows.

In Fig. 165 let A, B, and C be known points in plan or points subtending two observed angles at the camera station O. On the photograph draw any line perpendicular to the image of the plumb line, and project the images of A, B, and C upon it. Transfer the points a, b, and c so obtained to a paper straight-edge, and move the strip over the plot until a, b, and c simultaneously fall upon their respective rays OA, OB, and OC. The edge of the strip now coincides with the oriented picture trace, and a perpendicular OP upon it defines the position of P relatively to a, b, and c. The position of the principal line is therefore obtained, and OP should measure f.

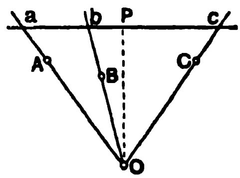


FIG. 165.

To find the position of the horizon line, the vertical angle α from the camera station, or the distance and relative elevation, of at least one pictured point A must be known. The ordinate h of the image of that point from the horizon line is computed from

$$h = \text{Oa} \tan \alpha,$$

and a point on the horizon line is obtained by setting off from the image of A the distance h parallel to the principal line. As a check, other points are obtained in the same way.

Alternatively, the horizon line may be located by sweeping a horizontal plane through the camera station with a level and noting definite points cut by it. These are subsequently identified and marked on the photograph.

Field Work. The entire control of a photographic survey may proceed simultaneously with the taking of the photographs, and the survey is then similar in principle to graphic triangulation. A photographic survey is, however, generally based upon a previously executed system of triangulation. Photographs are taken from such of the survey stations as are suitable for the purpose. Many other camera stations will be found necessary in order to obtain suitable views of the whole region, and these are located by subsidiary triangulation, the angles of which are

observed either by the photo-theodolite or by a separate theodolite. It is frequently convenient to locate camera stations by resection or interpolation from observations on three triangulation stations.

The selection of camera stations is an important part of the field work, involving consideration of the requirements of the subsidiary triangulation as well as the location of detail. All features to be plotted must be photographed from at least two stations so situated that the lines joining them to the point give a definite intersection there. The more important points should appear on three or more photographs. Sites should be chosen with a view to the avoidance of as much dead ground as possible so as to do away with the necessity for unnecessary extra camera stations.

For the orientation of the views, each should contain at least one point of known position, such as a triangulation station or a point whose direction is measured from the camera station. This is desirable even when the photo-theodolite is used and the orientation of the picture is read on the circle as well as being automatically recorded in terms of the magnetic bearing. As an additional check in plotting, adjacent views should overlap by an amount sufficient to make an easily recognised point appear in both. The elevation of each camera station is obtained by trigonometrical levelling, usually by observation of the vertical angles to two triangulation stations. In cases where no station appears on a photograph, the vertical angle to at least one conspicuous point in it should be observed.

In arranging the work at the camera stations, consideration should be given to the time of day at which each should be occupied so that the sun may be favourably situated with respect to the camera and the view. A certain amount of shadow is useful, but areas totally in shadow should not be photographed, and, as a rule, the best results are obtained during the middle part of the day.

In addition to keeping an angle book, the surveyor should make a sketch of the view embraced by each photograph. On these sketches are shown the approximate positions of triangulation stations included, the points to which angular observations have been taken, and the names of peaks, rivers, roads, etc.

Preparation of Photographs for Plotting. Needless to say, the negatives must be as sharp in detail and with as much contrast as can be secured. Rather slow isochromatic plates should be used, and the exposure made with a small aperture. The plotting may be performed either from the negatives or from prints. The direct use of negatives is the less convenient but more accurate method, since prints rarely correspond in size with the negatives. It is usual to work with enlargements of two to four times the linear dimensions of the originals in order to reduce errors arising in taking dimensions from the photographs. Great care in enlarging is necessary to avoid distortion, and the enlargements should be tested for perceptible distortion. The use of enlargements on glass has been recommended for the plotting of important features.

Orientation of Picture Traces. The accuracy of the plotting depends upon the correct placing of the picture traces upon the plan just as a plane table plot is dependent upon the orientation of the table. The conditions that the principal point P should be at the focal distance

from O, the plotted position of the station, and that the picture trace should be perpendicular to OP, define the trace as tangent to a circle with centre O and radius f. The laying down of the picture traces is independent of the scale of plotting, dimensions from the photographs and the focal length being drawn full size. When enlargements are used, the enlarged focal length is laid down.

The method of placing the picture traces on the plan depends upon the data available.

(1) When the photo-theodolite is used, and the horizontal circle is read at each exposure, the bearing of the principal plane is known, and it is only necessary to set off this bearing, mark the point P at a distance f from O, and draw the trace perpendicular to that of the principal plane. The result should, however, be checked by one of the following methods.

(2) When the photograph includes a point of known position already plotted on the plan or one of known direction from the camera station, the orientation may be performed with respect to it. In Fig. 166, let A be the position of a known station, and let its distance from the principal plane on the photograph be Pa. On OA set off $Oa_1 = f$, and from a_1 erect a_1a_2 perpendicular to OA, and such that $a_1a_2 = Pa$. Oa_2 is evidently the trace of the principal plane, and, on marking off $OP = f$, a perpendicular through P represents the picture trace. Alternatively, the length Oa is calculated from $\sqrt{f^2 + Pa^2}$, and triangle OPa is constructed upon it.

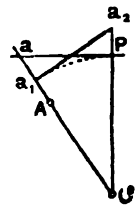


FIG. 166

(3) When the photograph includes two or more points of known position or direction, the orientation may be performed mechanically by means of a paper straight-edge on which are marked P, a, b, etc. The method for three or more points is given on page 471 and Fig. 165, the orientation being checked by the perpendicularity of the trace to OP and the length of OP. In the case of two known points A and B, the point P on the paper straight-edge must be kept upon the circumference of an arc with centre O and radius f, and the strip adjusted until a and b simultaneously fall upon the rays OA and OB.

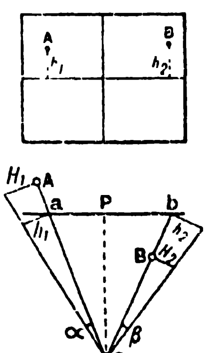


FIG. 167.

Plotting. Before plotting detail, the prints are carefully studied, and every salient point to be plotted is marked with a fine dot and numbered. The same points must be identified and similarly numbered on two or more photographs. To locate the marked points on the plan, a convenient method is to mark off their distances from the principal plane on the straight edge of a strip of paper, which is then fixed on the drawing with its edge along the appropriate picture trace. A similar strip is prepared from each photograph, and the lines through the station points and the points in the traces intersect at the required positions.

The elevations of the plotted points may be computed as on page 471 or may be determined graphically. Thus, in Fig. 167, to obtain the elevations of A and B

relatively to that of the camera, their ordinates h_1 and h_2 from the horizon line are taken from the photograph and set off at a and b on

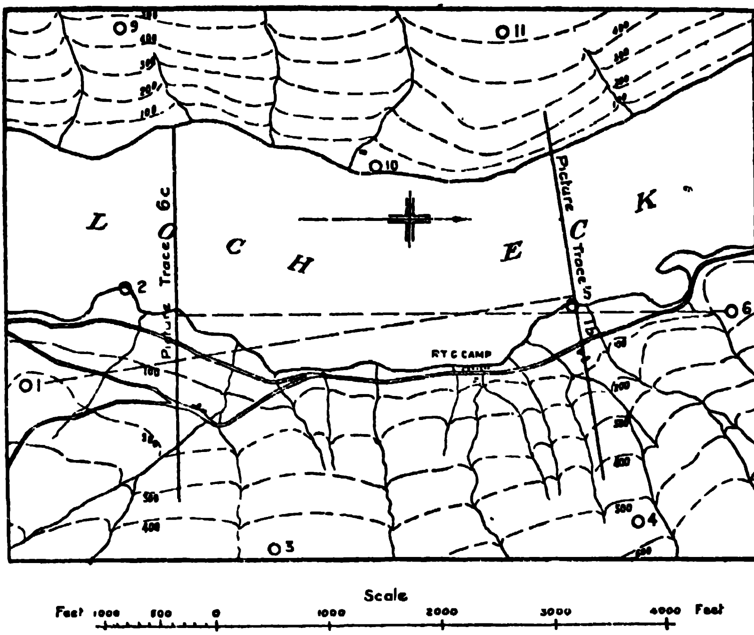


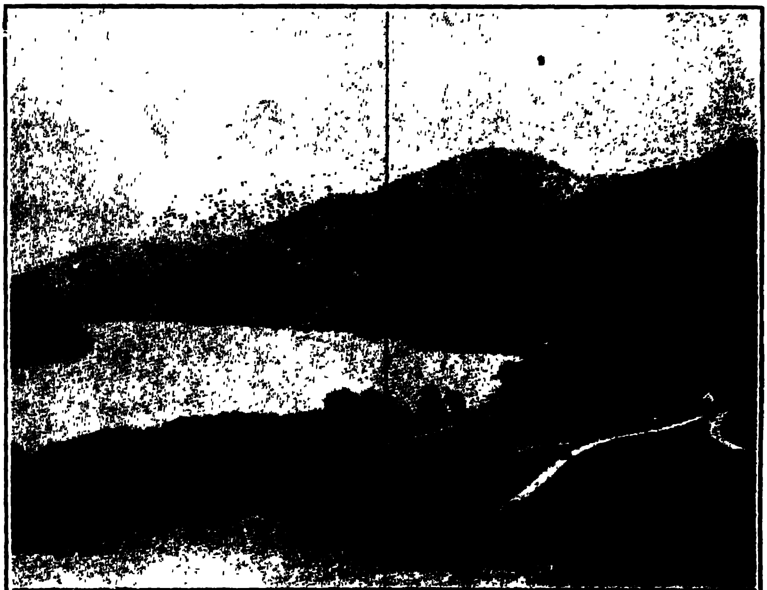
FIG. 168. PHOTOGRAPHIC SURVEY.

the plan as perpendiculars to Oa and Ob respectively. The angles α and β subtended at O are the true vertical angles to the points, and the intercepts H_1 and H_2 perpendicular to the rays from the plotted positions of A and B represent to the scale of the map the required differences of elevation. The results should be checked by the same construction with reference to the other stations from which the points are intersected.

Contours are plotted by interpolation between located points, the elevation of which have been determined as above. The sketching is facilitated by the circumstance that the horizon line passes through all points on the photograph having the same elevation as the camera. For contouring it is therefore useful to have several photographs of the same area from different elevations. It is to be observed that a line drawn parallel to the horizon line does not mark a contour, nor do points intersected by such a line have a uniform angle of elevation or depression from the camera.

Figs. 168, 169 and 170 show part of a trial photographic survey, and illustrate the foregoing principles.

Stereo-photographic Surveying. This comparatively recent development of photographic surveying enables the office work to be overtaken in considerably less time than is possible with the older systems. The method consists in taking photographs in pairs after the manner of ordinary stereoscopic photography. The two exposures are made with the



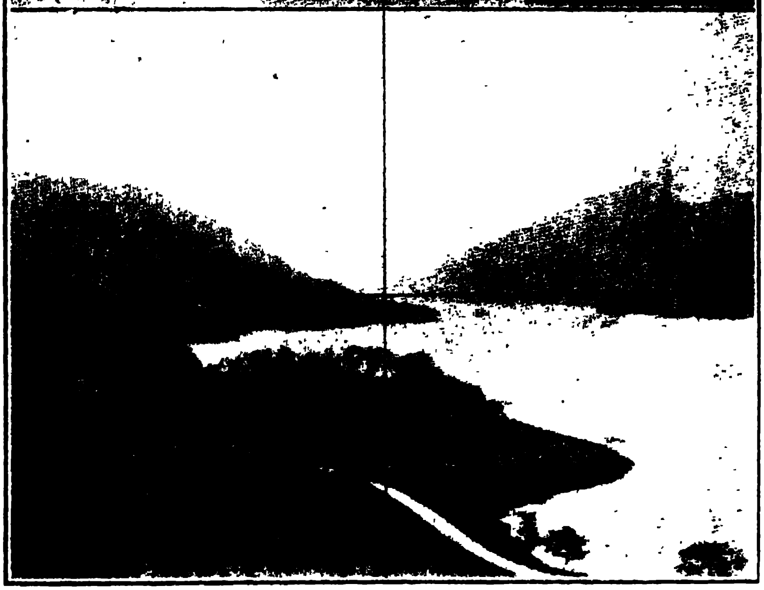


Fig. 169. Photograph 1 b.

Fig. 170. Photograph 6 c.

plates in the same vertical plane, but not necessarily at the same elevation. The stereoscopic base line, or horizontal distance between the parallel principal planes, usually lies between 100 and 400 ft., the necessary length being proportional to the square of the distance to the points being

located and the accuracy required and inversely proportional to the focal length. The base is measured tacheometrically or by taping. The photographs forming a pair when viewed through a special stereoscope show very bold relief because of the much greater distance between the camera stations than between the human eyes. The plotting is performed with the aid of such a stereoscope.

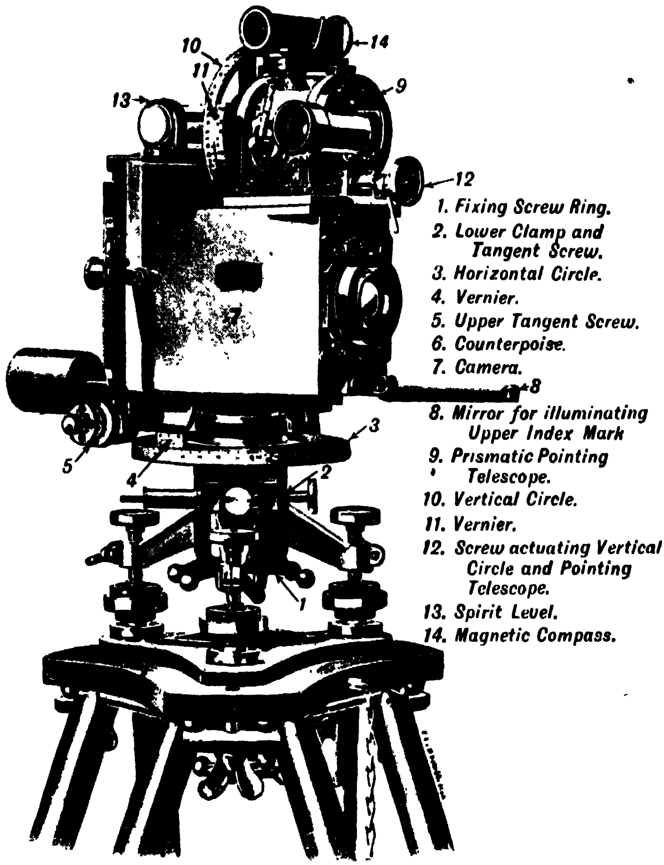


FIG. 171. ZEISS PHOTO-THEODOLITE.

Field Work. The instrument specially designed for stereo-photographic survey is the Zeiss photo-theodolite (Figs. 171 and 172). The method of locating camera stations by triangulation or resection is similar to that employed in the older systems. The difference in the field work arises in the photographing of every point twice from parallel instead of intersecting orientations. When the camera has been set to include any required view at a camera station, the stereoscopic base is set off at right angles to the principal plane by means of a transverse telescope attached to the camera. The exposure is made, the base measured, and the camera transferred to the other end of the base and set parallel to its

previous position by a backsight through the telescope to the point first occupied.

Plotting. The principle underlying the method of plotting is illustrated in Fig. 173. The points O_1, O_2 represent the optical centre of the lens, and P_1, P_2, the principal point on the negative when the camera is at either end of the stereoscopic base of length B. A point A is represented on the

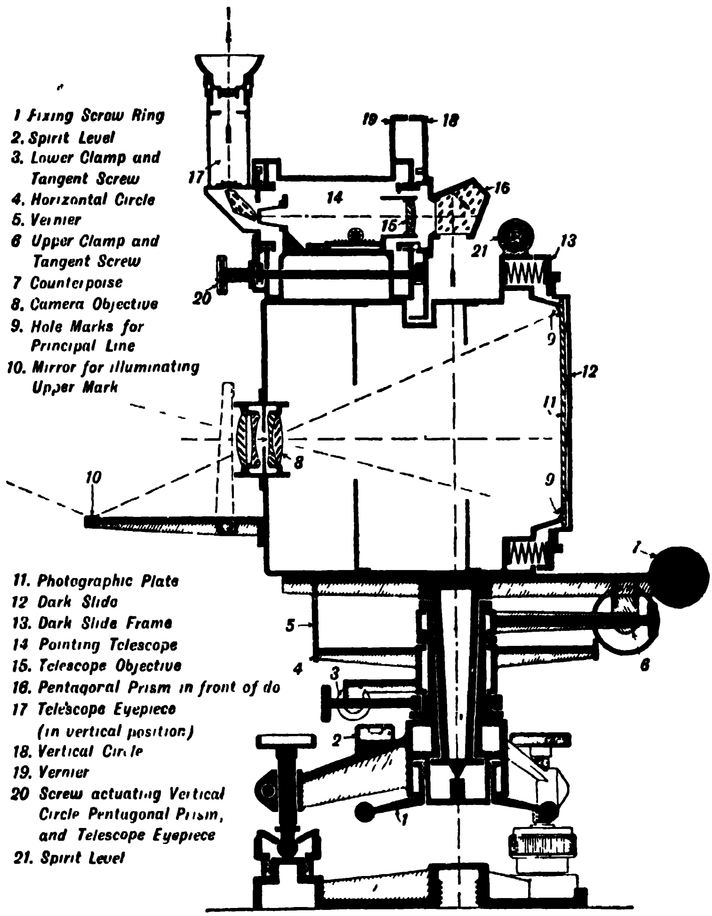


FIG. 172. ZEISS PHOTO-THEODOLITE DIAGRAMMATIC SECTION.

negatives P_1a_1, P_2a_2 at a_1 and a_2 respectively. Let $AC = D$ be the perpendicular distance of A from the stereoscopic base.

$$\frac{D}{CO_2 + B} = \frac{f}{P_1a_1} \text{ and } \frac{D}{CO_2} = \frac{f}{P_2a_2},$$

$$\therefore \frac{D}{B} = \frac{f}{P_1a_1 - P_2a_2},$$

$$\text{or } D = \frac{Bf}{P_1a_1 - P_2a_2}.$$

The quantity $(P_1a_1 - P_2a_2)$ is called the parallax of the point A. If *A* is a point of known position, the above expression can evidently first be used for the evaluation of *B*, and from the value so obtained those of *D* for unknown points can be derived. To fix the position of *A* in plan, its direction relatively to that of the principal plane is required, and is given by

$$\theta = \tan^{-1}\frac{P_1a_1}{f} \text{ or } \phi = \tan^{-1}\frac{P_2a_2}{f}.$$

The elevation of *A* relatively to that of either camera station is obtainable from the length *h* of its ordinate from the horizon line on the photograph, and has the value $\frac{Dh}{f}$.

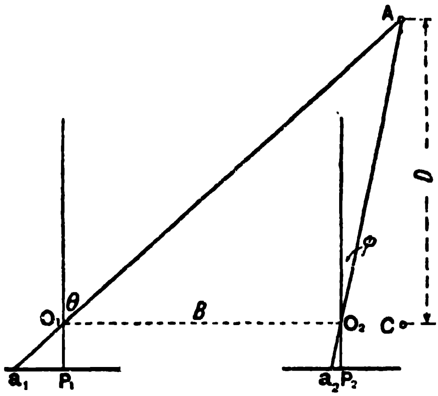


FIG. 173.

The practical value of the stereo-photographic method has been enhanced by virtue of the means designed for rapidly obtaining the data from the photographs. This is accomplished by a stereo-comparator (Fig. 174) which was first suggested in 1902 by Fourcade and was described and constructed later in the same year by Pulfrich of the German firm of Zeiss. This instrument consists of a special form of stereoscope in which magnified images of the two photographs are simultaneously examined. Both eyepieces are fitted with exactly similar indices, which can be made to combine stereoscopically with the point to be plotted by adjusting the distance between the two views. Stereoscopic combination occurs when the index appears to be as distant as the point, and the parallax is then given on a scale. The movements necessary to bring the point and the index together are also recorded on two scales, which are read for vertical angle and azimuth respectively. The plotting then proceeds in the ordinary manner, always with reference to the left-hand camera station, orientation being effected with reference to one or more known points.

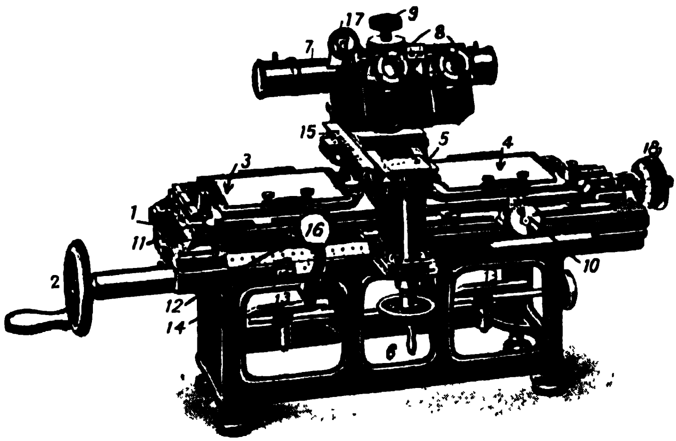
Plotting may be performed much more rapidly by mechanical means. The stereoautograph,* invented by von Orel of Vienna, has proved very successful. It makes the plotting almost entirely automatic, and is capable of tracing the contour lines upon the map. A somewhat similar instrument, called the stereo-plotter,† was designed at the Chatham School of Military Engineering, and has been employed on the Survey of India, etc.

Aerial Photographic Surveying. Aerial photographic surveying is a natural development of ordinary ground photographic surveying and was extensively employed during the war of 1914–18 for mapping on relatively large scales. Before the invention of the aeroplane, some work on the

* See *Engineering News*, Vol. 69, No. 13, 1913, and *The Geographical Journal*, Vol. 38, No. 4, 1911, Vol. 41, No. 4, 1913, and Vol. 59, No. 4, 1922.

† See *The Geographical Journal*, Vol. 31, No 5, 1908.

construction of maps from photographs taken from balloons or kites had already been done, notably by Laussedat in France in 1858, by Elsdale in Nova Scotia in 1883 and in England in 1886–7, by Woodbury in England in 1893 and by Adams in the United States. As a means of obtaining the photographs, however, the balloon has given way



1. Table carrying Plates
2. Handwheel for moving Table lengthwise
3,4. Plates under examination
5. Table carrying Stereo-microscope
6. Handwheel operating Upper Table
7. Stereo-microscope
8. Microscope Eyepieces
9. Microscope Fixing Screw
10. Compensating Screw for difference of Elevation between the two Camera stations
11. Adjusting Screws for Plates
12. Clamp fixing Slide carrying Plate 3.
13. Illuminating Mirrors
14. Abscissa Scale
15. Ordinate Scale
16. Reader for Abscissa Scale
17. Mirror and Lens for reading Ordinate Scale
18. Micrometer for measuring parallax

Fig. 174.

entirely to the aeroplane, and methods of compiling maps from photographs taken by special cameras suitably mounted on aeroplanes have now reached a high standard of efficiency which makes them of great practical importance in certain circumstances, particularly for rapid surveys of unmapped countries which are in a state of rapid development and for which maps are urgently needed if development is to take place on orderly and economical lines.

One great advantage of aerial methods which deserves special mention

is that the photographs may often serve many purposes besides those of ordinary mapping. Among these are assistance in the classification of land and soil, forestry and agricultural surveys, and use in connection with geological and archæological investigations, etc. Consequently, in many cases where air survey would not be nearly so economical as ordinary ground survey from a purely topographical point of view, the other uses to which the photographs may be put may make air survey much the more economical method in the end.

Among the engineering uses to which aerial photography has been applied are reconnaissance surveys for roads, railways, transmission lines, etc., town and country planning, land drainage, flood prevention, harbour works, river conservation, etc.

Aerial photographic surveying is very suitable for small-scale work, particularly in flat country, and, provided there is adequate provision of ground framework, it can also be used to construct plans on relatively large scales, such as 1/1,250, etc. In addition, it is well adapted for the revision of existing maps. Its great advantage over ordinary ground surveying is the speed with which it can be carried out when flying and photographic conditions are suitable. In general, however, it is not to be regarded as a method which is complete in itself but rather as one which fills in detail fitted to points that have already been established by ground methods to serve as horizontal and vertical control. Even after the photographs are available, it is often necessary to go on the ground to obtain names, etc., or to supply or identify detail which is missing or obscured or which cannot be properly identified on the photographs themselves.

In recent years, most, if not all, of the important aerial photographic surveys that have been carried out in the British Empire and the United States have been made by official organisations or by private companies formed to specialise in this class of work. The equipment needed for the work—including aeroplanes, cameras, interpretation and plotting apparatus, etc.—is, for the most part, highly elaborate and expensive, and hence, unless very large areas are involved, it is much cheaper and more satisfactory to get the survey made by an existing organisation than it is to attempt to do it on a purely private basis. In addition to this, the work itself—flying, photography, interpretation of photographs and plotting—is extremely technical and specialised and is best done by skilled and experienced personnel. Hence, a detailed description of the apparatus and methods employed would be beyond the scope of this book, and all that can be attempted is a brief sketch in most general terms. Those requiring further information will find it in books such as Hotine's *Surveying from Air Photographs* and Hart's *Air Photography Applied to Surveying*, or in the various publications of the Air Survey Committee and of the Directorate of Military Survey.

Photography. In the case of the revision of existing maps, it is sufficient to obtain photographs covering the whole area. For new surveys, each part of the area should appear on at least two photographs, so that stereoscopic views may be obtained of the whole ground. In addition, the positions and heights of a certain number of points in the area must be fixed by ground survey methods. These points, which must be such

that they will show clearly in the photographs, are called "ground control points" and it cannot be too strongly emphasised that, as in ordinary topographical surveying, unless a sufficient number of these points is available it will be impossible to produce an accurate map, and that the accuracy of the map will ultimately depend to a very large extent on the accuracy of the ground survey. In plotting, the most accurate results are obtained from vertical photographs; that is from photographs taken with the axis of the camera pointing vertically down. The "Radial Line" method, for instance, which enables the plotting to be done by relatively simple apparatus, is based on vertical photographs taken in a series of overlapping strips with a 60 per cent. overlap in the direction of flight between successive photographs. For small-scale work, however, time and expense are saved by the use of oblique as well as of vertical photographs. For such work, therefore, use is often made of multiple-lens cameras which take one vertical and up to six oblique photographs at one exposure, and considerable areas have been surveyed by this method in Canada and the United States. When taking photographs, the aeroplane should fly on a straight horizontal line, and keep on an even keel; otherwise, some of the area will not be covered by the overlap of the photographs, and the tilt of the camera will produce corresponding variations of scale in each photograph. An experienced pilot can generally avoid tilts greater than 2 degrees.

Interpretation of Photographs. The interpretation of photographs taken from the air requires a good deal of experience, and the information to be obtained from them depends on a number of factors such as angle of photograph, type of country, height of sun at time of exposure, scale, development and printing, etc. Experience can only be gained by dealing with large numbers of photographs and by examining them in the stereoscope, by comparing them with existing maps, and, when this is possible, by examining them both on the ground and in the drawing office.

The photograph is produced by light reflected from the surface of the earth, and, as the angle of reflection of the light received by an observer in an aeroplane will normally be very different from the angle of reflection of the light received by an observer on the ground, the appearance of ground objects will also usually be different for the two observers. For example, ploughed black earth, although seemingly dark to a ground observer, will often appear light on the photograph, whereas a field of corn will appear dark. This is because the ploughed earth will reflect a certain amount of bright light, especially after rain, but an observer in an aeroplane flying directly over the corn will see the countless shadows cast by the individual stalks and leaves.

Photographs should always be examined in such a manner that the direction of the light during examination is the same as that of the sun during exposure. This means that, as a general rule, the shadows should point towards the observer. If this rule is not observed, depressions and elevations may appear reversed, so that what is actually a depression may look like an elevation and *vice versa*. Correct orientation may be obtained by studying the direction of the shadows of high objects, such as telegraph poles, pylons, trees, houses, etc.

Plotting. If the photographs are not seriously tilted, and the country

is not very hilly, it may be assumed that the bearings of all points from the principal point of the photograph are virtually true. It is then possible by graphical methods to plot a framework of points to the mean scale of the photographs, each point being fixed by the intersection of two or more rays from principal points. Any ground control points that lie in the area are first plotted in their true positions by means of their rectangular co-ordinates, and the framework of intersected points is adjusted to fit the control points. The detail of the map is then traced from the photographs, and adjusted to fit into the framework of intersected points. The contouring of the map involves the use of a topographical stereoscope fitted with movable parallactic grids. If two ground control points, of which the heights are known, appear in the overlap which is under examination, then the heights of any other points in the overlap can be measured by use of the grid. When the heights of a sufficient number of points have been determined in this way, the draughtsman draws in the contours on one of the photographs of the pair. The stereoscope enables him to see the country in relief, and so to determine the shape of the contours; the heights already fixed control the positions of the contours. The contours are traced from the photographs in the same way as the detail. Topographical maps of considerable accuracy, on scales not larger than 1/20,000, may be compiled in this way from air photographs.

For larger scales, and more accurate surveys, it is necessary to take account of the height and tilt of the camera at each exposure. To determine the relative heights and tilts of the camera from a series of photographs a compound stereoscope must be employed. A number of these instruments have been made, to a variety of designs. The simpler form of the instrument provides angular measurements, from which the relative positions and tilts of the camera at each exposure can be computed. If three ground control points appear in the series of photographs, the absolute co-ordinates and height of any point of detail in the series can be calculated. In this way a rigid framework of points is provided for the map, but the computations are long and tedious. In the more elaborate instruments, computations are avoided by the provision of mechanism for automatic plotting. When a pair of photographs have been set in their correct relative positions in the instrument, the operator sees the ground in its true relief, and also a mark which appears to float in space. By manipulating the controls he can bring the floating mark down into contact with the ground at any desired point. The pencil of the automatic plotter then registers the position of this point on the map. By making the floating mark follow the line of a road, or a river, or any other detail, the feature is automatically drawn on the map. Contours may be drawn by fixing the floating mark at the required height, and moving it over the area, while keeping it in apparent contact with the surface of the ground. An instrument of this kind was made by Messrs. Barr and Stroude before the war from designs by Fourcade, Thompson and Frazer, but unfortunately it was destroyed during the bombing of Southampton. A description by Thompson of an early model of this instrument appears in the *Report of Proceedings, Conference of Empire Survey Officers*, 1935, and an illustration of it as finally constructed is included in Hart's *Air Photography Applied to Surveying.*

In recent years an instrument known as the "Multiplex" has come into extensive use for plotting. This consists of a number of wide angle projectors suitably mounted on a strong horizontal bar which enable reduced positives of the photographs, or "diapositives," to be set and adjusted in positions which, relative to one another and to the horizontal plane of the map, are similar to those which the plates occupied at the time of photography. Alternate projectors are fitted with red or blue-green filters and the superimposed images of overlapping diapositives, when viewed through spectacles with lenses of complementary colours, provide a stereoscopic effect. A small tracing table, adjustable for height and provided with a point of light used as a floating mark, can be brought and adjusted into contact with the projected image of any point, the position of the point on the map forming the horizontal plane being recorded by a pencil or needle fitted immediately underneath the floating mark, and the elevation of the point determined from the height of the table. With this instrument it is possible to bridge over gaps of several photographs in which there is no ground control of any kind.

Applications of Radar to Air Survey. In ordinary air survey work a good deal of ground control is necessitated by the fact that it is impossible, with ordinary equipment, to fix the position in space of the camera at the moment when a photograph is taken. This difficulty can now be overcome to a very considerable extent if two accurately and suitably located radar stations or beacons are available within about 200 to 250 miles of the area to be surveyed. Radar triangulation will give the fixing in the horizontal plane and accurate altimeter records will give the altitude. This enables the ground control which would otherwise be required to be very considerably reduced or even in some cases to be dispensed with altogether. Fixings thus obtained by radar are not, of course, so accurate as fixings obtained from suitable ground control points, so that methods depending mainly on radar fixings are best suited for work on small, or fairly small, scales. The method is, however, of great value in military operations as it enables maps to be made of country lying well behind the enemy lines.

In taking overlapping vertical photographs, the aircraft must be flown in parallel lines at constant distances apart, so that the photographs obtained will cover parallel strips of ground with overlaps at the sides, as well as along the line of flight, and with no gaps between strips. In the ordinary way, and especially when the ground is featureless and devoid of outstanding landmarks to guide the pilot, it is exceedingly difficult to navigate an aircraft to fly on parallel lines at the desired distances apart. With radar control, however, the aircraft can be flown fairly easily along a series of concentric circular arcs so that the photographs cover a series of concentric overlapping circular strips. When two radar stations are available, the aircraft is navigated along a circular arc at a pre-determined distance from one station and photographs are taken at pre-determined distances along the path as fixed by distances from the second station. Even when only one radar station is available, it can be used to ensure that the aircraft is flown at constant pre-determined distances from it.

Air Mosaics. It often happens that a comparatively rough map is required during the early stages of important development schemes before there is time to complete a proper survey by ground methods or to compile and plot an accurate map from air photographs. In such cases, immediate needs may be satisfied by the provision of air mosaics. These are overlapping air photographs, all on approximately the same scale, adjusted or cut to fit one another along the edges and then pasted together to form a single large composite air photograph of the area involved. When these mosaics are made simply by fitting together unrectified prints without reference to fixed ground stations, they are known as "uncontrolled mosaics." If, however, they are made from rectified prints, in which the major distortions due to tilt, flying height and height of ground, etc., have been removed as far as possible, and they are fitted to the plotted positions of fixed ground control points, accuracy is greatly increased and the mosaic is known as a "controlled mosaic."

In some ways mosaics may be more useful than surveyed plans since they show actual ground conditions and many details which cannot be shown on the plans. On the other hand, they can never be so accurate as a plan based on an accurate survey made by taking actual measurements on the ground or on one that has been carefully plotted from overlapping vertical photographs with good ground control as a basis, and, moreover, much important detail may be obscured by such things as trees in leaf, overhanging eaves, etc.

At the present time the Ordnance Survey of Great Britain has a very large programme of work in hand in surveying a number of towns and cities on the 1/1,1250 scale. Many of these towns, particularly those which have been severely damaged by bombing during the last war, are faced with the necessity for early expansion and development, and up-to-date plans are accordingly required for town-planning purposes. In the ordinary way, if these towns had to wait for accurate plans, it would take the Ordnance Survey a very long time to meet the need and in the meantime development would be held up. Accordingly, as a temporary expedient, and as a stop-gap measure before the final survey plans are available, arrangements have been made to supply air mosaics wherever they are required. It is recognised that important measurements should not be scaled from these mosaics and that they cannot take the place of good survey plans, but, provided their limitations are kept in mind, they are extremely useful as an aid to the town planner. They are made by rectifying and enlarging the original photographs to the 1/1,250 or other suitable scale and then, after cutting them to fit along lines representing the same detail on adjacent photographs, fitting and sticking them down together on an old map of the same area in such a way that features on the photographs coincide as far as possible with the corresponding features on the map. Hence, they can be considered to be controlled mosaics.

Air mosaics are also often used to obtain preliminary information in connection with such projects as the selection of possible routes for new roads and railways in unmapped country, or in connection with river improvements, geological developments, etc.

REFERENCES ON TOPOGRAPHICAL SURVEYING

ABRAHAMS, T. A. *Essentials of Aerial Surveying and Photo Interpretation.* London, 1944.

ANDERSON, R. O. *A.B.C.'s of Photogrammetry. Part II. Tilt and Control Extension.* Washington, D.C., 1949.

— *Applied Photogrammetry.* Chattagoona, Texas, U.S.A., 1937.

Aneroid Tables Based on a Standard Atmosphere and a Standard Lapse-Rate. H.M.S.O., London, 1935.

BROWN, MAJOR R. LL. "Optical Distance Measurements." *Empire Survey Review,* Vol. I, No. 4, 1932.

CHAMBERS, L. G. "Direct Use of Air Photos for Cadastral Purposes in Zanzibar." *Empire Survey Review,* Vol. X, No. 71, 1949.

CLOSE, SIR C. F., and WINTERBOTHAM, H. ST. J. L. *Text Book of Topographical and Geographical Surveying.* London, 1926.

COLLINS, C. B. "The Economic Aspect of Air Survey." *Report of Proceedings. Conference of Empire Survey Officers,* 1935.

CRONE, LIEUT.-COL. D. R. "Aerial Camera Calibration." *Empire Survey Review,* Vol. IV, No. 26, 1938.

— "Correction of Plane Table Heights for Curvature of the Earth and Atmospheric Refraction." *Empire Survey Review,* Vol. III, No. 20, 1936.

— "Heights by Aneroid Barometer." *Empire Survey Review,* Vol. IX, No. 69.

— "The Use of High Oblique Air Photographs for Topographical Mapping." *Report of Proceedings. Conference of Empire Survey Officers,* 1935.

DARE, MAJOR J. "The Topographical Survey of Sierra Leone." *Report of Proceedings Conference of Empire Survey Officers,* 1931.

DEVILLE, E. *Photographic Surveying.* Ottawa, 1895.

DIRECTORATE OF MILITARY SURVEY. "Some Methods of Map Making from Radar-Controlled Photography." Air Survey Research Paper No. 11. London, 1945.

DIXON, S. M. "Surveying with a Camera." *Min. Proc. Inst. C.E.,* Vol. CLXXXIV.

DOWSON, SIR ERNEST M. "Direct Use of Air Photographs for Cadastral Purposes in Zanzibar." *Empire Survey Review,* Vol. IX, No. 63, 1947.

EDEN, J. A. "Photographic Surveying of Flat Country." *Empire Survey Review,* Vol. III, No. 15, 1935.

FIELD, R. H. "The Aneroid Barometer and Altimeter." Topographical Survey of Canada. Bulletin No. 63. Ottawa, 1931.

FINCH, J. K. *Topographic Maps and Sketch Mapping.* New York, 1920.

FLEMER, J A. *An Elementary Treatise on Photographic Methods and Instruments.* New York, 1906.

FOURCADE, H. G. "A Projection Method of Mapping from Air Photographs." *Empire Survey Review,* Vol. V, No. 35, 1940.

— "The Rectification of Air Photographs." *Empire Survey Review,* Vol. III, No. 19, 1936.

GARDINER, LIEUT.-COL. R. A. "The Use of Air Photographs by the Ordnance Survey." *Empire Survey Review,* Vol. X, No. 76, 1950.

GRIBBLE, T. G. "Diurnal Atmospheric Variation in the Tropics and Surveying with the Aneroid." *Min. Proc. Inst. C.E.,* Vol. CLXXI.

GRUBER, O. VON. *Photogrammetry.* Translated from the German by G. T. McCaw and F. Z. Cazalet. London, 1932.

HART, C. A. *Air Photography Applied to Surveying.* London, 1948.

HART, LIEUT.-COL. C. A. "Air Survey: The Modern Aspect." *The Geographical Journal,* Vol. CVIII, Nos. 4–6, April, 1947.

— "Surveying from Air Photographs Fixed by Remote Radar Control." *Empire Survey Review,* Vol. IX, No. 64.

— "Utilisation of Radar and Altimetric Control for Multiplex Bridging." Directorate of Military Survey. Air Survey Research Paper No. 16. London, 1945.

HINKS, A. R. *Maps and Survey.* Cambridge, 1942.

— "Stereographic Survey: the Stereoautograph." *The Geographical Journal,* Vol. 59, No. 4, 1922.

HOLDITCH, SIR H. T. "Geographical Surveying in India." Prof. Papers of the Corps of Royal Engineers, Vol. 24, No. 11, Chatham, 1898.

HOTINE, COL. M. "Calibration of Surveying Cameras." Prof. Paper of the Air Survey Committee, No. 1.
— "Extensions of the Arundel Method." Prof. Papers of the Air Survey Committee, No. 6.
— "The Fourcade Stereogoniometer." Prof. Papers of the Air Survey Committee, No. 7.
— "Simple Methods of Surveying from Air Photographs." Prof. Papers of the Air Survey Committee, No. 3.
— "The Stereoscopic Examination of Air Photographs." Prof. Papers of the Air Survey Committee, No. 4.
— *Surveying from Air Photographs.* London, 1931.
HOWE, J. A. "Geology and Topographic Survey." *Report of Proceedings. Conference of Empire Survey Officers,* 1928.
HUNTER, J. DE G. "Notes on the Use of Thermometers, Barometers and Hypsometers with Tables for the Computation of Heights." Survey of India, Miscellaneous Publication No. 14, 1911.
— "Two Notes on Short Tertiary Bases." *Empire Survey Review,* Vol. I, No. 1, 1931.
HUTCHISON, MAJOR D. A. "The Application of Air Photos to Ordnance Survey Revision." *Report of Proceedings. Conference of Empire Survey Officers,* 1928.
IVES, H. E. *Airplane Photography.* Philadelphia, 1920
LAUSSEDAT, A. *Recherches sur les Instruments, les Méthodes, et le Dessin Topographiques.* Paris, Pt. I, 1901; Pt. 2, 1903.
LOXTON, J. W. "Height Control in Air-Photo Survey." *Empire Survey Review,* Vol. VI, No. 41, 1941.
McCAW, G. T. "New Aneroid Tables." *Empire Survey Review,* Vol. III, No. 16, 1935.
— "The Survey Aneroid: The Possibility of its Improvement." *Report of Proceedings. Conference of Empire Survey Officers,* 1931.
— "The Trigonometrical Survey of Fiji. Report on." Vol. I (Viti Levu). London 1916. Vol. II (Vanua Levu and Taveuni), London, 1917.
McNEIL, GOMER T. *A.B.C.'s of Photogrammetry. Part I. Fundamentals.* Washington, D.C., 1949.
MÜLLER, G. A. G. "The Stereoscopic Method of Surveying, and a First Trial of its Application to a Railway Survey in China." *Trans. Am. So. C.E.,* Vol. LXXIX.
NELLES, D. H. "Photogrammetry for Taking Topography of Watershed." *Eng. News,* Vol. 76, No. 19, 1916.
PETERS, F. H. "Surveying Half a Continent—Canada." *Report of Proceedings. Conference of Empire Survey Officers,* 1935.
PHILLIMORE, COL. R. H. "Cadastral Surveys by Air Photography in India." *Report of Proceedings. Conference of Empire Survey Officers,* 1931.
RAYNER, W. H. "Allowable Use of Small Angles in Surveying." *Eng. Record,* Vol. 68, No. 16, 1913.
REEVES, E. A. *Hints to Travellers,* Vol. I, Royal Geographical Society, London, 1935.
ROGERSON, G. C. *Instructions for the Topographical Branch.* Gold Coast Survey Dept., Accra, 1931.
— "Notes on the Use of the Aneroid Barometer in the Gold Coast." *Empire Survey Review,* Vol. II, No. 14, 1934.
ROTHERY, E. A. "The Range-Finder in Tropical Topographical Surveying." *Empire Survey Review,* Vol. II, No. 13, 1934.
SALT, LT. J. S. A. "A Simple Method of Surveying from Air Photographs." Prof. Papers of the Air Survey Committee, No. 8.
SWORDER, G. H. *Handbook of the Topographical Branch, Survey Department.* F.M.S. Survey Dept., Kuala Lumpur, 1940.
THOMAS, W. N. "Sources of Error in Photographic Surveying." *Min. Proc. Inst. C.E.,* Vol. CCX.
THOMPSON, LT. E. H. "An Automatic Plotting Machine for Use with Air Photographs." *Report of Proceedings. Conference of Empire Survey Officers,* 1935.
— "The Seven-Lens Air-Survey Camera." *Empire Survey Review,* Vol. IV, No. 26, 1937; No. 27, 1938.
THOMPSON, F. V. "Stereo-Photo Surveying." *The Geographical Journal,* Vol. 31, No. 5, 1908.

TOLLEY, B. B., and ROBBINS, P. H. *Photographic Surveying.* New York, 1945.
TOPOGRAPHICAL SURVEY OF CANADA. "The Use of Aerial Photographs for Mapping." Bulletin No. 62, 1932.
TROREY, LYLE G. *Handbook of Aerial Mapping and Photography.* Cambridge, 1950.
TROREY, MAJOR L. G. "A Multiplex Technique for Radar-Controlled Air Photographs." Directorate of Military Survey. Air Survey Research Paper No. 18. London, 1945.
TYMMS, F., and PORRI, C. "Flying for Air Survey Photography." Prof. Papers of the Air Survey Committee, No. 2.
"Uganda-Congo Boundary Commission, 1907–8. Official Report of the British Section." Colonial Office, 1909.
WAINWRIGHT, D. B. "A Plane Table Manual." U.S. Coast and Geodetic Survey, Report for 1905, Appendix No. 7 (revised 1916).
WHEELER, A. O. "Photographic Methods Employed by the Canadian Topographical Survey." Department of the Interior, Ottawa, 1905.
WILSON, H. M. *Topographic Surveying.* New York, 1910.
WINTERBOTHAM, BRIG. H. ST. J. L. *A Key to Maps.* London, 1939.
— "Organised Plane-Tabling." Ordnance Survey Training Series, No. 1, Southampton, 1931.
— "Revision by Air Survey." Ordnance Survey Professional Papers New Series No. 16.
— "Topography in the Tropical Forest Belt by Traverse and Aneroid." Ordnance Survey Training Series, No. 2, Southampton, 1931.

CHAPTER VIII

MAP CONSTRUCTION

IN this chapter are considered the various ways usually adopted for the plotting of extensive topographical surveys and the preparation of the final map or series of maps. Large scale maps are generally plotted by means of a system of rectangular co-ordinates, but, in small scale work, it is more usual to use geographical co-ordinates as a basis for plotting. Many different methods of plotting small scale work are available, each one with its own particular advantages and disadvantages, and a study of these different methods forms the subject of map projections, which is the first one now to be considered.

MAP PROJECTIONS

We have already seen that, when we attempt to define the positions of points, or to plot these positions, by means of any system of rectangular co-ordinates, we must, if the area involved is large, be prepared to accept the difficulty of representing a large area of a curved surface—the surface of the earth—on a flat one, and this leads to distortion of shape and variation of scale, the amount of which increases rapidly as the area to be represented increases. Exactly the same thing occurs when we proceed to plot work on a small scale. In this case, the first thing to be done is to plot a "graticule," or system of lines or curves to represent meridians of longitude and parallels of latitude, and the positions of the control points on which the drawing in of the detail depends are plotted with reference to the lines which form the graticule. The fundamental object of the study of map projections is therefore concerned with the different ways in which the geographical graticule of meridians of longitude and parallels of latitude can be represented on the map, and with the relative advantages, disadvantages and limitations of each method.

Classification of Projections. Since it is impossible to represent correctly any considerable portion of the earth's surface on a plane, there can be no perfect projection. The perfect projection would show the meridians as equally spaced straight lines converging correctly towards the poles. The parallels would intersect them at right angles at the correct intervals, and would be parallel to each other. The scale would, in consequence, be constant over the map, and we should have a correct representation throughout the map of distances and directions, and therefore of outlines and their contained areas.

Although constancy of scale is unattainable, many projections have been devised to make the resulting map correct in certain particulars. Thus, the scale may be constant along certain lines; directions from one point to any other point on the map may be correctly represented; or outlines, although distorted, may contain their correct areas. Other projections do not preserve any property of the spherical surface exactly, but are designed to give a minimum of distortion over the map or at least a fair general representation.

In the manner of their construction, map projections range from geometrical projections in the ordinary sense of the word to purely conventional systems of representation. They may be classified as :

(1) *Perspective,* in which the portion of the earth's surface is drawn as it would be seen from a definite point.

(2) *Conical,* in which the meridians and parallels may be supposed to be drawn on the surface of a cone, which is then developed.

(3) *Cylindrical,* in which a cylinder takes the place of a cone.

(4) *Zenithal,* in which the lines are drawn on the same system as in the preceding two classes, but upon a plane.

(5) *Miscellaneous.*

Projections may also be classed according to special properties they possess as :

(1) *Azimuthal* or *Zenithal,* in which the azimuths from the centre of the map to all other points on it are correctly shown. This property is peculiar to the perspective and zenithal projections.

(2) *Orthomorphic* or *Conformal,* in which the scale, although varying throughout the map, is the same in all directions at any point, so that very small areas are represented of correct shape. This property cannot be extended to large areas, the shapes of which are sometimes best represented by a projection which has not the orthomorphic property.

(3) *Equal-Area* or *Equivalent,* in which equal areas on the ground are represented by equal areas on the map. This property cannot be possessed by an orthomorphic projection, as the representation would then be perfect.

The names given to projections are descriptive first of the method of construction and then of the most important property, *e.g.* Conical Equal-Area, Cylindrical Orthomorphic, etc. Many projections are, however, better known by the name of their inventor.

In the following pages a brief description is given of a sufficient number of the more common projections to indicate the features of the different classes. The factors involved in the selection of a projection suited to a particular case can then be considered.

PERSPECTIVE PROJECTIONS

Projections of this class are formed by ordinary geometrical projection upon a plane. As the name implies, the meridians and parallels are represented as they would be seen from a particular view-point, the distance of which from the centre of the earth controls the properties of the projection. The plane of projection is normal to the line joining the view-point and the centre of the earth, its position on that line affecting only the scale of the resulting graticule. According to the situation of the view-point, the projection plane may be parallel to the plane of a meridian, and the result is a meridian perspective : if the plane is parallel to the equator, we have a polar perspective, and in any other position, a horizontal perspective.

Perspective projections are unsuitable for the mapping of comparatively small areas, but are of some importance for the representation of a hemisphere or more.

Orthographic Projection. In this case the viewpoint is infinitely distant (Fig. 175), and the projection is that employed in geometrical drawing. The regions near the margin of the map are so greatly contracted as to make the projection of little use.

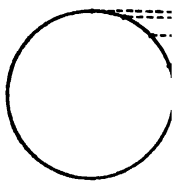


Fig. 175.

Stereographic Projection. The view-point is situated on the surface of the earth opposite the hemisphere to be mapped (Fig. 176). Features towards the edge of the map are expanded, but the projection is orthomorphic.

Gnomonic or Central Projection. The view-point is situated at the centre of the earth (Fig. 177), and the projection shows an enormous exaggeration remote from the centre of the map. The projection has the special property that, assuming a spherical earth, great circles are represented by straight lines, and, in consequence, this projection has been applied in the preparation of navigational charts for great circle sailing.

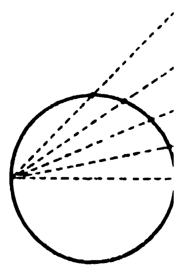


Fig. 176.

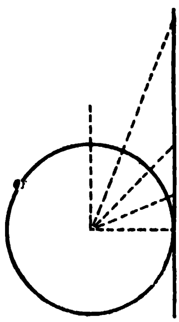


Fig. 177.

La Hire's Projection. Since the orthographic projection contracts areas near the margin and the stereographic expands them, these effects may be neutralised by the selection of an intermediate view-point. In La Hire's projection its distance from the centre of the earth is 1·71 times the radius.

Clarke's Minimum Error Projection. Col. Clarke investigated the position of the view-point to give a minimum of total misrepresentation. In his minimum error projection the point is so placed that the sum of the squares of the errors of scale in two directions at right angles to each other at all points on the map is a minimum. Its distance from the centre of the earth depends upon the extent of surface to be mapped, and has the values 1·625 R for an area contained by a small circle of 40° radius, 1·47 R for the hemisphere, and 1·367 R for a radius of 113½°. This last (Fig. 178) enables about two-thirds of the earth's surface to be represented, and is known as Sir Henry James' projection, for which he originally proposed a distance of the view-point from the centre of the earth of 1·5 R.

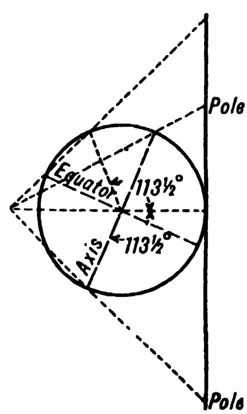


Fig. 178.

CONICAL AND MODIFIED CONICAL PROJECTIONS

These form an important group of projections, and are greatly used in topographical and atlas maps, except for very high latitudes. They are of no value for the mapping of a complete hemisphere.

To follow the construction of the conical projection in its simplest form, let it be supposed that a cone, the axis of which coincides with that of the earth, touches the earth along a parallel of latitude ϕ (Fig. 179). The meridians are projected from the centre of the earth upon the surface of the cone, and, on developing the cone they appear as equally spaced

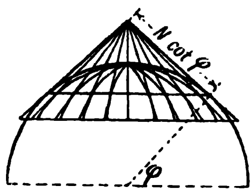


FIG. 179.

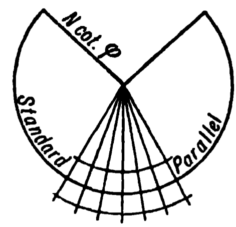


FIG. 180.

straight lines radiating from a point (Fig. 180). The parallel of contact, or standard parallel, is represented of correct length by a circular arc, the radius of which is R cot ϕ for a sphere, or N cot ϕ for the spheroid, where N is the length of the normal. The parallels on either side of it are constructed on the development.

Simple Conical Projection with One Standard Parallel. The standard parallel is one passing through the middle of the area to be mapped. Its radius is computed from N cot ϕ, and the arc is drawn either by compasses or by locating a number of points on it by co-ordinates. The arc is divided off into equal parts representing to scale the linear equivalents of degrees or parts of a degree of longitude, the values of which for the parallel in question are extracted from tables. The meridians are drawn through these points towards the centre, and along any one of them the linear values of arcs of the meridian are set out. Through the points so obtained the parallels on either side of the standard parallel are drawn as circular arcs concentric with the standard.

Features reproduced correctly in the projection are the scale along every meridian, and along the standard parallel, as well as the perpendicularity of the meridians and parallels. The scale along the parallels on either side of the standard is too great, the error becoming greater as the distance from the standard parallel increases. The projection is much used, and is suitable for the mapping of a region of any extent in longitude but narrow in latitude.

Conical Projection with Two Standard Parallels. In this projection, sometimes known as the modified secant conical, the errors of the simple conical projection at a distance from the standard parallel are reduced by the adoption of two standard parallels, one towards the top and the other towards the bottom of the map. These are constructed as concentric circular arcs of correct length and at their true distance apart. Each is divided to scale in the same manner as the single standard parallel in the last case. The meridians are straight lines passing through these points of division, and therefore radiate from the centre of curvature. The radii of the arcs representing the standard parallels are readily deduced from

these data, for, in Fig. 181, if

m = the true distance between the standard parallels,

p_1, p_2 = the lengths of, say, 1° on each,

r_1, r_2 = their radii,

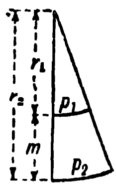


Fig. 181.

$$\text{then } \frac{r_1}{p_1} = \frac{r_2}{p_2} = \frac{m}{p_2 - p_1},$$

$$\text{or } r_1 = \frac{mp_1}{p_2 - p_1},$$

$$\text{and } r_2 = \frac{mp_2}{p_2 - p_1}.$$

The concentric circular arcs representing the intermediate parallels are truly spaced as before.

The use of two standard parallels makes the scale along the parallels too small between the standards and too great beyond them, but the error is kept within smaller limits than in the simple conical projection. The limits of latitude embraced by the map being known, the standard parallels may be selected conventionally as the parallels situated at a fourth of the range of latitude from the top and bottom of the map respectively. Otherwise they are selected so that the central and the extreme parallels may have the same absolute or proportional errors, or such that the mean length of all the parallels may be correct.

Neither of the above two projections is equal-area or orthomorphic, but, while retaining one or two standard parallels, they may be constructed either as equal-area or orthomorphic. For maps having a small range of latitude there is little difference between such conical projections and those which have been described. The conical equal-area projection is seldom constructed, more especially as the equal-area property may be obtained with less difficulty by means of certain of the modified conical projections. The orthomorphic property, however, is used in the conical orthomorphic projection, the main properties of which have been described in Chap. V in connection with systems of linear co-ordinates.

Simple Polyconical Projection. This is a modified conical projection in which each parallel has the same characteristics as the standard parallel of a simple conical projection. The radii of the parallels are given by $N \cot \phi$. The straight line representing the central meridian is intersected by the parallels at the correct intervals, and therefore the arcs representing the parallels are not concentric, but diverge from each other on either side of the central meridian. Each parallel is divided truly, and the meridians, other than the central, are curves through the points of division, and do not intersect the parallels at right angles.

The scale along each parallel and along the central meridian is correct, but is too great along any other meridian, the error increasing with distance from the central meridian. The projection is neither equal-area nor orthomorphic, but is suited for the mapping of countries having a small range of longitude. The ease with which it may be constructed from tables has led to its wide adoption, notably by the United States Coast and Geodetic Survey, for the graticules of topographical map sheets which need not fit exactly along the edges.

Rectangular Polyconical Projection. This is a modification of the simple polyconical projection. The parallels are constructed in the same way, but only one is divided truly. The meridians are curves passing through these points and intersecting all the parallels at right angles. The right angle intersections are therefore obtained by sacrificing the scale along all but one of the parallels. The projection is neither equal-area nor orthomorphic.

The International Map Projection. A modified polyconical projection has been adopted for the International Map on the scale of 1/1,000,000. The sheets normally cover an area of 4° in latitude and 6° in longitude. The top and bottom parallels of each are constructed as in the simple polyconical projection, and are divided truly. The distance between them, however, is such that the meridians situated at 2° on either side of the centre are straight lines of correct length. The other meridians are straight lines joining corresponding points on the extreme parallels, and the intermediate parallels divide the meridians into equal parts.

Bonne's Projection. This is a modified conical projection designed to represent areas correctly. One standard parallel is drawn as in the simple conical projection. The centre meridian is a straight line divided truly, and through the points of division the remaining parallels are drawn as concentric circular arcs with the same centre as the standard. Each parallel is divided truly, and the meridians become curves passing through corresponding points on each.

The scale is correct along the central meridian and along each parallel, and the projection is evidently equal-area since the perpendicular distance between the parallels is correct. The scale along the meridians increases towards the sides of the map, and the projection is not suitable for a wide range of longitude. It has been used to a considerable extent on European surveys, and was employed until recently by the Ordnance Survey for one-inch and smaller scale maps of Scotland and Ireland. New editions of these maps are, however, being produced on the Transverse Mercator projection.

Sinusoidal Equal-Area or Sanson-Flamsteed Projection. This is a special case of the last in which the equator is the standard parallel. The parallels become straight lines, correctly spaced and perpendicular to the straight central meridian. The other meridians, obtained by dividing each parallel truly, would become sine curves for a spherical earth. The projection has all the properties of Bonne's, and is used for the mapping of areas extending on either side of the equator.

CYLINDRICAL PROJECTIONS

These may be conceived as drawn on the development of a cylinder, and may be regarded as conical projections in the limiting case where the apical angle of the cone is zero. With the exception of Mercator's projection, and the Cassini and Transverse Mercator projections, which are modified forms of a true cylindrical projection, they are of little or no practical value.

Simple Cylindrical or Square Projection. The tangent cylinder touches the earth along the equator, which is therefore the standard parallel.

The meridians project into parallel straight lines perpendicular to the equator, and are spaced at their equatorial intervals. The parallels on either side of the equator are parallel straight lines at their correct distances from it, so that the graticule consists of a series of rectangles which are practically squares. The exaggeration of scale along the parallels becomes very great away from the equator, and the projection could be applied only to the case of a very narrow strip along the equator. By conventionally applying the system to the mapping of a narrow strip along any other parallel with that parallel as a standard, the rate of increase of distortion on either side is, of course, greater.

Cylindrical Projection with Two Standard Parallels. The distortion of the simple cylindrical projection can be reduced by making the scale correct along two parallels, one on either side of the equator and equidistant from it. Large errors of scale, however, occur along the other parallels.

Cylindrical Equal-Area Projection. The equal-area property may be obtained for the cylindrical projection by drawing the parallels at $R \sin \phi$ from the equator, where R is the radius of the earth, assumed spherical, and ϕ is the latitude. The interval between the parallels therefore decreases north and south of the equator, and the projection could be usefully applied only in the case of a narrow strip along the equator.

Mercator's or Cylindrical Orthomorphic Projection. To secure orthomorphism, the spacing of the parallels must be increased as we depart from the equator, in order to keep pace with the increase of scale along the parallels common to all cylindrical projections. For a sphere, this requires that the distance from the equator of a parallel in latitude ϕ shall be $= 2{\cdot}30259\, R \log \tan (45° + \frac{1}{2}\phi)$, where $2{\cdot}30259 = 1/M$, M being the modulus of the common logarithms.

The projection shows an increasing exaggeration of areas away from the equator, and, as is evident from the formula, the representation of the poles is infinitely distant from the equator. The projection is, however, of great value for the construction of navigational charts, by virtue of the property that a line of constant bearing, known as a loxodrome or rhumb line, appears as a straight line on the chart. The rhumb line course between two points is longer than that along a great circle, but on account of its simplicity rhumb line sailing is preferred except for very long voyages. To find the course to be followed by a ship sailing on a constant bearing between two points represented on a Mercator chart, it is only necessary to draw a straight line between them and protract the angle it makes with any meridian. The compass bearing is then obtained by applying the magnetic declination to the true bearing as protracted.

ZENITHAL PROJECTIONS

The essential feature of this class of projections is the correct representation of azimuths from the centre of the map. The same property is possessed by the perspective projections, but is obtained in a different manner in those to be described, which are therefore classed by themselves. They may be regarded as the other limiting case of the conical projection. As the standard parallel of the simple conical projection

increases in latitude, the tangent cone becomes flatter, and in the limit becomes a plane, tangent at the pole. A projection drawn on this plane by the same system as was used for the cone has the property of being azimuthal.

Zenithal Equidistant Projection. In the case strictly analogous to the simple conical projection the pole forms the centre of the projection, and the meridians radiate from it at their true angles. The parallels become complete circles correctly spaced along the meridians and with the pole as their common centre. The resulting projection enables distances as well as azimuths from the pole to be shown correctly, and is suitable for the mapping of the polar regions, but the scale along the parallels increases away from the centre.

The same principle may be applied to the construction of a projection for any portion of the earth's surface by considering the projection as drawn upon a plane, tangent at the centre of the area to be included. Equally spaced rays from the centre would then represent great circles passing through the point of tangency and making equal angles with each other. The concentric circles equally spaced along the rays would represent equidistant circles on the earth. These lines are not required, and the meridians and parallels must be constructed. They are not simple curves and are best drawn by plotting a number of their intersections by the use of tables or by computing their polar co-ordinates from the centre of the projection, as follows.

Let ϕ_c = the latitude of the selected centre of the projection,
ϕ = the latitude of the point to be plotted,
δL = the longitude difference between the centre and the point,
A = the azimuth of the ray from the centre to the point,
l = its length,
α = the angle it subtends at the centre of the earth.

Then, for a sphere, $\cos\alpha = \sin\phi_c \sin\phi + \cos\phi_c \cos\phi \cos\delta L$

$$\sin A = \cos\phi \operatorname{cosec}\alpha \sin\delta L$$

$$l = \frac{\pi R\alpha^\circ}{180^\circ}.$$

The intersections are plotted by rectangular co-ordinates from the computed polar co-ordinates.

Zenithal Equal-Area Projection. By sacrificing the equidistant property we may construct an equal-area projection by making $l = 2R\sin\frac{1}{2}\alpha$. This projection, as well as the last, is useful for the mapping of large areas, and is much used for atlas maps.

Airy's Zenithal Projection. This projection was designed by Sir George Airy to give a zenithal projection with a minimum of misrepresentation.

MISCELLANEOUS PROJECTIONS

The following conventional projections cannot properly be classed under any of the foregoing heads.

Globular Projection. On account of its simplicity, this projection is frequently used for the representation of a hemisphere. The equator

and the central meridian are equal straight lines at right angles, and each is divided into equal parts. The other meridians are circular arcs passing through the poles and the points of division of the equator. The meridian forming the circumference of the map is equally divided, and the parallels are circular arcs through the points so obtained and the corresponding points of division of the central meridian.

Polyhedric Projection. This projection, used for the topographical maps of several European surveys, may be regarded as a series of orthographic projections upon a number of different planes, one for each sheet. The plane of projection for a sheet is that passing through the four points on the earth's surface which form the corners of the sheet.

Projection by Rectangular Co-ordinates or Cassini's Projection. The system of rectangular spherical co-ordinates described in Chap. V, when used for plotting a map, gives the Cassini projection. This may be considered to be a form of cylindrical projection in which the cylinder touches the sphere along a meridian, instead of along the equator, and the generating lines of the cylinder are the great circles at right angles to this meridian, instead of meridians. In this projection, distances along the central meridian and along the great circles perpendicular to it are preserved, but away from the central meridian distances parallel to it are increased in length.

To draw the meridians and parallels, a sufficient number of their intersections are laid down by rectangular co-ordinates, and in the resulting graticule the parallels and the meridians, other than the central one, are curved lines. The scale is correct along the central meridian, but is too great along the others, the error increasing towards the sides of the map. The projection is unsuitable for a map covering a wide range of longitude, but is useful for the mapping of comparatively small areas. It was adopted by the Ordnance Survey for the maps of England on scales of from one to four miles to an inch and for the six-inch map of Great Britain and Ireland, but it is now being replaced by the Transverse Mercator projection. Tables for its construction for the latitudes of Great Britain are published in the official "Account of the Methods and Processes adopted for the Production of the Maps of the Ordnance Survey of the United Kingdom."

The Transverse Mercator Projection. Transverse Mercator co-ordinates, when used for plotting a map, also give a map projection called the Transverse Mercator, or Gauss Conformal, projection. This is simply the Cassini projection modified to become conformal and is a transverse form of the ordinary Mercator projection. Thus, in the ordinary Mercator projection, the spacing of the parallels of latitude is modified in accordance with the relation, Mercator meridian distance $= R/M.\log\tan(45°+\frac{1}{2}\phi)$. In the Transverse Mercator projection, distances along the great circles perpendicular to the central meridian are modified in accordance with the relation, $Y_M = R/M.\log\tan\left(45° + \frac{Y_c}{2R}\right)$.

In the Transverse Mercator projection, the meridians of longitude and the parallels of latitude are families of orthogonal curves—that is, curves intersecting one another at right angles—the central meridian being a straight line. The graticule can be plotted by rectangular co-

ordinates by laying down the points of intersection of various meridians and parallels, using the formulæ given in Chap. V. for the computation of rectangular co-ordinates from geographicals. Magnification of distance and area becomes greater as distance from the central meridian is increased but, of course, *very small* areas retain their true shape.

Trapezoidal Projection. In this purely conventional projection both meridians and parallels are straight lines. The truly divided lines are the central meridian and two of the parallels, one at about a fourth of the range of latitude from the top of the sheet, and the other the same distance from the bottom. The projection is quite unsuitable for other than very small areas.

Field Sheet Projections. These projections are used for the plotting of trigonometrical points controlling detail to be mapped in the field. Owing to the limited area embraced by each field sheet, the projection may be of an approximate character designed for ease of drawing. The construction is described on pages 498–500.

The Choice of a Projection. Several considerations have to be taken into account by the cartographer in selecting the projection best suited to a particular case, and definite rules cannot be laid down. The choice must be largely governed by : (1) the region to be mapped, particularly as regards its extent, general shape, and position on the earth ; (2) the purpose of the map ; (3) whether the map is contained on (*a*) a single sheet or a series of sheets which can be fitted together, or (*b*) a series of independent sheets.

(1) The choice of a projection for very large areas, such as a continent or a hemisphere, is a matter for the atlas maker. A fair general representation is all that can be secured. For the largest areas, such as a hemisphere, a projection having the azimuthal property, such as the zenithal equidistant or equal-area, or Airy's and Clarke's minimum error projections are the most generally applicable, although the conventional globular is very commonly used.

As the area decreases, the choice becomes wider, and the projection may be selected to suit the shape of the area and its position on the earth's surface. Zenithal projections are suited for roughly circular areas. It has been seen that a simple conical projection is suitable for areas having a small range of latitude, while other projections, such as Bonne's and the Transverse Mercator, are better adapted for areas narrow in longitude.

Many projections, such as the conical, are ill adapted for the mapping of high latitudes, for which the zenithal projections are suitable. For the mapping of equatorial regions the choice is wider, and for large areas the zenithal projections and Sanson-Flamsteed's are most commonly used.

(2) The choice of a projection may be entirely controlled by the special purpose for which the map is intended to be used. Thus, the deficiencies of the Mercator and gnomonic projections are over-ruled because of their suitability for navigational charts. In the case of maps made for statistical purposes the equal-area property is all important, while atlas maps made to exhibit trade routes from a centre should be constructed on the zenithal equidistant projection.

(3) The whole area surveyed may be represented on a series of sheets which may, or may not, be so projected that they can be exactly fitted

together.' In the former case, there is one graticule, of which the individual maps contain different parts. Some of the maps are, in consequence, more distorted than others, and this system should not be used if the series of maps embraces a very large area, since each sheet must be sensibly correct. A one-projection series is, however, quite suitable for the topographical maps of a comparatively small country, provided the projection is well chosen. It has the merit that by fitting together neighbouring sheets a smaller scale series may be reproduced by photography. The British Isles have been mapped on this system, the individual sheets being rectangular and not bounded by meridians and parallels. It is, however, generally desirable that the edges of sheets should coincide with meridians and parallels for convenience in taking bearings from the map. A conical projection with straight meridians is suitable, and by using two standard parallels the scale error is kept small. The latter was originally proposed for the 1/1,000,000 maps of India before the International Map projection was adopted.*

When all the individual sheets are not meant to fit together exactly, the choice of a projection is not so difficult, since many projections are indistinguishable from each other within the small area embraced. Ease of construction is then an important factor. The polyconic projection, simple or rectangular, is that most frequently used, since the radii of the parallels depend only upon their latitudes and are tabulated once for all. Adjacent sheets will fit along the parallels: the fit along the meridians, although not exact, is sufficiently good to enable field sheets to be joined.

MAP DRAWING

Plotting by Rectangular Co-ordinates. In the case of comparatively small surveys the plotting of all control points is most simply performed by rectangular co-ordinates from a central meridian. Their values may be worked out from the lengths of the triangle sides by plane trigonometry, the curvature of the earth being disregarded. Otherwise, the co-ordinates are computed from the geographical co-ordinates of the stations. No graticule is required for the purpose of plotting, but it can be constructed on the final map.

In large topographical surveys all control points, both on the field sheets and the fair map, are plotted by geographical co-ordinates with reference to a graticule.

Construction of the Field Sheet Graticule. The projections used for field sheets are approximately polyconical, but, to simplify their construction, each curve is drawn as a series of straight lines. The intersections of meridians and parallels are plotted either by rectangular co-ordinates or by the intersections of arcs. The data required in their construction will be found in such publications as the British War Office Projection Tables, the Survey of India Auxiliary Tables, the United States Coast and Geodetic Survey Tables, and in the official *Text Book of Topographical and Geographical Surveying* by Close and Winterbotham.

A graticule may be constructed without the aid of graticule tables

* "On the Projection for a Map of India and Adjacent Countries on the Scale of 1/1,000,000," by Col. St. G. C. Gore, Survey of India Professional Paper No. 1.

by computing for the graticule interval the linear distances along the meridians and along each parallel and plotting each compartment of the graticule as a trapezium with the meridians equally inclined to the parallels. This system, known as the Survey of India, or Col. Blacker's, projection, is sufficiently accurate for limited areas, and is largely used for field sheets. Tables of the quantities used in its construction are given in the Indian Auxiliary Tables, Thuillier and Smyth's *Manual of Surveying for India*, and the Royal Geographical Society's *Hints to Travellers*, but the plotting may easily be performed with the aid of a table of linear values of arcs of latitude and longitude.

As an example, let it be required to construct a graticule of 30′ × 30′ with 15′ intervals between latitudes 40° 45′ and 41° 15′N.

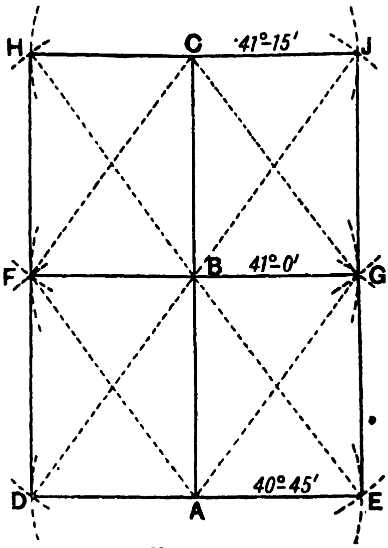


FIG. 182.

Taking the straight line ABC (Fig. 182) as the central meridian, mark off AB and BC to represent on the required scale the linear distances along the meridian from latitude 40° 45′ to 41° 0′ and from 41° 0′ to 41° 15′ respectively. From the tables, these distances in miles are AB = 17·251 and BC = 17·252. With centre A and radius representing the linear value of 15′ of arc of parallel in latitude 40° 45′, *viz.* 13·120 miles, describe short arcs at D and E. The corresponding radii for the latitudes of B and C are 13·071 and 13·021 miles respectively, and similar arcs are drawn at F and G from centre B and at H and J from centre C. The corners of the trapezia are located by diagonals, the lengths of which are obtained from the tables or are calculated from

$$d = \sqrt{m^2 + p_1 p_2},$$

where m = the length of the meridian in the trapezium,

p_1, p_2 = the lengths of the parallels.

Having obtained the length of the diagonals in the two lower figures as 21·658 miles and that for the upper figures as 21·629 miles, points D, E, F, G, H, and J are fixed by intersecting the previously drawn arcs with arcs swept from centres A, B, and C. The points are then joined with straight lines. Each line is commonly divided into three equal parts, so that, by joining corresponding points, the graticule is divided into compartments of 5′ of latitude and longitude.

The War Office tables are arranged for plotting as follows. The line ABC (Fig. 183) is drawn as before, and perpendiculars are erected through its points of division. Along these lines the abscissæ Ad, Ae, Bf, Bg, Ch, and Cj are set off equal to the tabulated values corresponding to the respective latitudes. The positions of the corner points d, e, h, and j are

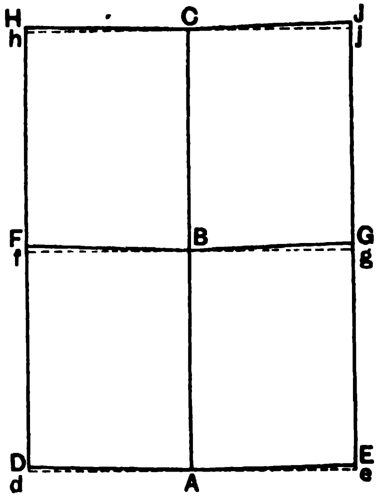


FIG. 183.

checked by scaling the diagonals dj and eh, which are also tabulated. From d, e, f, g, h and j are erected perpendiculars to the construction lines, on the side next the pole, and on them are marked off the ordinates dD, eE, etc., the values of which are tabulated for the respective latitudes and distances from the central meridian. The points so obtained are the intersections of the required meridians and parallels.

Plotting Trigonometrical Points. The mapping of detail on the field sheets being based upon the station positions, the latter must first be plotted with reference to the meridians and parallels of the graticule by using the computed latitudes and longitudes of the stations. Each point is plotted from the nearest meridian and parallel by co-ordinates parallel to them. The lengths of these co-ordinates represent the linear equivalents of the angular distances of the point from the graticule lines, and are computed by the use of tables giving the linear values of arcs of meridian and of parallel in different latitudes.

In place of scaling linear co-ordinates, the points may also be plotted by proportional division of the angular distances between the graticule lines. This is best done by diagonal scale. Although less accurate than plotting by linear distances, this method is sufficiently so for small scale work, and when many points have to be plotted it proves the quicker method. It has the merit that points may be placed in their correct relative positions on a graticule which has suffered expansion or contraction through change of humidity.

Fig. 184 illustrates the plotting of a station of latitude 1° 10′ 43″·75 N. and longitude 30° 11′ 14″·73 E. on a graticule which has been subdivided to 5′ sections.

The angular dis-

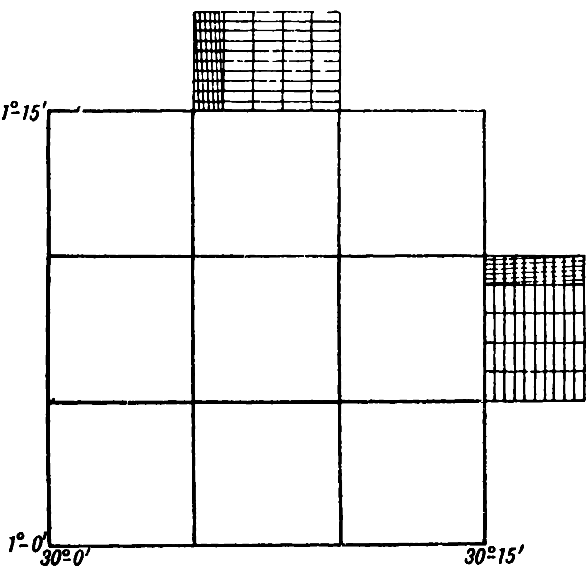


FIG. 184.

tances of the point from the nearest graticule lines are 43″·75 N. and 1′ 14″·73 E. To convert these to linear distances for the first method of plotting, we find from the tables that in mid-latitude 1° 10′ 21″.87, say 1° 10′, the value of 1″ of arc of meridian is 100·7658 ft, and in latitude 1° 10′ 43″·75, say 1° 11′, the value of 1″ of arc of parallel is 101·4323 ft. Therefore the required co-ordinates are

$$43''\cdot 75 \times 100\cdot 77 = 4{,}409 \text{ ft. N. of the } 1^\circ\ 10' \text{ parallel,}$$

$$\text{and } 74''\cdot 73 \times 101\cdot 43 = 7{,}580 \text{ ft. E. of the } 30^\circ\ 10' \text{ meridian.}$$

In applying the second method, two diagonal scales are constructed for the subdivision of a minute of meridian and of parallel respectively. The scales illustrated read to single seconds, but dimensions can be taken off to the nearest 0·1 sec. In high latitudes the value of 1 min. of parallel changes rapidly, and the same diagonal scale should not be used for all parallels: at least two, one near the top and the other near the bottom of the sheet, are required.

In either method the position of the plotted point should be checked by measurements from the farther sides of the graticule compartment.

Completion of Field Sheets. In the case of a small topographical survey for a special purpose, the field sheets may form the final maps. When, as in the production of maps for publication, they are used as material from which the final map is compiled, they must still be completed with the same refinement, since their accuracy cannot be improved upon by the cartographer.

During the field work, information collected on auxiliary sheets is inserted in the field sheets as soon as possible, and the latter are inked up daily. It is important that the sheets should not be overburdened with detail, particularly if the scale is larger than that of the final map. To promote clearness, it is almost essential to substitute numbers for names on the field sheets, the names being set out in a reference table or book. Before field sheets can be passed as complete, the edges should be compared with those of the adjoining sheets, and any discrepancies adjusted out or resurveyed.

Construction of the Final Map. The data from which the fair map sheets are prepared are the survey records of the framework and the field sheets and field books. The projection having been selected, the graticule is constructed by plotting a number of intersections of meridians and parallels by means of rectangular co-ordinates. Great care is necessary to ensure perpendicularity of the co-ordinates, and the operation is facilitated by the use of the Coradi co-ordinatograph. If a table of co-ordinates is not available, the projection must be calculated, and, unless for the smallest scales, it is necessary to take account of the spheroidal form of the earth. A sufficient number of points should be plotted so that they can be joined by straight lines without spoiling the appearance of the graticule, but a set of pearwood curves of large radius is useful. The stations of the framework are plotted as on the field sheets and carefully checked.

The transfer of detail from the field sheets is best accomplished by squaring in. In this process a series of small squares is drawn to the appropriate scales on both the field sheet and fair map, their positions

relatively to the control points being exactly the same on both sheets. The detail is then plotted on the map with reference to the squares. The method is particularly useful when the field scale has to be reduced. When a general topographical map is being constructed on a small scale, the cartographer must be competent to make a selection of the features to be included, so that the physical character of the country may be clearly represented.

Representation of Relief. The manner in which the relief of the ground is shown contributes largely to the quality of the map. The principal methods of representation are by means of: (*a*) Contours or Form Lines; (*b*) Hachuring; (*c*) Shading. These are used singly or in combination, but a map is of minor utility unless contours, or at least form lines, are shown. Although the other methods are sometimes used by themselves, they are best regarded as aids to the contour line system.

Contours. Contour lines give by far the most precise delineation of relief, and are indispensable for engineering and other purposes for which the values of the elevations are required. They should be numbered, and the vertical interval should have a constant value throughout the map. They do not obscure the detail as much as other systems, but to avoid any tendency to confusion they are best shown in brown. In rugged country, where the contours run close together, the nature of the relief may be presented without the necessity for close inspection of the map by drawing every fifth or tenth contour with a bolder line than the others. Form lines, or sketched contours, although of inferior precision, possess the merits of contours as a means of representing form, and are superior to hachuring or shading unaccompanied by contours.

To the inexperienced map user at least, the contour system alone does not impart to a map such a pictorial effect of relief that the general character of the country embraced by the sheet can always be understood at a glance. The difficulty may be overcome by the addition of hachures or hill shading but more easily by adopting the layer system of colouring the areas between adjacent contours. A range of graded tints is selected, each of which is applied between particular contours. There is no universally adopted colour scale. That on the Ordnance Survey half-inch maps embraces thirteen tints ranging, as the elevation increases, from green through buff to brown. Yellow and orange may be inserted between green and the paler browns, and the deepest browns sometimes merge into red, or are followed by blue greys and finally by white for the highest peaks. The result is sometimes very effective, but it frequently happens that the deeper tints are made so dark that the detail is obscured. If the map includes a wide range of elevations, it is difficult to obtain an entirely satisfactory system of colours for the higher ground, and it is then preferable to make one tint embrace several contours.

Hachuring. Hachuring is a method of indicating relief in which short lines, called hachures, are drawn at right angles to the contours, *i.e.* along the direction of steepest slope. On the assumption of vertical illumination of the features mapped, the flatter the ground the lighter does it appear, and the relative steepness of the slopes is so indicated in drawing the hachures. On flat slopes the lines are rather widely spaced, and are drawn lightly so that the white intervening spaces predominate. As the

contours approach each other, the hachures are drawn closer together, and the lines are heavier, giving an increasingly dark tone as the slope increases. Definite scales of ratios between black and white for different angles of slope have been worked to, but, as they are difficult of application by those unaccustomed to their use, the variations are commonly made arbitrarily.

It is impossible to secure that every irregularity which can be shown by contours will be faithfully represented by hachuring. Hachures should therefore be accompanied by contours. The pictorial effect of relief may be improved by drawing the hachures in such a manner as to suggest oblique illumination. The light is arbitrarily assumed to come

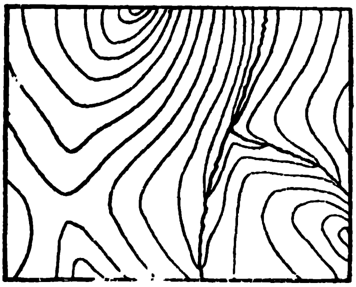


FIG. 185. CONTOURING.

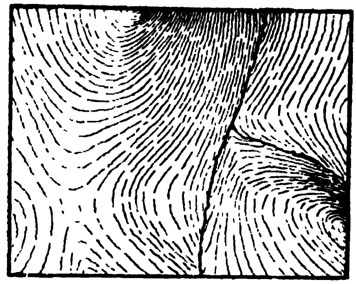


FIG. 186. HORIZONTAL HACHURING.

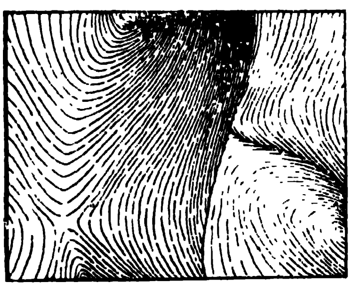


FIG. 187. HORIZONTAL HACHURING—OBLIQUE ILLUMINATION.

FIG. 188. VERTICAL HACHURING.

from the north-west or upper left-hand corner of the sheet and at 45° elevation. Slopes facing it are shown lighter, and those in shadow darker, than under vertical illumination. The result gives an entirely false impression of the steepness of the ground, and the addition of contour lines is more than ever necessary.

The drawing of hachures is tedious, and requires to be very carefully performed in order to produce the required effect of relief. They must be exactly perpendicular to the contours, and in consequence are curved between non-parallel contours. The lines should preferably be slightly wavy, and should break joint on the contours or, where the ground is very flat, on intermediate contours sketched in pencil.

Hachures possess the serious disadvantage that, unless perhaps when drawn in brown, they tend to obscure the other detail. In map reproduc-

tion, on other than the smallest scales, hachuring is now being superseded by the layer system, but it is sometimes employed in combination with the latter. Examples of hachuring in black may be studied on the older one-inch Ordnance Survey maps.

The above system is called vertical hachuring to distinguish it from the horizontal system. In the latter, the strokes are drawn parallel to the contours, and they therefore resemble broken form lines. By introducing the same number between each pair of contours, the tone is darkened as the slope increases, and the effect of relief may be emphasised by drawing thicker lines on steep slopes. Horizontal hachuring lends itself to oblique illumination, but it has not found the favour which has been accorded to the vertical system.

For the piece of ground represented by contours in Fig. 185, horizontal hachuring is shown with vertical illumination in Fig. 186, and with oblique illumination in Fig. 187. Fig. 188 represents vertical hachuring with vertical illumination.

Shading. Shading differs from hachuring in that different intensities of slopes are indicated by shading applied with a stump or by colour wash instead of by lines. It has the merit of being less laborious than hachuring, but does not exhibit as much detail of form. Shading is performed in grey or in brown, and the illumination may be vertical or oblique. The latter gives an effective result if the shading is not so heavy as to obscure the contours and other detail.

Except on atlas scales, shading is of little value unless the contours are drawn. In modern map reproduction the method is frequently used in combination with contours, the shading being printed in half tone. Hachures or layer tints may also be added, but in the latter case shading interferes with the colours, and, to minimise the confusion, very pale Indian ink should be used for the shade washes.

Conventional Signs. As the scale of the map decreases, it becomes increasingly difficult to show clearly all the information it is desired to convey. A full code of symbols is necessary to avoid overcrowding of the map with written descriptions. In the case of official maps, a sheet of conventional signs is prepared, and these must be strictly adhered to by the draughtsmen, and should be understood by the user of the map.

The civil engineer engaged in mapping should be guided in his selection of symbols by referring to official maps on a similar scale. It will usually be unnecessary to symbolise as much detailed information regarding existing artificial features as is required on general maps used for varied purposes. Information relating to circumstances likely to influence the design and construction of the proposed works must, however, be shown as clearly as possible. The character of lines of communication is always important. Railways are usually shown in black, and distinction is made between double and single lines, and standard and narrow gauges. Roads, as well as railways, must be drawn of exaggerated width for clearness, and indication should be given of their quality, particularly with regard to suitability for mechanical transport. Different classes are conventionally represented by differences in breadth, or in the boldness of the lines, or by colouring the better roads. All three conventions may be combined.

Triangulation and traverse stations should be shown with their reference numbers, different symbols being used to indicate the system to which they belong. Elevations shown should also be distinguished by different styles of figures according to the manner of their determination.

All the symbols adopted should be illustrated in a list of conventional signs drawn in a convenient position on the map or on a separate sheet.

Finishing the Map. Unless for indicating relief, colour washes should be used sparingly, but water areas should be tinted blue. Woods may be shown by a pale tint of green with or without tree signs, and boundaries are commonly emphasised by a band of colour.

In lettering the map, all names should be placed to the right of the point to which they refer and parallel to the top of the sheet, except in the case of such items as rivers, ranges of hills, or railways. Different kinds of features should have distinctive, but not elaborate, styles of lettering both for clearness and to avoid monotony.

The border should be of simple form, and should be divided in latitude and longitude. If the map extends over several sheets, it is an advantage in using the separate sheets if narrow overlaps are repeated on adjacent sheets.

In the case of maps intended for photographic reproduction, it is usual to perform the plotting to about twice the scale of the reproduction and the drawing is then reduced to the proper scale by photographic methods. The advantage of this procedure is that slight blemishes in draughtsmanship are reduced and become almost imperceptible. When drawing for photographic reduction it is important to see that the lettering is large enough to be clearly legible on the reduced scale. If the map is to be printed in more than one colour, drawings of the features to appear in each colour are prepared for photography.*

Models. The civil engineer may be called upon to prepare a model of the site of large works to illustrate the topographical conditions in the most expressive manner possible and one which readily appeals to persons unaccustomed to map reading.

The model is constructed from the data furnished by a contoured map. At the outset the vertical scale must be selected. This will be the same as the horizontal scale for large scale models, but for scales smaller than about one inch to a mile the relief may be emphasised by exaggerating the vertical scale two or three times. For models on atlas scales greater ratios are adopted.

The model is built up of sheets of paper, cardboard, or thin wood, each having a thickness proportional to the vertical interval between the contours. The individual contours are traced on these sheets from the map by means of transfer paper, and each sheet is then neatly cut along the outline of its contour. The shaped sheets are superimposed and glued together in their proper order and position, the correct placing of one sheet on another being facilitated if the upper contour is traced on the lower sheet. The resulting model is terraced, but is of value in that it exhibits the positions of the contours. For most purposes for which models are required, the steps should be filled in with wax or modelling

* See *Text Book of Topographical and Geographical Surveying*, edited by Col. Sir C. F. Close and Col. H. St. J. L. Winterbotham.

clay, with due regard to topographical form. The model is finally painted, detailed, lettered and varnished.

REFERENCES ON MAP CONSTRUCTION

ADAMS, O. S. "General Theory of the Lambert Conformal Conic Projection." U.S. Coast and Geodetic Survey, Special Publication No. 53, 1918.

— "General Theory of Polyconic Projections." U.S. Coast and Geodetic Survey, Special Publication, No. 57, 1919.

— "A Study of Map Projections in General." U.S. Coast and Geodetic Survey, Special Publication No. 60, 1919.

BRAZIER, H. H. "A Skew Orthomorphic Projection with Particular Reference to Malaya." *Proceedings of the* 1947 *Conference of Commonwealth Survey Officers.* London, 1949.

CLOSE, SIR C. F., and WINTERBOTHAM, COL. H. ST. J. L. *Text Book of Topographical and Geographical Surveying.* London, 1925.

— and CLARKE, A. R. "Map." *Encyclopædia Britannica,* 14th edition, 1929.

CRAIG, J. I. "Map Projections." Egyptian Survey Dept., Technical Lecture No. 3, 1909.

— "The Theory of Map Projections with Special Reference to the Projections used in the Survey Department." Egyptian Survey Dept., Paper No. 13, 1910.

CRAIG, T. "A Treatise on Map Projections." U.S. Coast and Geodetic Survey, 1882.

DEETZ, C. H. "Lambert Conformal Conic Projection with Two Standard Parallels, including Comparison of Lambert Projection with Bonne and Polyconic Projections." U.S. Coast and Geodetic Survey, Special Publication No. 47, 1918.

DEETZ, C. H., and ADAMS, O. S. "Elements of Map Projection with Applications to Map and Chart Construction." U.S. Coast and Geodetic Survey, Special Publication No. 68. Fifth Edition. Washington, 1945.

DUCHESNE, C. *Les Projections Cartographiques.* Brussels, 1907.

FINCK, J. K. *Topographic Maps and Sketch Mapping.* New York, 1920.

GERMAIN, A. *Traité des Projections des Cartes Géographiques.* Paris, *circ.* 1865.

HENDRIKZ, D. R. "The Theory of Map Projections." Trigonometrical Survey of South Africa. Special Publication No. 4. Cape Town, 1947.

HINKS, A. R. *Maps and Survey.* Cambridge, 1942.

— *Map Projections.* Cambridge, 1921.

— Article on "Map Projections" in *Dictionary of Applied Physics,* Vol. III, London, 1923.

HOTINE, BRIGADIER M. "The Orthomorphic Projection of the Spheroid." *Empire Survey Review,* Vol. VIII, No. 62; and Vol. IX, Nos. 63, 64, 65 and 66.

JURISCH, C. H. L. M. *A Treatise on Map Projections.* Cape Town, 1890.

KELLAWAY, G. P. *Map Projections.* London, 1946.

MELLUISH, R. K. *An Introduction to the Mathematics of Map Projections.* Cambridge, 1931.

RAISZ, E. *General Cartography.* New York, 1939.

ROYAL GEOGRAPHICAL SOCIETY. *New Geodetic Tables for Clarke's Figure of* 1880, *with Transformation to Madrid* 1924 *and other Figures.* London, 1927.

SMITH, R. S. *Manual of Topographical Drawing.* New York, 1894.

STEERS, J. A. *An Introduction to the Study of Map Projections.* London, 1929.

STUART, E. R. *Topographical Drawing.* New York, 1917.

STUBBS, G. C. "Malayan Map Projections and Military Grids." *Proceedings of the* 1947 *Conference of Commonwealth Survey Officers.* London, 1949.

"Tables for Polyconic Projection of Maps, based upon Clarke's Reference Spheroid of 1866." U.S. Coast and Geodetic Survey, Special Publication No. 5, 1917.

TISSOT, M. A. *Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques.* Paris, 1881.

WAINWRIGHT, D. B. "A Plane Table Manual." U.S. Coast and Geodetic Survey, Report for 1905, Appendix No. 7 (revised 1916).

WILSON, H. M. *Topographic Surveying.* New York, 1910.

YOUNG, A. E. "Some Investigations into the Theory of Map Projections." Royal Geographical Society, Technical Series, No. 1, 1920.

CHAPTER IX

RADAR RANGING AND TRIANGULATION

THE application of radar to surveying is a development of its use in air navigation and precision bombing during the later stages of the 1939–1945 war. In 1943 the first experiments were initiated by Major C. A. Hart, R.E., now Professor C. A. Hart of University College, London, and were carried out in co-operation with the Air Ministry and the Ministry of Aircraft Production, the Telecommunications Research Establishment, the Air Warfare Analysis Section, the Royal Aircraft Establishment, and with the support of Survey Units of the Royal Engineers, Royal Canadian Engineers and the U.S. Corps of Engineers and others, Major Hart being responsible for the research work on the survey side. Compared with the later apparatus for survey work, the original apparatus was somewhat crude, as it was designed primarily for air navigation and bombing work. As time went on, both apparatus and methods were improved and it is now clear that radar techniques can be made sufficiently accurate to allow them to have important applications in geodesy as well as in air and hydrographical surveying. On the American side, much important work has been done in recent years on the application of radar methods to geodesy by Col. Carl I. Aslakson of the 7th Geodetic Control Squadron, U.S. Army Air Force.

The introduction of radar has led to a form of triangulation, known as "radar triangulation," in which the lengths of the sides of the triangles are measured direct by radar. This form of triangulation can be used to fix the position of an aircraft at the moment when a photograph is being taken, thus enabling the amount of ground control required in air survey to be very considerably reduced. It can also be used to fix the position of a surveying vessel engaged on sounding work at a considerable distance offshore, but, so far as geodesy is concerned, its greatest use is to establish fixed points at very considerable distances apart, and to enable networks of ordinary triangulation which are separated by very wide water or other gaps, where it is impossible to establish a connection by ordinary triangulation or traverse, to be connected together.

Principles of Radar Measurements. The principle on which all radar measurements depends is based on the measurement of the time taken for a short, intense pulse of short-wave radio energy to travel from one point to another and then return to the sending station after reflection or re-transmission at the distant station. If the time interval, $2t$, between emission and reception is measured, and the velocity, v, of propagation of radio waves is known, the single distance is given by $d = v \, . \, t$. Since the velocity of radio waves is that of light and is very great, roughly 186,000 miles per second, it is obvious that, to be of any use, the time intervals to be observed are very small and very accurate methods of measuring them are needed. At present, it is possible to measure time intervals to the nearest tenth of a micro-second (0·1 micro- or μ-seconds, or $1/10^7$ seconds), that is, to a ten millionth part of a second, and no

finer single measurement is yet possible. A radio wave will travel about 30 metres in 0·1 μ-seconds and hence a single distance over which the sum of the times of the outward and return journeys is observed cannot at present be measured to a greater accuracy than about 15 metres. In practice, of course, other uncertainties such as the true, as opposed to the assumed, velocity of propagation of the waves, time lag in apparatus, etc., enter into the problem, so that, at the present stage of development, the limits of accuracy of a *single* measure of a distance by radar are about 17 to 20 metres, and these values are, for all practical purposes, independent of the length of the line.

Non-Co-operative and Co-operative Radar. There are two forms of distance measurement by radar—non-co-operative and co-operative radar. In non-co-operative radar the waves are reflected directly from the object or station whose distance is being measured. This form of measurement, commonly used in conjunction with directional wireless to determine a rough bearing as well as distance, is employed for military purposes in the detection of strange aircraft and for anti-aircraft gunnery, but its accuracy is not so great as that of co-operative radar, and its range is more limited. Consequently, co-operative radar is preferred for survey work. In this form, a special receiving instrument receives the signals at the distant point and, after a short delay of known duration, re-transmits them on a different frequency to the sending station, generally boosting up the strength of the signal in the process. The main transmitting and receiving apparatus may either be in the aircraft or at the ground station.

Gee-H and Other Radar Systems. In the British Gee-H system of fixing the position of an aircraft in space, two ground stations, fixed by ground methods and about 100 to 120 miles apart, are used. The aircraft carries a pulse transmitter and receiver and pulses are sent out to each ground station in turn and are re-transmitted on a different frequency to the aircraft, where they are received and the time interval between emission and reception is recorded or measured. Thus, in Fig. 189, A and B are the ground stations or "beacons" and C is the aircraft. The measurements give the lengths of the lines AC and BC. These lines are inclined lines well above sea level, but, as the heights of A and B are known from the ground survey data, and that of C is obtained from altimeter readings in the aircraft, the inclined lengths can easily be reduced to the equivalent arc lengths A′C′ and B′C′ at sea level, where A′, B′ and C′ are the points where verticals from A, B and C meet the surface corresponding to the surface of mean sea level. Accordingly, the lengths of the sides of the spherical triangle A′B′C′ become known

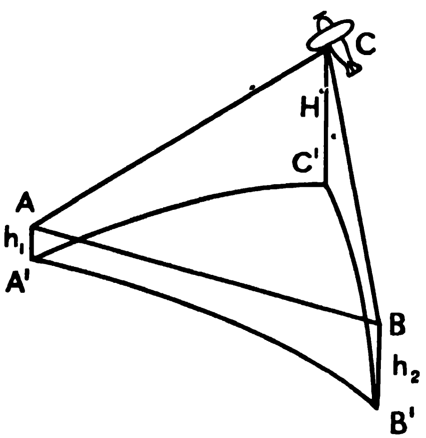


FIG. 189.

and the triangle can, if necessary, be solved to give the angles at A′, B′ and C′.

The British Oboe system differs from the Gee-H mainly in the fact that the transmitters are at the ground stations, and the signals, after reception and re-transmission at the aircraft, are received back at the same stations. This system is more accurate than the Gee-H system because it employs a much higher radio frequency, resulting in greater accuracy of distance measurement, and, as the receiving and measuring apparatus is at the ground stations, this apparatus can be heavier and more complex than can one carried in an aeroplane. On the other hand, the radio information is distributed between two ground stations, while other important information (altimeter readings, etc.) is in the aircraft, so that recording has to be done at three points and the times correlated. For this and for other reasons, the Gee-H is the system which British surveyors have had to accept as the only type of equipment so far satisfactorily adopted for field survey work.

In the British Gee and in the American Loran system, which is used mainly in navigation, the measured quantity is the difference between AC and BC. This does not fix the position of the aircraft but defines an hyperbola, with A and B as foci, on which the aircraft must lie. Consequently, a third ground station D is required, and the observations of the difference of distance from it and from one of the other stations to C define another hyperbola on which the aircraft must be situated. Hence, the position of the aircraft is given by the point of intersection of the two hyperbolæ.

The Decca system, first developed in Great Britain for navigational purposes, has now also been applied to hydrographical surveying. It differs from the Loran and Gee systems in that phase differences and not difference in time are measured, the position lines again being hyperbolic. Ranges up to 300 miles have been obtained over water in day-time, but the reliable range diminishes to about 100 miles at night. The accuracy varies with the position of the point to be fixed relative to the ground stations, but it is claimed that, at a distance of 30 miles over water, where conditions are more static than over land, and with sufficient repeat observations, an accuracy of the order of about ± 10 yards can be obtained in favourable conditions. Recent tests have shown that it is suitable for both inshore and offshore hydrographical work and it is also now being used satisfactorily for the tracking of strips of air photographs (page 483) for the purposes of Ordnance Survey large scale mapping. It operates on a wave length of about 3,000 metres.

In the Gee system, the ground stations are not independent of one another, as, when a pulse is transmitted to the aircraft C from the "master" station A, another pulse is sent to the "slave" station B. This causes a pulse to be released at B and sent to C so that the actual time difference recorded includes the time taken for the signal from A to reach B, and this time has to be allowed for. The necessity for "interlocking" the ground stations in this way means that the choice of sites for the latter is much more limited.

The American Shoran system, used in the U.S.A., Canada, and elsewhere, is operated on much the same principles as the Gee-H system

but works on a frequency of 210 to 260 megacycles per second, whereas the British Gee and Gee-H systems work on frequencies of about 30 to 43 megacycles per second (*i.e.*, wave lengths of 10 to 7 metres). When the rays consist of lines a very short distance above the earth's surface, as in the case of rays from a ship to stations on shore, the operation of the Shoran system, like that of the Gee-H system, is confined to the limits of ordinary visual observation. The Loran system can be used for much greater distances, up to about 1,400 nautical miles at night and 750 miles by day, but, although this is useful for ordinary navigational purposes, Loran is not accurate enough in itself for survey work. Consequently, the Coast and Geodetic Survey has evolved an " Electronic Position Finder " which combines the main features of the Shoran (Short Range) and Loran (Long Range) systems so that it is now possible to fix the position of a surveying ship working about 200 to 300 nautical miles from the shore accurately enough for purposes of hydrographical surveying.

Radar Ranging. In the methods described above, the distances measured are between ground stations and an aeroplane, not between two ground stations. Distances between two ground stations can be measured direct by radar provided the range is limited to the distance of visual observation, or not much beyond it, but near the ground abnormal atmospheric conditions may introduce errors in the assumed value of the velocity of propagation of the radio pulses. Even in a distance of 100 kilometres, near the limit of visual observations, the uncertainty of ± 20 metres in observing the time intervals means an error of 1/5,000 in the length of the line, which would be too great for ordinary geodetic purposes. Very long distances, say up to 500 or 600 kilometres, can, however, be measured with the aid of an aeroplane, an error of 20 metres then corresponding to errors of 1/25,000 or 1/30,000 of the length of the line. For this purpose, the plane flies across the line, approximately at right angles to it and near the centre, a series of simultaneous observations being taken at close intervals of time to each end of the line, at which suitable receiving and transmitting or re-transmitting apparatus has been fixed. Let p, q, r and s in Fig. 190 be different positions of the aircraft as it flies across the line AB. At p the distances pA and pB are measured, and at q the distances qA and qB, and so on for a number of points on each side of AB. In this way, the sum of the distances A to plane and plane to B is obtained for a number of points on the flight. The results are plotted as a curve against time and the minimum sum for all possible positions of the aircraft on the line of flight is obtained. This minimum sum will be the length of the line AB.

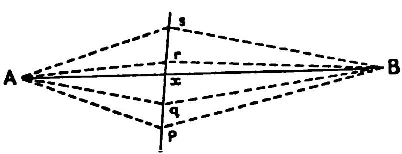


Fig. 190.

By repeating the observation for a number of flights, the precision of the determination can be increased and an accuracy of something like 1/50,000 finally obtained. This method of measuring the lengths of long lines is known as " radar ranging."

The same method can also be used for calibrating the apparatus.

Here the flights are made across a line whose length has been determined by ground methods. The difference between the length obtained by radar ranging and that determined by ground methods will give a correction which can be applied to other measurements made with the same apparatus.

Radar Triangulation Over Wide Gaps. It will be seen that radar ranging can be used to bridge very wide gaps, greater than can be bridged by ordinary or flare triangulation, such as the gap separating England and Norway. Thus, in Fig. 191, assuming that the points A and B are points fixed by accurate ground survey, the length and azimuth of the side AB are known or can be calculated and the distances AC, AD, BD, BC and CD can be found by radar ranging, the lengths of these sides being anything up to 300 to 600 miles. The lengths of all the sides in each triangle being now known, the angles can be calculated and the points C and D fixed. The same process could be extended to establish a "Grand Triangulation" with very long sides over a very large area of country, as is now actually being done over certain parts of Canada not yet reached by the ordinary geodetic triangulation.

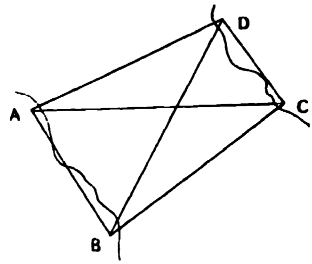


FIG. 191.

Variations of Velocity of Radio Waves with Atmospheric Changes and Height Above Sea Level. It is obvious that the accuracy of radar measurements of distance depends on an accurate knowledge of the velocity with which the pulses of energy are propagated along the path from transmitter to receiver. This velocity varies with variations in the value of the index of refraction of the atmosphere, which, in turn, is dependent on the values of atmospheric pressure, temperature and humidity. These values vary with height above sea level and so the velocity of propagation of radio waves varies with height above sea level, the values ranging from about 186,208 miles per second at sea level to about 186,248 miles per second at 25,000 feet above sea level. If C is a constant represented by the velocity of transmission of waves in free space, *i.e.*, in a vacuum, or say 186,271 miles per second, μ is the index of refraction for the atmosphere and V is the velocity of transmission in the atmosphere,

$$V = \mu C.$$

A good value of μ may be calculated from the formula of Englund, Crawford and Mumford, which is as follows :—

$$(\mu - 1) \times 10^4 = \frac{p}{2T}\left[2\cdot11 + \frac{100w}{p}\left(\frac{101\cdot59}{T} - 0\cdot00293\right)\right],$$

where T = absolute temperature Centigrade,

p = total atmospheric pressure in mms. of mercury,

w = partial pressure of water vapour in mms. of mercury.

This formula, however, would be inconvenient for computing a mean velocity for the path, so a number of approximate formulæ, assuming

normal atmospheric conditions, have been devised and are in use. The simplest is based on the assumption that, as a first approximation,

$$\mu_h = \mu_0\left(1 - \frac{h}{4R}\right),$$

where

μ_h = index of refraction at h feet above sea level,
μ_0 = index of refraction at sea level,
R = mean radius of the earth in feet for the area under consideration.

Then, if v is the mean velocity along the path,

$$v = v_0\left(1 + \frac{H+h}{8R} - \frac{L^2}{64R^2}\right),$$

where

v_0 = velocity at sea level under normal atmospheric conditions,
L = approximate slant distance ($v_0 \times t$),
H = height of aircraft above sea level,
h = height of ground station above sea level.

Reducing Observed Slant Distances to Surface Distances. If x in Fig. 190 is the point on the line of flight of the aircraft which lies on the straight line AB, the distances Ax and Bx that are actually obtained from the measurement are, of course, the slant distances from ground station to aircraft and these have to be corrected to bring them to "horizontal" lines at sea level. Consequently, the altitude of the aircraft above sea level must be known at the moment of reading and this is found from readings on the altimeter (a form of aneroid barometer) carried by the aircraft. Let this altitude be H feet above sea level and h be the height of the ground station above sea level. Let A in Fig. 192 be the ground station, C the position of the aircraft, O the centre of the earth and let the curved arc EF be mean sea level. The measured length is AC = L and the required length is the arc EF = s. Let K be the chord length EF. Then it is easy to show that, if R is the radius of the earth (supposed spherical),

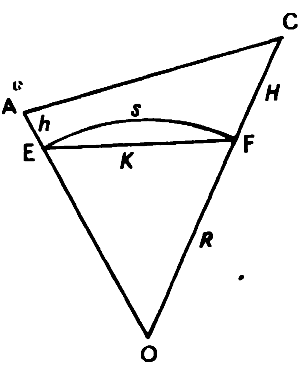


FIG. 192.

$$K = \sqrt{\frac{[L-(H-h)][L+(H-h)]}{\left(1+\frac{h}{R}\right)\left(1+\frac{H}{R}\right)}},$$

and the arc distance may now be computed from the equation

$$s = K + \frac{K^3}{24R^2} + \frac{3K^5}{640R^4} + \cdots.$$

Approximate Formulæ for Reduction of Recorded Slant Distance to Geodetic Distance. Tables have been prepared by the Air Warfare Analysis Section (A.W.A.S.) for reducing recorded times of transmission

to arc distances at sea level and these have been published as Appendix A to Part II, Section 7, of Air Survey Research Paper No. 11 (H.M.S.O.). These tables are based on somewhat more accurate formulæ for μ and v than those given above. Where the tables are not available, use can be made of one or other of a number of approximate formulæ for the direct conversion of the slant distances actually recorded in the apparatus (*i.e.*, time of transmission multiplied by an arbitrary calibration velocity which is assumed to be 186,218 miles per second) to the true geodetic distance at sea level, any one of which will give results that agree to within a metre or two with the A.W.A.S. tables. One of these, for example, is :—

$$A = \frac{2{\cdot}152}{10^8} M (H + h) + \frac{1{\cdot}794}{10^8} \frac{(H - h)^2}{M} - \frac{0{\cdot}2477}{10^8} M^3$$

where

A = Slant distance minus arc distance in miles,
M = approximate arc distance in miles (say recorded slant distance, true slant distance, chord distance or arc distance),
H = height of aircraft above sea level,
h = height of ground station above sea level.

Choice of Wave Length of Transmitted Pulses. For radar work it is necessary to use very short-length waves and to confine observations to cases where no land masses intervene between ground station and aircraft. In ordinary radio broadcasting, the comparatively long waves which are employed may travel between transmitter and receiver either directly parallel, or nearly parallel, to the ground or else by reflection from the ionosphere.* The direct wave ordinarily travels but a short distance beyond the limits of ordinary visual observation, although when refraction is exceptional it may travel very much further. Hence, long distance reception depends almost entirely on the waves received after reflection from the ionosphere, and, if this were not so, long distance reception would generally not be possible. It would be very difficult, if not impossible, to trace the exact path of a ray to the ionosphere and back and to calculate its length sufficiently accurately for survey work. Hence, we must use waves that travel direct from transmitter to receiver in a straight line or along a path which is only slightly curved by refraction, and avoid interference by rays reflected from the ionosphere. High frequency waves, *i.e.*, waves of short wave length, are not reflected back to the earth from the ionosphere as those of low frequency are, because they penetrate it instead of being reflected. In the case of ordinary short waves of lengths over about 5 to 15 metres there are indications of bending by diffraction † in addition to the effects of ordinary refraction, and on this account these waves will travel somewhat beyond the limits of visual observation, but ultra-short waves of lengths up to say 1 to 2 metres will not be so much bent by refraction and diffraction and will

* The ionosphere is a layer of negatively charged ions which surrounds the ordinary atmosphere at a height of from 60 to 180 miles above sea level and which has the property of reflecting waves of certain frequency.

† Diffraction is a bending of the rays of radio energy caused by the lower part of the wave front being slowed down relatively to the upper part as it travels along the surface of the earth, where the velocity of propagation is less than it is in air.

follow a visual path whose length can be more accurately calculated. Hence, for the most accurate work, ultra-short waves which travel along an optical path must be used, but range can be increased at the expense of accuracy by using waves of slightly longer length, though not of such a length as to be reflected from the ionosphere.

Although straight line transmission means that an aircraft must be visible from the ground station and ranges are thus governed by the earth's curvature, this does not mean that radar measurements are confined to distances corresponding to those of ordinary visual observations. In such observations the stations are at comparatively low elevations, and their intervisibility is controlled by the neighbouring topography and the earth's curvature. When aircraft are used, however, they raise one end of the " line of sight " far above the heights commonly met with on the ground and so greatly extend the range where direct transmission along straight-line or slightly curved paths is possible.

Another factor which makes short wave transmission preferable is that the rate of growth of the pulse depends upon frequency of transmission, and short waves give a sharper and narrower impression of the pulse on the recorder than long waves do, thus giving a more accurately defined commencement of the pulse-front from which measurements can be taken. Hence, on this score, short waves give the more accurate results. Electronic difficulties, on the other hand, are increased as wave length decreases, so that the wave lengths used so far lie mainly between 1 and 10 metres. Ultra-short waves of length less than a metre travel along an optical path and give the most accurate results.

Method of Operation of Radar. In the foregoing we have spoken of the transmission of " pulses " of energy and by this we mean intense bursts of energy which last a very short time, only a few micro-seconds, and are then repeated after a relatively long period, about 0·001 second. The object of the comparatively long delay between bursts is to give the radiated energy time to travel to the distant target and then return to the transmitter receiving station before the next burst is emitted. In this way, no confusion or interference in reception is likely to arise between the outward and returning waves. The interval between bursts is still shorter than the time interval during which the screen retains the flourescent image or the eye retains an image after the disappearance of the object viewed. Hence, with suitable apparatus and a large number of repetitions of the bursts, the emitted and received signals can be made to form images which can be seen by the eye or can be photographed as a continuous picture.

The signals are displayed on the screen of a large cathode ray tube forming part of a cathode ray oscillograph. The picture on the screen is formed by a stream of very fast negatively charged particles, or electrons, shot off from a heated cathode, C in Fig. 193, so as to impinge on a thin layer of flourescent material F on the inside of an air-exhausted glass tube, where the point of impact shows as a bright spot. This stream is concentrated into a narrow pencil or beam and focussed on F by passing through holes in the plates D, A_1 and A_2 and the metal cylinder A_3, where D is negatively charged with respect to C and A_1, A_2 and A_3 are positively charged, A_3 being at a lower potential than A_1

and A_2.* From A_2 the pencil passes between two vertical and two horizontal plates, V and H, and is deflected horizontally or vertically when

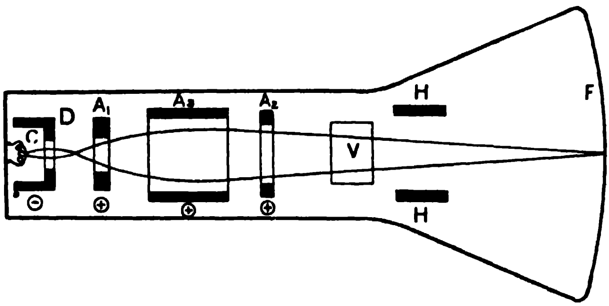


FIG. 193.

opposite plates are at different electrical potentials. It is deflected very rapidly to and fro horizontally when the vertical plates are connected to a high frequency oscillatory circuit known as a "time base" which is operated by a crystal oscillator. During one of these oscillations the bright spot marking the end of the pencil on the screen travels from one side of the latter to the other in a very short but known time, the motion being directly proportional to time, and, when it reaches the far end of the screen, it is released to return instantaneously to the starting point. These oscillations are repeated extremely quickly, so that, thanks to the flourescence of the screen and the tendency of the eye to retain an image for an appreciable fraction of a second, the motion of the spot appears as a steady bright horizontal line. In practice, in the case of the Gee and Gee-H systems, this line is marked by calibration "pips" at regular intervals corresponding to single "Gee-Units" or tenths, a Gee-Unit being the distance travelled by the radio waves in 66·66 μ-secs : in other systems straight time or distance measurements may be used. The emission and reception pulses cause variations in a circuit connected to the horizontal plates and these variations, which are synchronised with the time base so as to occur always when the moving spot is at the same point on its path across the screen, are registered as vertical displacements or "blips" on the time scales, and hence the measurement consists in determining the distance in Gee-Units between the blips made by the emitted and received pulses. Actually, there normally are six scales on each tube, of which one set is used only for navigational purposes and one set for distance measurements. Each set consists of an upper, or main, scale, 25 Gee-Units in length, which gives the whole number readings, and two lower, or expanded, scales, slightly longer than the main scale, and extended to represent about 1·2 Gee-Units, so that the first, second and third decimal places are obtained from these scales, the emission pulse appearing on one scale and the "reflected" pulse on the other.

* The parts D, A_1, A_2 and A_3 thus form what is virtually an electric lens, the function of which is to bring the emitted electrons to a focus on the flourescent screen.

Fig. 194 shows one set of three scales, the upper being the main, or slow, time base and the two lower the extended, or fast, time base. The length of the main base is 25 Gee-Units, so that each of the smaller divisions is one Gee-Unit. In the lower extended base, the large pips

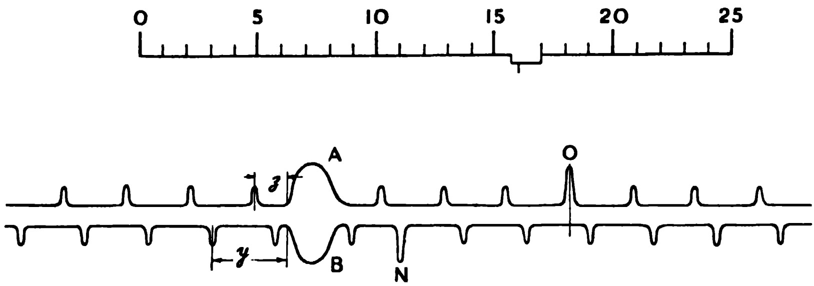


Fig. 194.

at O and N denote the whole numbers and the smaller pips are at intervals of 0·1 Gee-Units. The blip caused by the emission pulse is at A on the upper extended base and the blip caused by the reception of the reflected pulse is at B on the lower extended base. These two blips are brought into coincidence by the operation of a knob and the reading is then the whole number indicated by the displacement on the main base, plus the number of tenth pips lying on the upper extended scale between O and N, plus $(y - z)$, where z is the distance of the beginning of pulse A to the nearest pip on the left and y is the distance from the beginning of blip B to the pip on the lower scale which lies to the left of the pip on the upper scale to which z is measured. The small quantity $(y - z)$ is measured by means of a measuring microscope or Leitz magnifier on the original negative of the photographic record. In the illustration, the reading is $16 + 0{\cdot}2 + (0{\cdot}122 - 0{\cdot}055) = 16{\cdot}267$.

In order to facilitate readings and have a permanent and conveniently measured record of them, a second cathode ray tube is fitted in the automatic observer in parallel with the one in the cockpit, the extended or high speed time base being photographed synchronously with the survey photographs or emission pulse, and the main time base giving whole numbers displayed immediately afterwards. Accurate measurements of the last two or three decimal places may then be made at leisure on the negatives with a micrometer.

Errors and Accuracy of Radar Measurements. The principal sources of error in radar measurements are :—

(i) Errors in the measurement of time intervals giving distance.

(ii) Errors of recording.

(iii) Errors in allowance for the time taken in transmission through the electrical circuits.

(iv) Unknown variations from standard in the assumed or calculated velocity of propagation of the radio waves largely owing to changes in atmospheric pressure, temperature and humidity near the ground and to changes in the profile of the latter.

The first three are errors in the determination of the time and are

independent of the length of the line, but the fourth is, in general, proportional to the length of the line. The error in the measurement of the time interval is inherent in the apparatus and the degree of accuracy to which it can be read. It is not easy to judge on the record the exact point of beginning of the pulses and it is well for the interval to be measured by three different individuals. The errors in delay in the electrical apparatus may be found and allowed for by taking a number of sets of readings as the aircraft flies across a line of known length. At present, with careful work, it may be assumed that the random errors in the case of Oboe may amount to a total probable error of about ± 20 metres for a single determination and with Gee-H to about ± 30 metres, the errors for all practical purposes being independent of the length of the line. The accuracy of the final determination can, of course, be greatly improved by taking a large series of measurements.

It is as yet rather difficult to give definite figures for the accuracy to be expected of radar ranging because much of the experimental work so far has been done with a modified apparatus intended originally for bombing operations, and not with apparatus specifically designed for survey work. In Italy, immediately after the war, the mean of the results of 22 flights across a base of 618 kilometres gave an estimated accuracy of about 1/30,000. Other, more recent, measurements in the United States and Canada, involving a comparison of lengths measured by ordinary geodetic methods with lengths measured by radar ranging using the Shoran apparatus, show discrepancies from about 1/20,000 to 1/100,000, or better. As matters stand, however, it may be assumed that radar ranging over long lines cannot be guaranteed to give results much better than, say, 1/20,000, but this figure may be considerably improved in future as apparatus is improved and more is known about the paths of the emitted pulses and the velocity of propagation. Recent experiments in Canada, Australia and the U.S.A. are proving very encouraging.

Other Applications of Radar to Survey Work. In addition to fixing the position of an aircraft in space and to radar ranging and triangulation, there are several other important applications of radar to survey work. One is controlling the navigation of the aircraft when it is taking strips of photographs with fore and aft and lateral overlaps, as described on page 483. This method is now in use in the Colonial Empire for topographical mapping. Radar is also used to control the navigation of the aircraft in flare triangulation and to indicate to the pilot when he is near a position in which a flare should be dropped.

In hydrographical work, radar has important applications in fixing the position of a ship when it is making soundings at some distance from the coast line. Here, if the distance is not much beyond the distance of ordinary visual observation, the ship's position may be found by direct radar triangulation from fixed beacons on shore. For distances much beyond that of ordinary visual observation, it is possible from shore stations to fix an intermediate string of buoys parallel to the coast line. Each buoy is fitted with a reflector beacon and a ship equipped with a precise radar apparatus can fix its position by radar measurements to the buoys. A second string of buoys parallel to the first may also be fixed and used to extend the survey still further out to sea.

It is obvious that radar opens up great possibilities when applied to extensive survey operations but at present it is in the early stages of development and a good deal of research work is in hand or needs doing. There is little doubt that accuracy, and possibly range as well, will be considerably increased before long. We have already seen (page 338) that the application of radar has led to a re-examination of certain geodetic formulæ, which, in their present form, are not applicable with sufficient accuracy to the lengths of lines now capable of measurement by radar, and additional small terms have had to be introduced into these formulæ to make them applicable to lines up to about 500 miles in length. (See Appendix V., pages 541-546).

In the above paragraphs it has only been possible to give a very general description of the present applications of radar to surveying and of the the main principles underlying them, but, of course, there are many details which are outside the scope of this book. For those interested, however, much useful information is to be found in a series of "Air Survey Research Papers" now being published in lithographic form by the War Office and obtainable from H.M. Stationery Office, most of which have been written by Professor C. A. Hart, to whom the early development of the methods now in use owe so much. Principal among these pamphlets may be mentioned No. 11, "Some Methods of Map-Making from Radar-Controlled Photography," and No. 19, "Mapping by Remote Control with the Aid of Radar." An interesting and fairly technical account of the principles and use of geodetic applications of radar by the same author is published in the *Bulletin Géodésique*, No. 10, 1er Octobre 1948, under the title "Some Aspects of the Influence on Geodesy of Accurate Range Methods by Radar Methods, with Special Reference to Radar Techniques."

APPENDIX I

MECHANICAL COMPUTING

By L. J. Comrie, Ph.D., F.R.S.

The highly developed calculating machines that are now available are frequently employed in surveying calculations, not only in the field, but also in cases where calculations can be concentrated in centralised offices. A great variety of machines, both hand and electric, has been

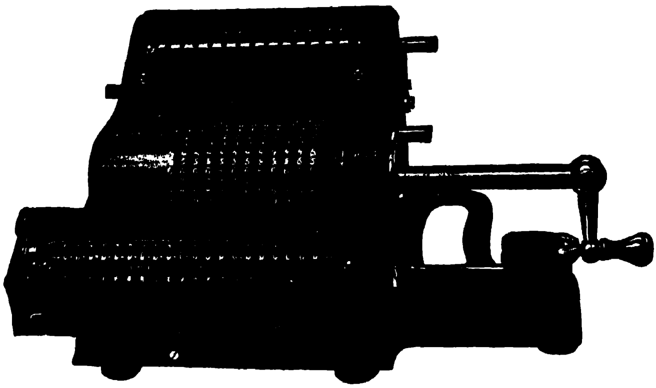


Fig. 195. Brunsviga Calculating Machine, Model 20.

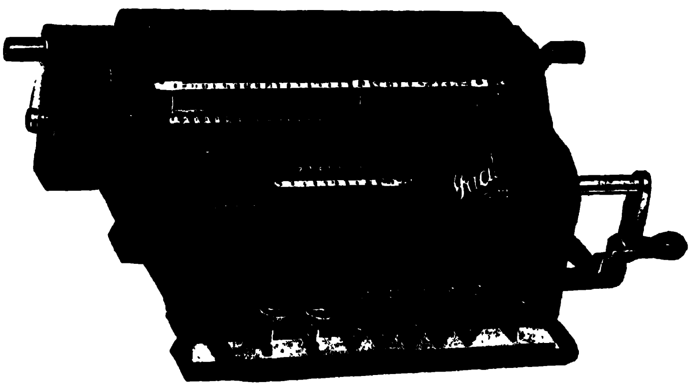


Fig. 196. 10-Key Facit Calculating Machine, Model LX.

produced for commercial applications. A machine for the particular needs of the surveyor should fulfil the following requirements:

(1) It should be a hand machine that can be easily transported, so that the same machine can be used either in the office or in the field.

(2) It should not be subject to mechanical breakdowns, as it may frequently have to be used a long way from large towns where expert repair service is available.

(3) It should be able to stand the abuse to which it may be subjected by junior or native clerks.

(4) It should be an up-to-date machine embodying the latest and best developments of the calculating machine art.

(5) It should serve the true purpose of a calculating machine, namely to lessen fatigue, increase accuracy and speed up the work. Its operation should not be tedious or require great concentration or skill.

(6) The price should be reasonable.

The above considerations rule out electric machines, as they require more upkeep, and current is not usually available in the field; also full keyboard machines on account of their weight and size, and also key-driven machines such as the Comptometer. The choice therefore falls on barrel-type machines, such as the Brunsviga (Fig. 195), Marchant, Odhner and Facit (Fig. 196). Of these the Brunsviga has perhaps been more widely applied in the past than any of the others, and will, therefore, be described in detail. The Facit, which is similar in principle, is a close second.

A calculating machine is really simple, and may be learned in an hour, as no new arithmetical conceptions are required. It will perform the four fundamental processes of arithmetic—addition, subtraction, multiplication and division—and since all arithmetic consists of combinations of these four processes, it may be said that it can be used for any arithmetical problem. Consider first addition. It is necessary to have a means of conveying to the machine the number to be added; this is done by means of setting levers, each of which may be set in any one of the positions 0, 1 . . . 9. The right-hand lever represents units, the next tens, and so on. The number set is shown in a sight dial, situated immediately above the levers, so that the setting is easily verified. The next requirement is a register for showing the results of additions; this is situated in the front of the machine, and is known as the product register. A forward turn of the operating handle adds whatever number is set on the levers to whatever number is already in the product register. Needless to say, the product register can be cleared or zeroised, which is done by a lever on its right. A lever on the left of the machine will clear the setting levers, or they may be altered at will by hand. For subtraction the handle is turned backwards.

Multiplication is really only continued addition. If 6 be added seven times, the same result, namely 42, is obtained as by multiplying 6 by 7. Consider this principle applied to the multiplication of 789 by 456. The ordinary hand or schoolboy method results in a "sum" as shown. If 789 be set on the levers of the machine, and the handle turned 6 times, the product 4734 will appear in the product register. In order to eliminate the risk of error in counting the handle turns, a revolution counter or multiplier register is provided at the top of the machine. We now require the product of 789 and 5, but wish the units of this product to be added in the tens wheel of the product register. This is easily effected, because the product register is mounted in a movable carriage. By means of the carriage spacer in front the carriage may be stepped one position to the right, so that its second

$$\begin{array}{r} 789 \\ \underline{456} \\ 4734 \\ 3945 \\ \underline{3156} \\ 359784 \end{array}$$

or tens wheel is aligned with the units setting lever. At the same time a moving indicator in the multiplier register has moved one position to the left, indicating that turns of the handle will now be effective in building up the tens figure of the multiplier. The operator makes five turns, but the new product 3945 is not separately shown; it has been added to 4734, so that the product register now shows the product of 789 and 56, namely 44184. In this respect the machine is superior to the schoolboy; it does not write its separate "partial products", to be added at the end, but adds them as it goes along. The problem proposed is completed by stepping the carriage once more to the right, and making four more turns. The two factors and the product, namely 456, 789 and 359784, are now all visible; the two former may be verified, and the latter copied on to the computing sheet.

A slight variation, known as short-cutting, is usually applied to the digits 6, 7, 8 or 9 in a multiplier. Thus a multiplier 19 may be treated as 9 in the units position and 1 in the tens position, but two forward turns in the tens position and one backward or subtractive turn in the units position will yield exactly the same result, with a saving of 7 turns, i.e. 3 instead of 10. This process soon becomes automatic and involves no mental effort; on the average it reduces the number of handle turns required by forty per cent, especially if the process is extended to include 5's that are preceded or followed by a larger digit.

Mechanical division is equally simple, and, like multiplication, merely follows the schoolboy theory. Suppose 54321 is to be divided by 23; the normal pen and paper method is shown alongside.

```
23)54321(2361
   46
   --
    83
    69
    --
    142
    138
    ---
      41
      23
      --
      18
```

The dividend 54321 is set on the setting levers, and the handle turned, so that the dividend appears in the product register (it is usually put in the left of this register). Having cleared setting levers and multiplier register, the divisor 23 is set on the levers, choosing levers in such a way that the 23 is aligned with the 54 of the dividend. As the first figure of the quotient is 2, we require to multiply the divisor by 2, and to subtract the product from the first two figures of the dividend. It has already been seen that multiplication by 2 is effected by making two turns, and that subtraction is effected by backward turns; hence the operation required is evidently two backward turns, after which the remainder 8 (together with the unused figures 321 of the dividend) appears in the product register. An apparent difficulty arises with the multiplier register, which now becomes the quotient register, counting, as in multiplication, the number of handle turns. It was observed that in short-cutting two backward turns produced an 8, whereas now a 2 must be recorded. Actually the multiplier register has two sets of figures on each wheel, and a sliding window exposes one set or the other. The multiplication set consists of white figures 0, 1 . . . 9, and the division set of red figures 9, 8 . . . 0. The machine senses whether the operator is multiplying or dividing, by virtue of the fact that the first turn of the handle is forward in one case and backward in the other, and automatically exposes the desired set of figures. At this stage there is a red 2 in the multiplier register. The next operation in the

hand method is to bring down another figure ; on the machine this is done by stepping the carriage one position to the left, so that instead of having 8 under 23, we now have 83. Three backward turns will produce 3 in the quotient, and leave the remainder 14 (together with the unused 21) in the product register. Step the carriage again. Instead of estimating that the next figure of the quotient will be 6, and making six backward turns, the operator merely turns backwards, making the remainder 119, 96, 73, 50, 27 and 4 with successive turns. As soon as the remainder is less than the divisor the process is naturally finished, and the correct figure (here 6) of the quotient appears in the multiplier register. The carriage is stepped, and one turn found to be sufficient in the next position. Should the operator inadvertently turn too many times, he is informed of the fact by the ringing of a bell, and corrects the over-turn by a forward turn.

On the Brunsviga a transfer lever on the left of the machine enables

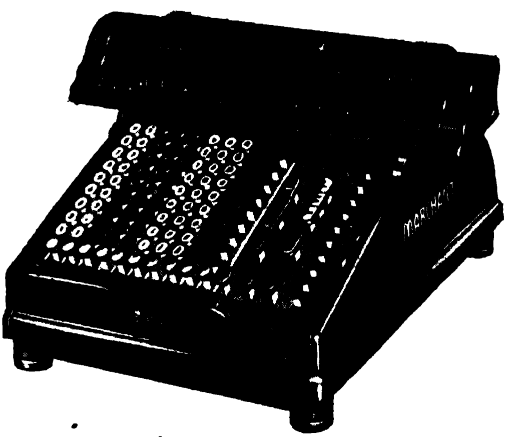


FIG. 197. MARCHANT ELECTRIC CALCULATING MACHINE.

any number in the product register to be transferred to the setting levers while the product register is being cleared. This feature is useful when the product of three or more factors is required, or when the complement of a negative result in the product register is required ; in the latter case the negative result is transferred to the levers, and the handle turned backwards.

In the Facit machine, shown in Fig. 196, the setting is done by ten setting keys, which are operated like the keys of a typewriter. Thus to set 64407 we touch the 6 key, the 4 key twice, the 0 key and the 7 key. This machine, which possesses all the virtues enumerated on page 519, is described more fully in a book listed on page 529.

Simple as these processes are, both in theory and practice, many electric machines, such as the Frieden, Madas, Marchant and Monroe make them automatic. The Marchant, illustrated in Fig. 197, is a full keyboard machine, with a row of auxiliary keys on the right to act as multiplier keys. When one of these keys—say the 6—is depressed, the machine makes six turns, and the carriage is displaced to the next

position. If, during this turning, another key is depressed, the machine will finish the first digit of the multiplier, move to the next position, turn in the second digit of the multiplier, and move to the third position. In other words, the setting of the multiplier can always be one step ahead of the multiplication. As the machine makes 1300 revolutions a minute, or 22 a second, even the highest digit (9) is absorbed in four tenths of a second. Hence the machine keeps pace with the setting of the multiplier, and yields its answer not more than half a second after the last digit is set. It is, indeed, so fast that short-cutting does not pay! Division is fully automatic once the dividend and divisor are set, and takes about half a second for each figure of the quotient. Machines like this, although unsuitable for field use, are of great value where computations can be concentrated. They have been extensively used in recent years for the adjustment of triangulations, in which systems of more than a hundred normal equations frequently have to be formed and solved.

The principal advantages offered by the use of calculating machines are:

(1) They increase the speed with which computation may be done, both directly, and also indirectly, by virtue of the fact that there are fewer errors to be removed than in the case of hand or logarithmic calculations.

(2) They relieve the computer of much mental effort. In particular they remove the *fear* of making errors, which is one of the most fatiguing elements in computing.

(3) They will perform all four arithmetical processes, in any desired combination, e.g. $ab + cd$, whereas logarithms may be used for multiplications and divisions only.

(4) They permit the employment of junior grades of labour for routine calculations.

(5) They are used with natural values of the trigonometrical functions instead of with logarithmic values. Every computer knows that the differences of the log sin or log tan of a small angle, or of the log cos or log tan of a large angle, are large and troublesome, whereas in a natural table the differences of sines or cosines are never troublesome, while the tangent is difficult only in the case of large angles, and even then trouble can usually be avoided by using the cotangent.

In addition to easier interpolation, natural values of the trigonometrical functions offer a more equable sensitivity for inverse interpolation, as will be seen from the following table.

DIFFERENCES FOR 1″ IN A 7-FIGURE TABLE

Angle	sin	cos	tan	cot	log sin	log cos	log tan	log cot
0°	48	0	48	∞	∞	0	∞	∞
15	47	13	52	724	78	6	84	84
30	42	24	65	194	37	12	49	49
45	34	34	97	97	21	21	42	42
60	24	42	194	65	12	37	49	49
75	13	47	724	52	6	78	84	84
90	0	48	∞	48	0	∞	∞	∞

For the cosines of small angles or the sines of large angles, the sensitivity of the natural table is more than twice that of the logarithmic table; in other words the differences for a small variation are more than twice as great, so that the uncertainty in an angle interpolated from an accurate function is only half as great. In logarithmic work an angle less than 45° is often determined from its sine, and one greater than 45° from its cosine, in order to avoid insensitive parts of the table. It will be seen that the natural sine of an angle of 60° is more sensitive than the logarithmic sine of an angle of 45°. Again a tangent formula is frequently used in logarithmic work, especially with auxiliary angles whose magnitude cannot be controlled, to avoid insensitive regions, although this often leads to heavy interpolation near 0° and 90°. Since a tangent is usually computed as the quotient of two numbers, the machine user can always divide the smaller of his two numbers by the greater, thus obtaining a quotient between 0 and 1, whose sensitivity is always ample yet never embarrassingly large.

Calculating machines are particularly useful in least square adjustments, in which they save a great deal of writing and adding, as quantities ab entering into a summation $[ab]$ are automatically added on the machine as each product is formed. The Brunsviga Model 20 (Fig. 195) is also capable of evaluating $[abc]$, which arises when unequal weights are assigned to observations. The clearing of the 20-figure product register may be controlled by a partial clearing lever, whose effect is to prevent the left-hand ten figures from being cleared. If a_1b_1 is formed in the usual way in the right of the product register, it may be transferred by means of the transfer lever to the setting levers; multiplication by c_1 gives $a_1b_1c_1$, which again may be transferred to the setting levers. The carriage is now moved to the extreme right, and the product turned into the left of the product register for storing. The operator proceeds to form $a_2b_2c_2$ in the same way, the left-hand half of the product register, containing $a_1b_1c_1$, remaining uncleared, and showing $a_1b_1c_1 + a_2b_2c_2$ and finally $[abc]$.

The use of proportional parts in interpolation may be dispensed with if a calculating machine is available. Thus to interpolate sin 12° 34′ 56″·78 from

x	$\sin x$	Δ'	$\cos x$	Δ'
12° 34′ 50″	0·217 8120		0·975 9907	
		+ 473		− 105
12 34 60	0·217 8593		0·975 9802	

set 0·217 8120 on the setting levers, and multiply by 1·000. Clear the setting levers and multiplier register and set 473; multiply by 1·000 and check both the settings by verifying that the product register shows 217 8593 followed, of course, by three ciphers. Change the multiplier to 0·678, when the product register will show the required result, namely 0·217 8441, which can, if desired, be immediately transferred by means of the transfer feature to the setting levers for further multiplication.

Twin machines (described below) may be used for the simultaneous interpolation of the sine and cosine of an angle. The same is true also of any machine with sufficient keyboard capacity. Thus to get both the

sine and cosine of 12° 34′ 56″·78, with a 10-column machine, we would enter into the product register (in two stages if necessary) 8120 000 9907 000,

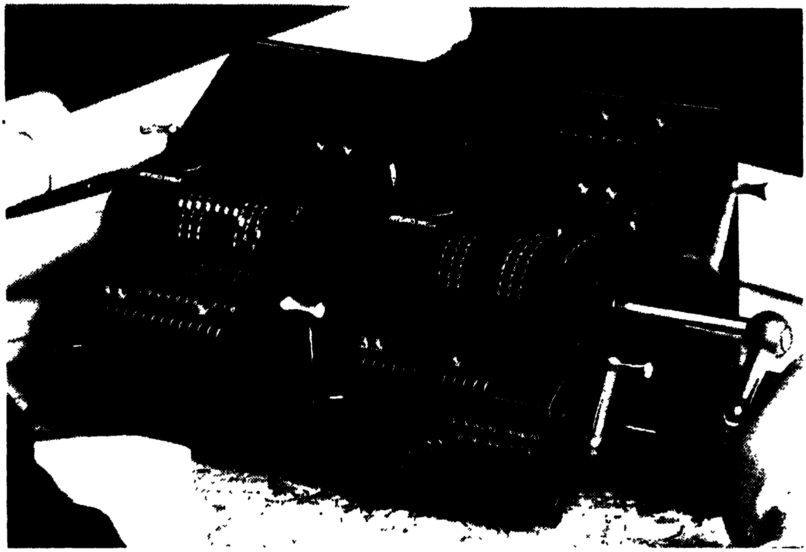


FIG. 198. TWIN BRUNSVIGA CALCULATING MACHINE.

and then set on the setting levers 472 999 9895, which is obviously equivalent to + 473 in the sine position and − 105 in the cosine, as is easily seen by turning the handle and producing 8593 000 9802 000. The remainder of the interpolation is as in the previous case, when the product register shows 8440 694 9835 810 so that

$$\text{sine} = 0{\cdot}217\ 8441 \quad \text{cosine} = 0{\cdot}975\ 9836$$

Fuller descriptions of mechanical interpolation, including the once-dreaded problem of inverse interpolation, are given in *Chambers's Six-figure Mathematical Tables*, Volume II, and in the book on the Twin Marchant listed on page 529.

Another type of machine, specially developed for survey calculations, consists of two single machines mounted together, so that both can be turned by the single crank handle. One machine always rotates in the same direction as the handle, but, by means of a control lever, the other can be controlled so that it rotates in the same or in the opposite direction ; or it may remain stationary. The first such machine was the Twin Brunsviga (about 1930). English attempts (1940) to produce a similar machine were superseded (1941) by the coupling of hand Marchants in pairs. The Twin Marchant was discontinued in 1945, but the Twin Brunsviga, as shown in Fig. 198, is now available again.

The obvious application of twin machines is to cases where two numbers are multiplied by a common multiplier, as when the sine and cosine of an angle are multiplied by a length, which occurs when resolving bearing and distance into rectangular co-ordinates. But, just as the

logarithmic computer develops methods suited to the means at his disposal, so does the machine user develop techniques with the powerful means at his disposal. As an illustration suppose we wish to evaluate b from

$$b = \frac{a \sin B}{\sin A}$$

If we divide a by $\sin A$ on the right-hand machine, while at the same time $\sin B$ is set on the left-hand machine, and this machine is switched to go in the opposite direction to the right-hand machine, the desired quantity b will be in the product register of the left-hand machine, and $a \div \sin A$ will be in the multiplier register. In other words, a single operation has evaluated the complete expression.

The field in which extensive use has been made of these machines is in military surveying, especially with rectangular co-ordinates, familiarly known as eastings and northings. The classical problems of bearing and distance, intersections, resections and inaccessible bases, so tedious with logarithms, lose all their terrors with a twin machine. Full descriptions of these machines and of these applications have been given elsewhere (see page 529) by the present writer. They are now also used extensively for traverse calculations.

Another great success that has been scored by twin machines is in the interpolation of pairs of double-entry tables, such as those giving eastings and northings from latitude and longitude (or *vice versa*), or for converting one system of rectangular co-ordinates to another. If A and B are both functions of two independent variables, and if n and m are the fractions of the tabular interval in these two variables, the tables can be arranged for the following interpolation formulæ :—

$$A = a_0 + a_1n + a_2m + a_3nm + a_4n^2 + a_5m^2$$
$$B = b_0 + b_1n + b_2m + b_3nm + b_4n^2 + b_5m^2$$

The quantities $a_0 \ldots a_5$ and $b_0 \ldots b_5$, which are tabulated, are set on the two components of a twin machine and multiplied simultaneously by their common multipliers. A check is provided, not by the weak process of repetition, but by having a similar inverse table, which is entered with the result of the first calculation, and should produce the original data.

The laborious calculations necessary to convert geographicals to rectangular co-ordinates, or *vice versa*, are known to many. They can now be replaced by the use of two pairs of double-entry tables, one for each conversion ; each checks the other. A further advantage can also be secured. If in the two sets of formulæ certain of the higher-order or elliptical terms are neglected, small inconsistencies arise. These are now avoided by computing first, from an agreed or adopted formula, the table for converting a geographical graticule to a rectangular grid—by far the easier table to compute since, for any fixed latitude, it is usually a power series in the difference between the longitude of the point and that of the central meridian. The inverse table is formed, not from the fearsome formulæ of the textbooks, but by numerical inversion. With such pairs of direct and inverse tables and a twin machine, conversions can now be made in a fraction of the time formerly taken.

The following list shows the tables that may be used for natural values of the trigonometrical functions. The full titles are given in the references at the end of this Appendix.

No. of Decimals	Interval of Argument	Author
4	$1'$ and $0^{\circ}{\cdot}1$	Milne-Thomson and Comrie
4	$10''$	Comrie
5	$1'$	Comrie (with a decimal dropped)
5	$10''$	H.M.N.A.O.
5	$0^{\circ}{\cdot}01$	Lohse
5	$0^{g}{\cdot}01$	Steinbrenner
6	$10''$	Brandenburg or Peters
6	$1'$	Comrie
6	$0^{\circ}{\cdot}01$	Comrie, Peters
6	$0^{g}{\cdot}01$	Peters
6	$0^{g}{\cdot}001$	Peters
7	$10''$	Brandenburg
7	$0^{\circ}{\cdot}001$	Peters
7	$0^{g}{\cdot}001$	Peters
7	1^{c}	Comrie (H.M.S.O.)
8	$1''$	Peters or U.S. Coast and Geodetic Survey
8	$0^{\circ}{\cdot}01$	Buckingham
8	$0^{\circ}{\cdot}001$	Peters
8	$0^{g}{\cdot}01$	Roussilhe and Brandicourt
15	$10''$	Andoyer

The 7-figure tables (not mentioned above) for every $10''$ by Benson and by Ives are marred by errors. Gifford's 8-figure tables at interval $1''$, which also contain a considerable number of errors, are, unfortunately, not arranged semi-quadrantally, so that two openings are necessary to find the sine and cosine of an angle. Gifford's tables, of which the tangents were pirated in U.S.A. in 1948 by Huey, were superseded in 1939 by the tables of Peters (the copy for which was prepared by Comrie), in which all four functions of any angle appear on a single line. With this excellent table in existence, it is difficult to see why the U.S. Coast and Geodetic Survey produced a table of sines and cosines only, awkwardly arranged; there is no preface to explain the motive, or the method of calculation.

The use of the decimal division of the quadrant has become increasingly popular in France, and has now been compulsory in Germany for some time. It suits the metric system of length, since, approximately,

$$1^{g} = 100 \text{ km.} \qquad 1^{c} = 1000 \text{ metres} \qquad 1^{cc} = 10 \text{ metres}$$

In other words, there is a one-to-one relation between units of length and units of angle. The same advantage occurs when feet and the sexagesimal system are combined, since $1'' = 100$ feet nearly. Inconveniences occur with metres and seconds, since $1'' = 30$ metres nearly. The difficulties of degrees, minutes and seconds can often be avoided by working entirely in seconds.

Surveyors have long been familiar with *Chambers's Seven-figure*

Mathematical Tables. After seventy years the type for these became so worn that new tables, known as *Chambers's Six-figure Mathematical Tables*, have been prepared by Comrie, in two volumes of 600 pages each. Volume I contains logarithmic values, and Volume II natural values. The decision to give six rather than seven figures was partly due to the fact that this was the greatest number of decimals that could be made linear in a *collection* of tables. These tables provide for those numerous cases in scientific work generally where four decimals just fail. For five-figure work, this six-figure table (with a decimal dropped) has the advantage of a small interval of argument. For survey use, six decimals suffice for angles measured to the nearest second of arc, which is ample in all but geodetic work, for which the excellent eight-figure tables to 1″ by Peters are available. Each volume contains a helpful bibliography of tables to more than six decimals. The legible figures, the attention that has been paid to typography, the many tricks of tabulation, and the generous explanation will appeal to all surveyors, whether engaged in field or in office work. An abridged edition of 400 pages includes the tables in degrees, minutes and seconds—to which surveyors are still tied by the divisions on their theodolites.

The impossibility in each survey office of having all the machines and tables described, and many others, as well as the cost of expert labour abroad, are factors that have led to the establishment in London of a central calculating bureau (known as *Scientific Computing Service*), with all known mechanical and library facilities. One table-making machine (not described above) will integrate from finite differences up to the sixth (a process used in subtabulation or systematic interpolation to smaller intervals) or will difference functions to the fifth difference, printing all differences. It is well known that differencing of functions that should be smooth is the most powerful means of detecting accidental (but not systematic) errors. The kind of work that lends itself to centralised computing, and is being done thus on an increasing scale, is figural adjustment by least squares, reductions of large triangulations, the preparation of conversion and other tables for adopted projections, and the reduction of refined and extensive series of astronomical observations for position or azimuth.

It must be emphasised that the best result from a calculating machine cannot, in general, be obtained by simply taking the formulæ and methods developed for logarithmic work, and applying the machine to them. The expert computer will start *de novo*, and build up, from fundamental formulæ and first principles, methods that utilise to the full the new facilities afforded by even the simple machines described. One simple illustration suffices to illustrate the possible advantages of a machine. If we require a radius vector r from rectangular co-ordinates x and y, the usual logarithmic method is based on

$$\tan\theta = \frac{x}{y}$$

$$r = x \operatorname{cosec}\theta = y \sec\theta$$

involving four entries in tables. The direct formula

$$r = \sqrt{x^2 + y^2}$$

would involve five entries and much writing, but may be evaluated directly on a machine without any tabular entries or writing, although the use of Barlow's *Tables* or Zimmermann's squares to 100,000 will shorten the labour of finding the square root.

REFERENCES

ANDOYER, H. *Nouvelles Tables Trigonométriques Fondamentales (Valeurs Naturelles).* In three volumes. Paris, Hermann et Fils, 1915, 1916 and 1918.

BARLOW'S *Tables of Squares, Cubes, Square Roots, Cube Roots and Reciprocals of all Integer Numbers up to* 12,500. Fourth Edition, edited by L. J. Comrie. London, Spon, 1941.

BRANDENBURG, H. *Siebenstellige trigonometrische Tafel für Berechnungen mit der Rechenmaschine.* Second edition. Leipzig, Alfred Lorentz, 1931. Not available.

— *Sechsstellige trigonometrische Tafel für Berechnungen mit der Rechenmaschine.* Leipzig, Alfred Lorentz, 1932. American reprint, 1945.

BUCKINGHAM, E. *Manual of Gear Design, Section One. Eight Place Tables of Angular Functions in Degrees and Hundredths of a Degree.* New York, Machinery, 1935.

CHAMBERS. *See* Comrie.

COMRIE, L. J. *Four-Figure Tables of the Natural and Logarithmic Trigonometrical Functions with the Argument in Time.* London, Scientific Computing Service, 1931.

— *Interpolation and Allied Tables.* London, H.M. Stationery Office, 1936.

— *On the Application of the Brunsviga Twin* 13z *Calculating Machine to Survey Problems.* Second Edition. London, Scientific Computing Service, 1940.

— *The Twin Marchant Calculating Machine and its Application to Survey Problems.* London, Scientific Computing Service, 1942.

— (Compiled by). *Seven-Figure Trigonometrical Tables for every Second of Time.* London, H.M. Stationery Office, 1939.

— *Chambers's Four-Figure Mathematical Tables.* Edinburgh and London, W. and R. Chambers, 1947.

— *Chambers's Six-Figure Mathematical Tables.* Edinburgh and London, W. and R. Chambers. Volume I, Logarithmic Values, 1948; Volume II, Natural Values, 1949. Abridged Edition, 1950.

COMRIE, L. J., and HARTLEY, H. O. (Translated and revised by). *Modern Machine Calculation with the Facit Calculating Machine Model LX.* London, Scientific Computing Service, 1939.

GIFFORD, E. *Natural Secants to every Second of Arc and Eight Places of Decimals.* London, Scientific Computing Service, 1929.

H. M. NAUTICAL ALMANAC OFFICE. *Five-Figure Tables of Natural Trigonometrical Functions.* London, H.M.S.O., 1947.

LOHSE, O. *Tafeln für numerisches Rechnen mit Maschinen.* Second Edition. Leipzig, Wilhelm Englemann, 1935. Not available.

MILNE-THOMSON, L. M., and COMRIE, L. J. *Standard Four-Figure Mathematical Tables.* Edition A, with positive characteristics in the logarithms; Edition B, with negative characteristics in the logarithms. London, Macmillan, 1931.

PETERS, J. *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Grades.* Leipzig, Teubner, 1938. There was an American reprint in 1942.

— *Sechsstellige Tafel der trigonometrischen Funktionen, enthaltend die Werte der sechs trigonometrischen Funktionen von zehn zu zehn Bogensekunden des in 90° geteilten Quadranten und die Werte der Kotangente und Kosekante für jede Bogensekunde von 0° 0′ bis 1° 20′.* Bonn, Dümmler, 1929.

— *Sechsstellige trigonometrische Tafel für neue Teilung.* (Die Werte der sechs trigonometrischen Funktionen für jede Minute des in hundert Grade geteilten Quadranten.) Berlin, Wichmann, 1930.

— *Sechsstellige Werte der Kreis- und Evolventen-Funktionen von Hundertstel zu Hundertstel des Grades.* Bonn, Dümmler, 1937.

— *Sechsstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Neugrades.* Berlin, Wichmann, 1938.

PETERS, J. *Achtstellige Tafel der trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten.* Berlin, Reichsamt für Landesaufnahme, 1939. Reprinted by Edwards Bros., Ann Arbor, 1943.

— *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Neugrades.* Berlin, Reichsamt für Landesaufnahme, 1941.

— *Achtstellige Werte der trigonometrischen Funktionen für jedes Tausendstel des Grades.* Bonn, Dümmler, 1950.

ROUSSILHE and BRANDICOURT. *Tables à 8 Décimales des valeurs naturelles des sinus, cosinus et tangentes dans le système décimal, de centigrade en centigrade.* Section de Géodésie de l'Union Géodésique de Géophysique Internationale. Publication Spéciale No. 1. Paris, Dorel, 1925.

STEINBRENNER, G. *Fünfstellige trigonometrische Tafeln neuer Teilung zum Maschinenrechnen.* Braunschweig, Grimme Natalis & Co., 1914. Not available.

U.S. COAST AND GEODETIC SURVEY, Special Publication No. 231. *Natural Sines and Cosines to Eight Decimal Places.* Washington, D.C., U.S. Government Printing Office, 1942.

ZIMMERMANN, L. *Vollständige Tafeln der Quadrate aller Zahlen bis* 100,009. Reprinted by Edwards Bros., Ann Arbor, 1946.

A review of the three 8-figure natural tables by Gifford, Peters and the U.S. Coast and Geodetic Survey is given in the *Empire Survey Review*, No. 47 (Vol. 8), p. 42 (January 1943). Since then the table by Peters, which is by far the most convenient of the three, has been reprinted in U.S.A., under licence.

Information about tables, both new and old, and, to a lesser extent, about calculating machines, may be found in a new quarterly, *Mathematical Tables and Other Aids to Computation*, produced under the sponsorship of the National Research Council of America. Each issue contains descriptions of recent mathematical tables, lists of errors in tables, notes on manuscript tables, etc. The British agents are Scientific Computing Service Ltd., from whom also the tables in the above list may be obtained.

This same firm has published a book of 450 pages entitled *An Index of Mathematical Tables*, by A. Fletcher, J. C. P. Miller and L. Rosenhead, of the University of Liverpool. Part I gives, in twenty-four sections dealing with various functions or groups of related functions, lists of the known tables of those functions, with particulars of the author, date, range, interval, number of decimals and facilities for interpolation. Part II gives, in alphabetical order of authors, bibliographical details of the tables each has published. This unique and monumental work is the standard reference book from which information about existing tables may be obtained.

APPENDIX II

PROOF OF FORMULA FOR SAG CORRECTION FOR A HORIZONTALLY SUSPENDED TAPE

CONSIDER a perfectly flexible tape suspended under a known tension with its two ends at the same elevation. Imagine this tape to be cut at a certain point where the tension or pull is T and the tangent to the curve at this point makes an angle ψ with the horizontal. Then the vertical component of the tension is $T \sin \psi$ and the horizontal component is $T \cos \psi$. The change in the vertical and horizontal components at the end of a very short element of length dl may be written $d(T \sin \psi)$ and $d(T \cos \psi)$ respectively. These changes in the components must balance the components of the external forces acting on the element. The only external force is $w \,.\, dl$, acting vertically downwards, where w is the weight of the tape per unit of length.

Hence,

$$d(T \sin \psi) = wdl.$$
$$d(T \cos \psi) = 0.$$

Integrating,

$$T \sin \psi = wl + A.$$
$$T \cos \psi = B.$$

where A and B are constants of integration.

Take the origin at the middle or lowest point of the span and measure all lengths from there. At this point, by principles of symmetry, the tape is horizontal so that $\psi = 0$ and $l = 0$. Let the tension there be T_0. Then it follows that $A = 0$ and $B = T_0$. This gives :—

$$T \sin \psi = wl.$$
$$T \cos \psi = T_0.$$
$$\therefore \tan \psi = \frac{wl}{T_0}.$$
$$\therefore \cos \psi = \frac{1}{(1 + \tan^2 \psi)^{\frac{1}{2}}} = \frac{1}{\left(1 + \frac{w^2l^2}{T_0^{\,2}}\right)^{\frac{1}{2}}}$$
$$= 1 - \frac{1}{2}\frac{w^2l^2}{T_0^{\,2}} + \frac{3}{8}\frac{w^4l^4}{T_0^{\,4}} - \ldots$$

and

$$\sin \psi = \frac{\tan \psi}{(1 + \tan^2 \psi)^{\frac{1}{2}}}$$
$$= \frac{wl}{T_0}\left(1 - \frac{1}{2}\frac{w^2l^2}{T_0^{\,2}} + \ldots\right).$$

We now have :—

$$dx = dl \cos \psi$$
$$= dl\left(1 - \frac{1}{2}\frac{w^2l^2}{T_0^{\,2}} + \frac{3}{8}\frac{w^4l^4}{T_0^{\,4}} - \ldots\right).$$
$$\therefore x = l - \frac{1}{6}\frac{w^2l^3}{T_0^{\,2}} + \frac{3}{40}\frac{w^4l^5}{T_0^{\,4}} - \ldots$$

the constant of integration vanishing since $x = 0$ when $l = 0$.

Let S be the total chord length and L the total length of the curve. Then, since the curve is symmetrical about the lowest point, $x = \frac{1}{2}S$ when $l = \frac{1}{2}L$. Consequently,

$$\tfrac{1}{2}S = \tfrac{1}{2}L - \frac{w^2L^3}{48T_0^2} + \frac{3}{1280}\frac{w^4L^5}{T_0^4}.$$

$$\therefore\ S = L - \frac{w^2L^3}{24T_0^2} + \frac{3}{640}\frac{w^4L^5}{T_0^4}.$$

Again, let F be the tension at the end of the tape where $l = \frac{1}{2}L$. At this point we have :—

$$F\cos\psi = T_0.$$

$$\therefore\ F\left(1 - \frac{w^2L^2}{8T_0^2} + \ldots\right) = T_0.$$

$$\therefore\ \frac{1}{T_0} = \frac{1}{F}\left(1 + \frac{w^2L^2}{8T_0^2} - \ldots\right)$$

$$= \frac{1}{F}\left(1 + \frac{w^2L^2}{8F^2}\right) \text{ approximately.}$$

$$\therefore\ \frac{1}{T_0^2} = \frac{1}{F^2}\left(1 + \frac{w^2L^2}{4F^2}\right).$$

$$\therefore\ S = L - \frac{w^2L^3}{24F^2} - \frac{11}{1920}\frac{w^4L^5}{F^4}.$$

The last term in this expression is negligible in practice.

To obtain the sag of the tape,

$$dy = dl\sin\psi$$

$$= dl\left(\frac{wl}{T_0} - \frac{1}{2}\frac{w^3l^3}{T_0^3} + \ldots\right).$$

$$\therefore\ y = \frac{wl^2}{2T_0} - \frac{w^3l^4}{8T_0^3},$$

the constant of integration again vanishing since $y = 0$ when $l = 0$. Again, the sag, Y, is the value of y when $l = \frac{1}{2}L$, and hence :—

$$Y = \frac{wL^2}{8T_0} - \frac{w^3L^4}{128T_0^3}$$

$$= \frac{wL^2}{8F} + \frac{w^3L^4}{128F^3}.$$

Finally, the slope at the ends is given by :—

$$\tan\psi = \frac{wL}{2T_0}$$

$$= \frac{wL}{2F}\left(1 + \frac{w^2L^2}{8F^2}\right),$$

and

$$\psi\ (\text{in seconds}) = \frac{wL}{2F\sin 1''}\left(1 + \frac{w^2L^2}{24F^2}\right).$$

APPENDIX III

PROOF OF THEOREM IN SPHERICAL TRIGONOMETRY *

To prove that, if PBC is a spherical triangle, right angled at C, and the angles P and $90° - B$ are very small, then :—

$$\eta = c - b = \frac{p^2}{2}\cot b - \frac{p^4}{24}\cot b\,(1 + 3\cot^2 b),$$

$$P = p\,\operatorname{cosec}\,(b + \tfrac{2}{3}\eta),$$

$$\frac{\pi}{2} - B = P\cos\,(b + \tfrac{1}{3}\eta).$$

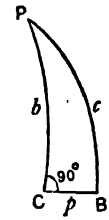


FIG. 199.

We have, by the ordinary rules for the solution of a right-angled spherical triangle :—

$$\cos c = \cos b\cos p.$$

$$\therefore\quad \cos\,(b + \eta) = \cos b\cos p.$$

$$\therefore\quad \cos b\cos\eta - \sin b\sin\eta = \cos b\left\{1 - \frac{p^2}{2} + \frac{p^4}{24} - \ldots\right\}.$$

$$\therefore\quad \cos b\left(1 - \frac{\eta^2}{2} + \ldots\right) - \sin b\left(\eta - \frac{\eta^3}{6} + \ldots\right)$$
$$= \cos b\left\{1 - \frac{p^2}{2} + \frac{p^4}{24} - \ldots\right\}$$

$$-\,\eta\sin b - \frac{\eta^2}{2}\cos b + \frac{\eta^3}{6}\sin b = -\frac{p^2}{2}\cos b + \frac{p^4}{24}\cos b - \ldots$$

The first approximation is $\eta = \frac{p^2}{2}\cot b$ and the second gives :—

$$\eta\sin b = \frac{p^2}{2}\cos b - \frac{p^4}{24}\cos b - \frac{\eta^2}{2}\cos b$$
$$= \frac{p^2}{2}\cos b - \frac{p^4}{24}\cos b - \frac{p^4}{8}\cos b\cot^2 b.$$

$$\therefore\quad \eta = \frac{p^2}{2}\cot b - \frac{p^4}{24}\cot b\,(1 + 3\cot^2 b).$$

Again,

$$\tan P = \tan p\,\operatorname{cosec}\,b$$
$$= \frac{1}{\sin b}\left\{p + \frac{p^3}{3} + \ldots\right\}.$$

* See page 329.

$$\begin{aligned}
\therefore\quad P &= \tan P - \tfrac{1}{3}\tan^3 P \ . \ . \ . \\
&= \frac{1}{\sin b}\left\{p + \frac{p^3}{3}\right\} - \frac{p^3}{3\sin^3 b} \\
&= \frac{1}{\sin b}\left\{p - \frac{p^3}{3}\frac{\cos^2 b}{\sin^2 b}\right\} \\
&= \frac{p}{\sin b}\left\{1 - \frac{p^2}{3}\cot^2 b\right\} \\
&= \frac{p}{\sin b}\left\{1 + \frac{p^2}{3}\cot^2 b\right\}^{-1} \\
&= \frac{p}{\sin b + \dfrac{p^2}{3}\cot b\cos b} \\
&= \frac{p}{\sin b + \frac{2}{3}\eta\cos b} \\
&= \frac{p}{\sin(b + \frac{2}{3}\eta)}.
\end{aligned}$$

We also have :—

$$\cos B = \cos b \sin P.$$

$$\therefore\quad \sin\left(\frac{\pi}{2} - B\right) = \cos b \sin P.$$

$$\begin{aligned}
\therefore\quad \left(\frac{\pi}{2} - B\right) &= \cos b\left\{P - \frac{P^3}{6} + \ . \ . \ .\right\} \\
&\qquad + \frac{1}{6}\cos^3 b\left\{P - \frac{P^3}{6} + \ . \ . \ .\right\}^3 \\
&= P\cos b\left\{1 - \frac{P^2}{6}(1 - \cos^2 b) + \ . \ . \ .\right\} \\
&= P\cos b\left\{1 - \frac{P^2\sin^2 b}{6} + \ . \ . \ .\right\}.
\end{aligned}$$

Using the approximation $P = p \operatorname{cosec} b$, this gives :—

$$\begin{aligned}
\frac{\pi}{2} - B &= P\cos b\left\{1 - \frac{p^2}{6}\right\} \text{ approximately} \\
&= P\cos b\left\{1 - \frac{\eta\tan b}{3}\right\} \\
&= P\left(\cos b - \frac{\eta}{3}\sin b\right) \\
&= P\cos(b + \tfrac{1}{3}\eta).
\end{aligned}$$

APPENDIX IV

ACCURATE MEASUREMENTS OF DISTANCES BY COMBINED OPTICAL AND ELECTRONIC MEANS

IN principle, the method devised by Bergstrand for the accurate measurement of distances by modulated light waves which is referred to on page 190 is somewhat similar to that employed by Fitzeau in 1849 to determine the velocity of light, although the details of the actual apparatus are very different to those of the apparatus used by Fitzeau.

In Fig. 200 L is a source of light with a concave mirror to concentrate the rays into a beam and P is an amplifying photo-electric cell which receives the rays of light after reflection by a mirror M situated at the distant station. Cr is a crystal controlled oscillator which transmits a series of electrical high frequency oscillations to L and P. These oscillations have the effect of varying the intensity of the light emitted at L and at the same time of affecting the sensitivity of the photo-electric cell at P, so that the sensitivity varies with the same frequency as the pulses of emitted light. After reflection at M, the light is received on the cathode of the photo-electric cell, where it causes variations in the intensity of the current corresponding to the variations in the intensity of the light received. The phase difference between the tension directly received from Cr and that due to the variations in the intensity of the reflected light will be proportional to the distance LM $= D$ and may be used to measure D. Maximum current will flow when $D = ND_n$, where N is a whole number and D_n a quantity inversely proportional to the frequency n.

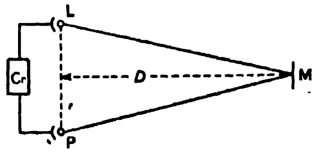


FIG. 200.

In practice, as maximum current does not give a sharply defined point, the receiving apparatus is somewhat modified so that a direct current galvanometer connected to P receives impulses of current in opposite directions and records zero when opposite impulses have the same numerical value.

The variations in the intensity of light from the source L in Fig. 200 are imposed during the passage of the light through a Kerr cell, Ke in Fig. 201, which is placed between two crossed Nicol's prisms, N_1 and N_2, the planes of polarisation of the prisms being at right angles to one another and making an angle of 45° with the direction of the applied field. The Kerr cell consists of a couple of conducting plates with a thin layer of nitrobenzene between them. These plates are connected to the high frequency oscillator Cr, which has a frequency of 8·3 megacycles per second, corresponding to a wave length of 36 metres, and an amplitude of 2,000 volts. This arrangement has the property of varying the amount of light emerging from the prism N_2, the amount passing at any time depending on the square of the tension of the applied field in accordance with the equation

$$J = J_0 \sin^2 kV^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $J =$ intensity, $J_0 =$ maximum intensity, k is a constant and $V =$ difference of potential between the plates. This equation is not of the simple harmonic type corresponding to the oscillations given out by

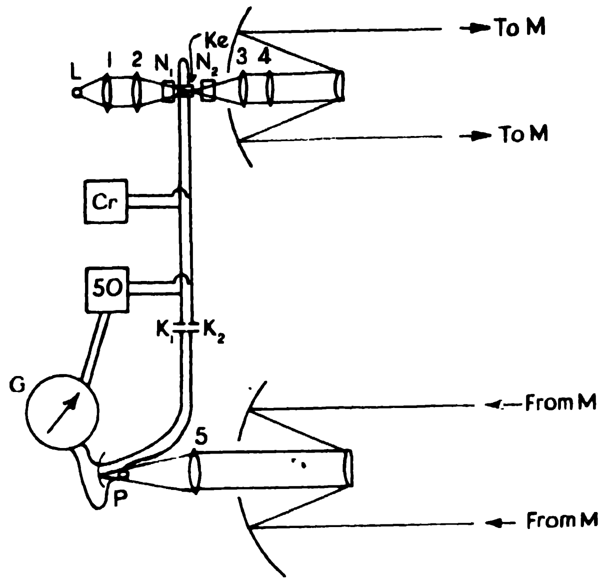


FIG. 201.

the oscillator, but, by adding these oscillations to those from a low frequency 50-cycle transformer with an amplitude of 6,200 volts which is also connected to the plates of the Kerr cell as shown in Fig. 201, the variations in the emitted light are very similar in form to the high frequency oscillations from the crystal. The 50-cycle transformer gives an approximately square wave of the type $\sqcap\sqcup\sqcap\sqcup$, so that the tension from it remains practically constant for half a cycle but changes in sign at the end of each half cycle. Thus, these changes occur 100 times per second. The intensity of the light emitted from N_2 is now assumed to be of the form

$$J_1 = C_1 + C_2 \sin\{\omega(t - t_1)\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\omega = 2\pi n$, n being the frequency of the high frequency oscillations, $t =$ time from the beginning of one of these oscillations, $t_1 =$ time taken between time of arrival of impulse from Cr and its effect on the light emerging from N_2, C_1 is the constant intensity imposed by the low frequency oscillations and C_2 is the amplitude of the variations.

If we suppose that this expression gives the intensity of the light emitted during the existence of the positive half of a 50-cycle oscillation, we can distinguish the intensity during the existence of the negative half of a 50-cycle oscillation by writing

$$J_2 = C_1 + C_2 \sin\{\omega(t - t_1) + \pi\}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the effect of a change of sign in the low frequency tension being to introduce a phase change of π in the high frequency oscillation.*

The time taken by light to travel from N_2 to the mirror M and back to the photo-electric cell at P will be $2D/c$, where c is the velocity of light, and hence the intensity of the light falling on the cathode of the photo-electric cell may be written in the forms

$$J_3 = C_3 + C_4 \sin\left\{\omega\left(t - t_1 - \frac{2D}{c}\right)\right\} \quad . \quad . \quad . \quad . \quad (4)$$

$$J_4 = C_3 + C_4 \sin\left\{\omega\left(t - t_1 - \frac{2D}{c}\right) + \pi\right\} \quad . \quad . \quad . \quad (4a)$$

The anode of the photo-electric cell is connected directly to Cr, which thus supplies the operating voltage, the 50-cycle oscillations being stopped by the condensers K_1 and K_2 and special chokes (not shown), but current will only flow during the positive half of a high frequency oscillation when the anode is positively charged. The illumination received at the cathode will produce charges at the anode which may be written.

$$Q_1 = A + B \,.\, \sin\{\omega(t - t_1 - t_2 - 2D/c)\} \quad . \quad . \quad . \quad (5)$$

$$Q_2 = A + B \,.\, \sin\{\omega(t - t_1 - t_2 - 2D/c) + \pi\} \quad . \quad . \quad (5a)$$

where t_2 = time of travel of the electrons in the photo-electric cell between cathode and anode, A and B, of course, being constants. A time interval, t_3, will be taken for the high-frequency electrical oscillations from Cr to reach the anode of the photo-electric cell. Hence, if time is reckoned from the commencement of a positive high-frequency oscillation at Cr, this oscillation will reach the anode at time t_3 and end at time $t_3 + \frac{1}{2n}$. Consequently, since current is quantity of electricity per second and there are $\frac{1}{2}n$ positive high-frequency oscillations in a second, the current leaving the anode will be

$$i_1 = \tfrac{1}{2}n \int_{t_3}^{t_3 + \frac{1}{2n}} [A + B \,.\, \sin\{\omega\,(t - t_1 - t_2 - 2D/c)\}]dt \quad . \quad . \quad . \quad (6)$$

* The reason for a change of phase of π may be explained as follows. During a positive half of the 50-cycle oscillation the total voltage applied to the plates of the cell will be of the form $V_1 + V_2 \,.\, \sin \omega t$, where V_1 is the positive voltage due to the 50-cycle oscillation, the total voltage then being positive since V_1 is always numerically greater than $V_2 \,.\, \sin \omega t$. During the negative half of the 50-cycle oscillation the total voltage applied to the cell will be of the form $-V_1 + V_2 \,.\, \sin \omega t$, which is negative. Thus, if t is the time when the positive half cycle suddenly ends and the negative half cycle begins, the numerical value of the applied voltage suddenly changes from $V_1 + V_2 \,.\, \sin \omega t$ to $V_1 - V_2 \,.\, \sin \omega t$, which is equivalent mathematically to a phase change of π in the angle ωt. The intensity of the emitted light cannot have a minus sign and is only affected by the numerical value of the applied voltage. Hence, corresponding to the intensity $C_1 + C_2 \sin\{\omega(t - t_2)\}$ at the end of a positive half cycle, the intensity at the beginning of the negative half cycle will be $C_1 - C_2 \sin\{\omega(t - t_1)\}$, which is equivalent to $C_1 + C_2 \sin\{\omega t - t_2) + \pi\}$.

for the positive half of a 50-cycle oscillation and

$$i_2 = \tfrac{1}{2}n\int_{t_3}^{t_3+\frac{1}{2n}} [A + \mathrm{B} \,.\, \sin\{\omega(t - t_1 - t_2 - 2D/c) + \pi\}]dt \quad . \quad . \quad (6a)$$

for the negative half of a 50-cycle oscillation.

Now suppose currents proportional to these two currents passed in opposite directions through a slow reacting direct-current galvanometer in a manner presently to be described. The effective current flowing through the galvanometer will be proportional to

$$i_2 - i_1 = \frac{\beta}{\pi}\cos\omega(t_3 - t_1 - t_2 - 2D/c) \quad . \quad . \quad . \quad . \quad (7)$$

and this will be zero when

$$\omega(t_3 - t_1 - t_2 - 2D/c) = \frac{\pi}{2} - N\pi \quad . \quad . \quad . \quad . \quad (8)$$

in which N can be any whole number whatever.

As $\omega = 2\pi n$ and $n\lambda = c$, where λ is the wave length of the oscillations from Cr, we have for zero current in the galvanometer :—

$$\begin{aligned} D &= \tfrac{1}{2}n(t_3 - t_1 - t_2)\lambda + \tfrac{1}{8}(2N - 1)\lambda \\ &= K + \tfrac{1}{8}(2N - 1)\lambda \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)^* \end{aligned}$$

in which K is a constant for the instrument which represents distance "lost" by time of travel in the circuits of the apparatus itself. This is the fundamental equation establishing D in terms of λ. The constant K can be calculated from the design and dimensions of the apparatus, or, better still, it may be found from observations over an accurately measured distance.

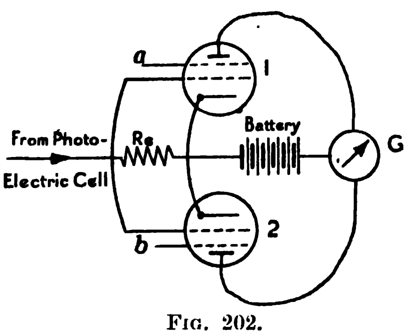


FIG. 202.

It remains to consider the separation of the currents during the positive and negatives halves of a 50-cycle oscillation. The current from the photo-electric cell is fed into a circuit consisting of the galvanometer, G, and two valves, (1) and (2) in Fig. 202. The grids **a** and **b** of the valves are connected through resistances (not shown) to the 50-cycle voltage in such a way that **a** has normal working potential when **b** has maximum negative potential. During the positive half of a 50-cycle oscillation, the potential of the grid **a** is positive and the valve 1 supplies current to G proportional to i_1 in equation (6),

* Equation (2) implies that the angle kV^2 in equation (1) is a small angle and that, when $(V_1 + V_2 \sin \omega t)^2$ is substituted for V^2, the term $V_2{}^2 \sin^2 \omega t$ is negligible in comparison with the first two terms in the expansion. While this assumption leads to some simplification in the explanation, it is not essential to the mathematical theory, as, by developing $\sin^2 k\{V_1 + V_2 \sin \omega t)^2$ in a series containing higher powers of $\sin \omega t$, it can be shown that the right-hand side of equation (7) becomes a series in which the general term is $K \cos p\omega\,(t_3 - t_1 - t_2 - 2D/c)$, where K is a constant and p an *odd* integer, and, from this, equation (9) follows as before.

the valve 2 being blocked during this time by high negative potential on grid **b**. During the negative half of the 50-cycle oscillation, valve 2 supplies current to G proportional to i_2 in equation (6a). Hence, currents in opposite directions flow through G during the course of each full 50-cycle oscillation and the net current affecting the needle is the difference of these two currents.

The method of measurement thus consists in finding values of D which satisfy the relation (9); that is, when the galvanometer registers zero current. With high frequency oscillations of 8·3 megacycles per second, corresponding to a wave length of about 36 metres, there would be zero reading on the galvanometer for distances of D differing by ·9 metres. In practice, the mirror M is not moved to obtain zero galvanometer deflections but instead the wave length λ is altered slightly by means of a valve, inductance, and variable condenser connected to the crystal oscillator as shown in Fig. 203, the change in frequency being read on a scale attached to the condenser. This scale is such that readings can be estimated to 2 cycles, corresponding to a difference in D of 4 mm. when $D = 18$ kilometres. An alternative method is to vary the time lag, t_3 (Eq. (9)), in the direct connection between Cr and the anode of the photo-electric cell in a known manner, in which case λ can always be maintained at the same constant value.

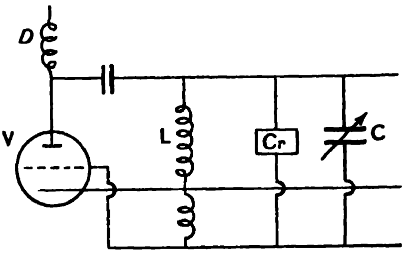


FIG. 203.

We must know N, "the crude count," before we can calculate D and this is found by increasing the wave length a second time by means of a second crystal to give one or more additional null points. Let λ_1 be the original wave length for which $N = N_1$, and let λ_2 be the second wave length giving x addition null points, so that $N = N_1 - x$. Then, from Eq. (9),

$$D = K + \tfrac{1}{8}(2N_1 - 1)\lambda_1,$$

and

$$D = K + \tfrac{1}{8}(2N_1 - 2x - 1)\lambda_2.$$

$$\therefore \quad (2N_1 - 2x - 1)\lambda_2 = (2N_1 - 1)\lambda_1.$$

$$\therefore \quad N_1 = \frac{2x\lambda_2 + (\lambda_2 - \lambda_1)}{2(\lambda_2 - \lambda_1)} = \frac{x\lambda_2}{(\lambda_2 - \lambda_1)} + 0{\cdot}5 \quad . \quad . \quad (10)$$

The accuracy of the method depends on an accurate knowledge of the velocity of light.* The frequency of the crystal, which can be determined for standard atmospheric conditions to within 1×10^{-7}, and which should be re-determined at intervals of about six months, depends on temperature

* By using the apparatus on a carefully measured base line 7 km. long, Bergstrand recently (1950) obtained an *in vacuo* value for this constant of 299 793·1 km./sec. (= 186 283·4 miles/sec.) with a mean square error of ± 0·25 km./sec. (p.e. = ± 0·17 km./sec.). This value may be compared with values for the velocity of radio waves of 299 792 ± 2·4 km./sec. obtained by C. I. Aslaskan in 1949 and 299 792·5 ± 3 km./sec. obtained by Dr. L. Essen in 1950.

but can be kept constant by means of a thermostat to within 1×10^{-6}. The velocity of light varies with atmospheric changes, the variation being $0{\cdot}9 \times 10^{-6}$ per degree Centigrade and $\cdot 4 \times 10^{-6}$ per millimetre of mercury. Good atmospheric conditions, usually found during windy weather, are essential for accurate results. These conditions can be tested by means of sights through telescopes along the line in each direction, steadiness of the image of the distant point being an indication that the temperature is probably uniform over the whole distance. All measurements must, of course, be made at night or in twilight and must be accompanied by observations for atmospheric pressure, temperature and humidity.

The apparatus has already been tested over a distance of 30 kilometres, using a light source of 30 watts. The existing experimental equipment for this distance weighs about 200 lb. and is operated from a motor generator developing 400 watts, but in later designs it will probably be possible to reduce weight to 100 lb. The apparatus is being manufactured and put on the market by AGA (Svenska ab Gasaccumulator) of Stockholm under the name of "geodimeter." A smaller battery-driven set is now being considered for the direct measurement of distances on secondary and tertiary triangulation.

It is difficult at present, in the absence of extensive tests by different experimenters in different parts of the world, to assess the extent to which this apparatus will be used in future, but it appears from the tests already made in Sweden that it possesses considerable potentialities. Since it has already proved its practicability in Sweden over a distance of 30 kilometres and results of the order of 1/300,000 have been obtained with it when observations were taken over a line of measured length, it seems that it may eventually allow the use of much longer base lines than are normally employed at present, and so do away with much of the work involved in extending from a short base, and it appears also that triangulation may be considerably strengthened by the introduction of more check bases than are now economically possible. In fact, it is not at all improbable that its principal use will be to enable the lengths of the sides of most, if not all, of the triangles in a chain or network to be measured direct, thus avoiding to a considerable extent the relatively rapid growth of linear error such as occurs in ordinary triangulation. Another possible use for the apparatus is to measure the lengths of the legs of precise traverses.

One great advantage of the method is that measurements may be made over rough country, much rougher than the comparatively flat country necessary for measurement with invar wires and tapes.

While Figs. 201, 202 and 203 show the general lay-out of the main components and electrical connections of the geodimeter, it must be understood that they do so only in broad outline. In practice, the apparatus includes a number of transformers, valves, condensers, resistances and chokes which have been omitted for the sake of simplicity. A more complete description and diagram of the electrical connections will be found in a reprint in English of a paper by E. Bergshand in the Swedish *Arkiv för Fysik*, Vol. 2, No. 15, 1950, entitled "A Determination of the Velocity of Light," which is obtainable from Messrs. H. K. Lewis & Co. Ltd., London.

APPENDIX V

LATITUDE, LONGITUDE AND REVERSE AZIMUTH FORMULÆ FOR VERY LONG LINES

On page 338 it was pointed out that the ordinary Clarke and Puissant formulæ for calculating geodetic positions and reverse azimuths from lines of given length and azimuth, which are given on pages 325–340, are only valid for lines that can be sighted over in the usual way, and, since the use of radar and flare triangulation has made it possible to measure the lengths of lines much longer than those used in ordinary triangulation, it has been necessary to work out other formulæ or to extend the existing ones to include higher order terms which have hitherto been neglected. Several sets of formulæ are now available, but those that follow, which have been adapted by J. E. R. Ross of the Geodetic Survey of Canada from a method originally outlined by Jordan, are easy to follow and reasonably easy to compute, although, like all similar sets, they are somewhat long.*

The data given are the latitude and longitude of the initial point and the geodesic azimuth and distance to the forward point. The geodesic distance is the distance measured by radar, and the geodesic azimuth is obtained by adding the geodesic angle to the geodesic azimuth of a line already fixed. The geodesic angle is obtained as described on page 315 by adding to the plane angle, obtained by solving a triangle of known sides as a plane triangle, the correction for spherical excess, including the second term given on page 315, plus the correction for the difference between the geodesic and the spherical angle (page 315). Since lines used in radar are too long to sight over, it is not possible to use an observed azimuth for the azimuth of the initial line. Instead, two very distant points established by ground survey must be used as the terminal points of one side of a radar-observed triangle and the length and azimuth of this line calculated.

In Fig. 204 (a), NAB is the triangle on the spheroid formed by the

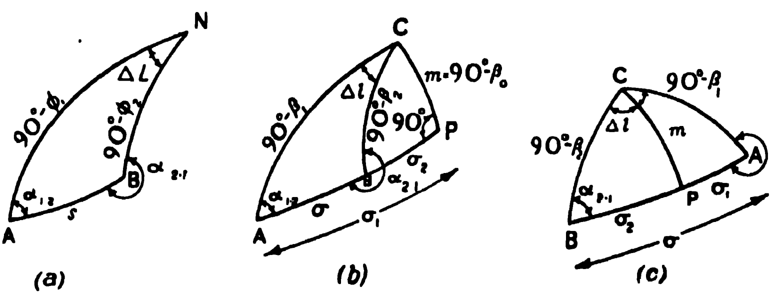


Fig. 204.

meridians through A and B and the geodesic line joining A to B. Let ϕ_1 be the latitude of A and ϕ_2 that of B and let ΔL be the difference in longitude

* See "Geodetic Problems in Shoran," by J. E. R. Ross. Canadian Geodetic Survey Publication No. 76, 1949.

between the two points. The values given are the angular values of ϕ_1 and $\alpha_{1\cdot 2}$, the geodesic azimuth in the direction AB, and the radar measured distance s. We need to solve this triangle for ϕ_2, ΔL and $\alpha_{2\cdot 1}$, the geodesic azimuth in the direction BA.

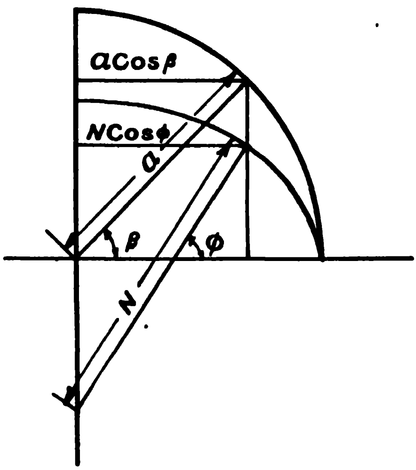


Fig. 205.

Take an auxiliary spherical triangle ACB in Fig. 204 (b) in which the points A and B have latitudes corresponding to the reduced latitudes β_1 and β_2 of A and B, and the side AB has the same azimuths as the side AB in Fig. 204 (a). The reduced latitude of a point is the eccentric angle of conic sections—that is, the angle between the semi-axis major of an ellipse and the radius vector to the point where the y-co-ordinate produced meets a circle, the auxiliary circle, having the major axis as diameter and the centre of the ellipse as centre (Fig. 205). This reduced latitude is given by

$$\tan \beta = \frac{b}{a} \tan \phi,$$

where b is the semi-axis minor of the ellipse and a the semi-axis major.

[The possibility of constructing a spherical triangle with the four specified elements derived directly or indirectly from the triangle on the spheroid arises from the fact that the geodesic satisfies the equation $N \,.\, \cos \phi \sin A =$ constant all along its length, so that

$$N_1 \cos \phi_1 \sin \alpha_{1\cdot 2} = - N_2 \cos \phi_2 \sin \alpha_{2\cdot 1}.$$

But it is easy to see from Fig. 205 that $N \,.\, \cos \phi = a \,.\, \cos \beta$, and hence, by substitution,

$$\cos \beta_1 \sin \alpha_{1\cdot 2} = - \cos \beta_2 \sin \alpha_{2\cdot 1},$$

which is the ordinary sine relation for the spherical triangle ABC in Fig. 204 (c).]

From the apex C of the spherical triangle draw CP $= m$ (Fig. 204 (b)) perpendicular to the side AB or AB produced and let AB $= \sigma$, BP $= \sigma_2$, AP $= \sigma_1$ and $M = 90° - \sigma_1$.

In the triangle ACB we know the angle $A = \alpha_{1\cdot 2}$, and the side AC $= 90° - \beta_1$ can be calculated from the given latitude ϕ_1. The side CB $= 90° - \beta_2$ and the angles ACB $= \Delta l$ and CBA $= 360° - \alpha_{2\cdot 1}$ can therefore be found if we know the value of σ, and this can be obtained from s, the measured radar distance between the points A and B, as follows :—

Calculate, or where possible obtain from special tables,

$$\tan \beta_1 = \frac{b}{a} \tan \phi_1,$$

$$\tan M = \tan \beta_1 \sec \alpha_{1\cdot 2},$$

$$\sin m = \cos \beta_1 \sin \alpha_{1\cdot 2} \text{ or } \cos m = \cos \beta_1 \cos \alpha_{1\cdot 2} \sec M,$$

$$e' = \sqrt{\frac{a^2 - b^2}{b^2}} = \frac{e^2}{1 - e^2},$$

$$k = e' \cos m,$$

$$K_1 = 1 + \tfrac{1}{4}k^2 - \tfrac{3}{64}k^4 + \tfrac{5}{256}k^6,$$

$$K_2 = \tfrac{1}{4}k^2 - \tfrac{1}{16}k^4 + \tfrac{15}{512}k^6,$$

$$K_3 = \tfrac{1}{128}k^4 - \tfrac{3}{512}k^6,$$

$$F = \frac{1}{K_1 \sin 1''},$$

$$G = \frac{K_2}{K_1 \sin 1''},$$

$$H = \frac{K_3}{K_1 \sin 1''}.$$

Then

$$\sigma'' = F\frac{s}{b} + G \sin \sigma \cos (2M \pm \sigma) + H \sin 2\sigma \cos (4M \pm 2\sigma).$$

Here σ enters into the second and third terms on the right so preliminary values must be obtained by approximation, first using the first term on the right, then substituting the approximate of σ so found in the second and third terms on the right, a third approximation seldom or never being necessary. The sign of σ in the second and third terms is positive if B in Fig. 204 (b) lies between A and P and negative if P lies between A and B.

When σ has been obtained, calculate σ_2 from

$$\sigma_2 = \sigma_1 \pm \sigma = 90° - M \pm \sigma$$

in which the sign of σ must be determined from the position of the point P with reference to the points A and B. Thus, in Fig. 204 (b) $\sigma_2 = \sigma_1 - \sigma$ and in Fig. 204 (c) $\sigma_2 = \sigma_1 + \sigma$.

To obtain β_2, we have

$$\sin \beta_2 = \cos m \cos \sigma_2,$$

and to obtain Δl and $\alpha_{2\cdot 1}$ solve the triangle ABC for the angles C and B by the ordinary rules of spherical trigonometry, thus :—

$$\tan \tfrac{1}{2}(B - C) = \sin \tfrac{1}{2}(b - c) \operatorname{cosec} \tfrac{1}{2}(b + c) \cot \tfrac{1}{2}A$$
$$\tan \tfrac{1}{2}(B + C) = \cos \tfrac{1}{2}(b - c) \sec \tfrac{1}{2}(b + c) \cot \tfrac{1}{2}A,$$

giving

$$B = \{\tfrac{1}{2}(B + C) + \tfrac{1}{2}(B - C)\}$$
$$C = \{\tfrac{1}{2}(B + C) - \tfrac{1}{2}(B - C)\},$$

where

$$b = 90° - \beta_1,\ c = \sigma,\ A = \alpha_{1\cdot 2},\ B = 360° - \alpha_{2\cdot 1},\ C = \Delta l.$$

Hence, having found B and C we get $\alpha_{2\cdot 1}$ and Δl, and finally

$$\tan \phi_2 = \frac{a}{b} \tan \beta_2,$$

$$\Delta L'' = \frac{\Delta l''}{l'} \{1 - (\lambda_1)\Delta\phi^2 - (\lambda_2)\Delta l^2 - (\lambda_3)\Delta\phi^4 - (\lambda_4)\Delta\phi^2\Delta l^2 - (\lambda_5)\Delta l^4\}.$$

in which a and b are now the semi-axis major and the semi-axis minor as before and

$$(\lambda_1) = \frac{\sin^2 1''}{24} \cdot \frac{\eta^2}{V^4} \{3t^2 + 1 + \eta^2 + 6\eta^2 t^2\}$$
$$(\lambda_2) = \sigma_2$$
$$(\lambda_3) = \frac{-\sin^4 1''}{1440} \eta^2 \{1 + 15t^2\}$$
$$(\lambda_4) = \frac{\sin^4 1''}{720} \eta^2 \{-1 - 10t^2 + 15t^4\} \cos^2\phi$$
$$(\lambda_5) = \frac{\sin^4 1''}{240} \eta^2 \{-3t^2 + t^4\} \cos^4\phi$$
$$t = \tan \tfrac{1}{2}(\phi_1 + \phi_2),$$
$$\eta^2 = e'^2 \cos^2\phi$$
$$V = \sqrt{\frac{N}{R}} = \sqrt{1 + \eta^2}$$

all the arguments being for $\phi_m = \frac{1}{2}(\phi_1 + \phi_2)$.

Much of the actual heavy calculation work can be avoided if special tables are computed or used which give values of F, G, H, V, η, (λ_1), (λ_2), . . . (λ_5) for convenient intervals of m and ϕ_m. Tables based on the Clarke 1886 spheroid and for values of ϕ_m from 40° to 85° at intervals of 10′ are given by Ross in "Geodetic Problems in Shoran." In some cases, however, accurate interpolation from these tables involves the use of second differences.

The reverse problem, to find s, $\alpha_{1\cdot 2}$ and $\alpha_{2\cdot 1}$ when we are given ϕ_1, ϕ_2 and ΔL, is easy, for, after β_1 and β_2 have been computed from

$$\tan \beta_1 = \frac{b}{a} \tan \phi_1$$
$$\tan \beta_2 = \frac{b}{a} \tan \phi_2,$$

Δl can be calculated from

$$\Delta l'' = V\Delta L\{1 + (\lambda_1)\Delta\phi^2 + (\lambda_2)\Delta L^2 + (\lambda_3)\Delta\phi^4 + (\lambda_4)\Delta\phi^2 \,.\, \Delta L^2 + (\lambda_5)\Delta L^4\},$$

where V, (λ_1), (λ_2), (λ_3), (λ_4) and (λ_5) have the same meanings as before. We can then solve the spherical triangle ABC for the angles A and B, thus,

$$\tan \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2}(\beta_2 - \beta_1) \sec \tfrac{1}{2}(\beta_2 + \beta_1) \cot \tfrac{1}{2}\Delta l,$$
$$\tan \tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}(\beta_2 - \beta_1) \operatorname{cosec} \tfrac{1}{2}(\beta_2 + \beta_1) \cot \tfrac{1}{2}\Delta l,$$

whence,

$$\alpha_{1\cdot 2} = A = \{\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B)\},$$
$$360° - \alpha_{2\cdot 1} = B = \{\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B)\}.$$

The side σ can now be found from the right-angled triangles ACP and BCP by

$$\tan \sigma_1 = \cos \alpha_{1\cdot 2} \cos \beta_1,$$
$$\tan \sigma_2 = \cos \alpha_{2\cdot 1} \cos \beta_2,$$

and

$$\sigma = \sigma_1 \pm \sigma_2,$$

with check

$$\sin \sigma \sin m = \sin \Delta l \cos \beta_1 \cos \beta_2,$$

where m has the same value as before.

When σ has been found, s can be computed from

$$s = \frac{\sigma}{U}\{1 - (\sigma_1)\Delta\phi^2 - (\sigma_2)\Delta L^2 - (\sigma_3)\Delta\phi^4 - (\sigma_4)\Delta\phi^2 \,.\, \Delta L^2 - (\sigma_5)\Delta L^4\},$$

or, in logarithmic form,

$$\log s = (\log \sigma - \log U) - \mu(\sigma_1)\Delta\phi^2 - \mu(\sigma_2)\Delta L^2 - \mu(\sigma_3)\Delta\phi^4 - \mu(\sigma_4)\Delta\phi^2 \,.\, \Delta L^2 - \mu(\sigma_5)\Delta L^4,$$

in which,

$$U = \frac{V}{a \sin 1''}$$

$$(\sigma_1) = \frac{\sin^2 1''}{24} \frac{\eta^2}{V^4}\{t^2 - (1 + \eta^2 + 6\eta^2 t^2\},$$

$$(\sigma_2) = -\frac{\sin^2 1''}{12} \eta^2 \sin^2 \phi = (\lambda_2),$$

$$(\sigma_3) = \frac{\sin^4 1''}{480} \eta^2 (1 - t^2),$$

$$(\sigma_4) = -\frac{\sin^4 1''}{720} \eta^2 (- 1 + 2t^2 + 15t^4) \cos^2\phi,$$

$$(\sigma_5) = \frac{\sin^4 1''}{720} \eta^2 (9t^2 - 5t^4) \cos^4\phi,$$

μ = modulus of common logarithms.

Here again, labour of computation is lightened if tables are used which give values of U, (σ_1), (σ_2), (σ_3), (σ_4) and (σ_5) for convenient intervals of ϕ_m, this being the argument to be used in taking out values of U and of the (σ)'s. Such tables are given by Ross for 10′ intervals of ϕ_m from 40° to 85°, but, once more, interpolation with second differences is necessary in some cases.

A rigorous set of formulæ for the inverse problem of finding length and azimuth from the latitude and longitude of the terminal stations, and which is valid for lines of some thousands of miles, was given by Clarke in his *Geodesy* * and other simpler formulæ for the same problem, but not of quite the same degree of accuracy, were given by W. D. Lambert in 1924.† In 1932–33, G. T. McCaw published formulæ for both the direct and the inverse solutions which are valid for lines of two or three thousand miles, but these, like those of Clarke, require the use of 10-figure logarithms for full scale accuracy (0″·001).‡ More recently, J. H. Cole

* *Geodesy*, 1880 Edition, pages 106–108. Note, however, that the azimuths in these formulæ are the azimuths of the plane curves, not the azimuths of the geodesic, which are the azimuths obtained from the formulæ given above.

† "The Distance Between Two Widely Separated Points on the Surface of the Earth." By Walter D. Lambert. *Journal of the Washington Academy of Sciences*, Vol. XXXII, No. 5.

‡ "Long Lines on the Earth." By G. T. McCaw. *Empire Survey Review*, Vol. I, No. 6 and Vol. II, Nos. 9, 12, and 14.

(1946) * has extended Puissant's formulæ to give results which are sufficiently accurate for all ordinary radar computation, and H. F. Rainsford (1949) † has produced other formulæ, in the form of long series, which are accurate up to about 500 miles or more and which, like Cole's formulæ, need only be computed with 8-figure logarithms. The exact limits beyond which the formulæ given above are no longer of sufficient accuracy have not yet been thoroughly worked out, but it would appear that satisfactory results may be expected for arcs up to about 20° in extent. These formulæ also need no more than 8-figure tables for their computation, but, when they are used for lines much longer than about 1,000 miles, it is advisable to use 10-figure tables.

* "Computation of Distances of Long Arcs for Radio Purposes" By J. H. Cole. *Empire Survey Review*, Vol. VIII, No. 59, and "Point to Point Computation of Geographical Co-ordinate of Long Lines." *Empire Survey Review*, Vol. IX, No. 66.

† "Long Lines on the Earth: Various Formulæ." By H. F. Rainsford. *Empire Survey Review*, Vol. X, Nos. 71 and 72.

TABLE OF CONSTANTS

	Number	Logarithm
π	3·14159 26536	0·49714 98727
$1 \div \pi$	0·31830 98862	9·50285 01273
e, base of natural logarithms	2·71828 18285	0·43429 44819
$\log_{10} e = M =$ modulus	0·43429 44819	9·63778 43113
$\log_e 10 = 1 \div M$	2·30258 50930	0·36221 56887
1 radian, in degrees	57·2957 79513	1·75812 26324
1 radian, in minutes	3437·74 67708	3·53627 38828
1 radian, in seconds	206264·80625	5·31442 51332
1 degree, in radians	0·01745 32925	8·24187 73676
1 minute, in radians	0·00029 08882	6·46372 61172
1 second, in radians	0·00000 48481	4·68557 48668
1 metre	39·37014 7 inches	1·59516 703
1 metre	3·28084 55 feet	0·51598 578
1 kilometre	0·62137 226 miles	9·79335 186
1 inch	25·3999 56 millimetres	1·40483 297
1 foot	0·30479 947 metres	9·48401 422
1 mile	1·60934 12 kilometres	0·20664 814
1 French legal metre	3·28086 93 feet	0·51598 893
1 United States legal metre	3·28083 33 feet	0·51598 417
1 kilogramme	2·20462 23 pounds	0·34333 420
1 pound	0·45359 243 kilogrammes	9·65666 580

Logarithms of numbers less than unity have been increased by 10.

Note.—The relations between the British and Metrical units of length given here are based on the determination of the ratio of the Imperial Standard Yard to the International Prototype Metre made by Sears, Johnson and Jolly in 1927, when they found that 1 metre = 39·370147 inches (*Phil. Trans.*, A Series, Vol. 227, 1928). The values given in the corresponding table in the second edition of this book were based on the conversion factor, 1 metre = 39·370113 inches, determined by Chaney and Benoît in 1896. For an important and interesting account of different determinations of the relations between British and Metrical units of length, and of their effects on geodetic operations, see an article by McCaw on "The Two Metres: The Story of an African Foot" in the *Empire Survey Review*, Vol. V, No. 32, page 96, April 1939.

Greek Alphabet

A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upsilon
E	ϵ	Epsilon	N	ν	Nu	Φ	ϕ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega

ϖ An alternative form for Pi

ANSWERS TO EXAMPLES

CHAPTER I (page 45)

1. – 12° 59′ 10″·6.
2. 55° 17′ 55″.
3. 12^h 36^m 12^s·9.
4. 10^h 08^m 33^s·7.
5. 21^h 12^m 49^s·2.
6. 22^h 39^m 45^s.
7. 12^h 06^m 00^s·7.
8. 21^h 44^m 00^s ; eastern.
9. 22^h 13^m 06^s·7 and 3^h 06^m 14^s·3 55° 30′ 41″.
10. 12^h 10^m 28^s·2.

CHAPTER II (page 136)

1. 4^m 11^s fast ; 5^m 00^s slow.
2. 1^m 15^s·5 fast.
3. 38^s·5 fast.
4. 6^m 12^s·3 slow.
5. 4^m 20^s·4 fast.
6. 90° 17′ 28″, 71° 57′ 28″ west of south.
7. 64° 32′ 42″ west of south.
8. 0° 55′ 52″ west of south.
9. 132° 26′ 50″ ; 21^h 44^m 43^s.
10. 55° 56′ 03″ N.
11. 51° 30′ 12″ N.
12. 2° 22′ 40″·8 N.
13. 49° 01′ 36″ N.
14. 23° 10′ 59″·0 E.
15. 17^s·34 gaining.
16. 30′ 19″·5, 49′ 34″·5, 1° 13′ 43″·5, 53′ 10″·5 west of A.
17. 6^s·55 losing ; 21° 03′ 36″·8.
18. 13′ 39″ W. or 88° 42′ 36″ W.

CHAPTER III (page 254)

1. 58 ft.
2. 812 ft.
3. Line of sight fails to clear C by 64 ft.
4. – 3·0046 ft.
5. 31,348·544 ft.
6. 1724·45 ft.
7. 16.532·37644 m.
8. + 0·00586 ft.
9. 97° 08′ 39″·88.
10. – 1″·32.
11. + 3″·9.
12. 71° 49′ 46″·22.

CHAPTER IV (page 310)

Answers to questions on angle adjustment are given in terms of the seconds only.

1. *a*, 11″·97 ; *b*, 50″·17 ; *c*, 36″·08 ; *a* + *b* + *c*, 38″·22.
2. *a*, 30″·60 ; *b*, 13″·11 ; *c*, 3″·31 ; *d*, 49″·47 ; *e*, 23″·51.
3. AOB, 32″·74 ; BOC, 21″·88 ; COD, 2″·54 ; DOE, 17″·57 ; EOA, 45″·27.
4. *a*, 39″·97 ; *b*, 15″·29 ; *c*, 32″·45.
5. *a*, 10″·99 ; *b*, 30″·42 ; *c*, 29″·20 ; p.e. of an observed angle = ± 0″·51, of an adjusted angle = ± 0″·36.
6. *a*, 22″·6 ; *b*, 44″·9 ; *c*, 52″·5.
7. ± 0″·51 ; ± 1″·44.
8. ± 0·039 ft.
9. 1 : 1·28 : 6″·00.
10. 32 miles.
11. 41″·01.
12. ± 0·01803 ft.
13. A, 27″·19 ; B_1, 35″·19 ; C_1, 57″·62 ; B_2, 2″·82 ; C_2, 14″·36 ; D, 42″·82.
14. *a*, 3″·28 ; *b*, 34″·48 ; *c*, 14″·81 ; *d*, 46″·50 ; *e*, 24″·21 ; *f*, 16″·73 ; *g*, 32″·56 ; *h*, 7″·43. p.e. of an observed angle = ± 1″·107, of an adjusted angle ± 0″·783 and of side CD ± 0·53 ft.
15. *a*, 3″·35 ; *b*, 34″·38 ; *c*, 14″·69 ; *d*, 46″·43 ; *e*, 24″·13 ; *f*, 16″·85 ; *g*, 32″·66 ; *h*, 7″·52.
16. AB, 45,367·24 m. ; BC, 37,700·48 m.

CHAPTER V (page 390)

1. 6,382,000 m.

2. 50° 25′ 11″·49 N.

3. 25,866·4 m.; 178° 30′ 56″·4.

4. 111,239 m.; 111,258 m.

5. 3″·86; log AB = 5·130 5994, log BC = 5·096 0424.

6. 22° 00′ 04″·80 S; 14° 59′ 40″·13 E; 41° 22′ 52″·15.

7. $X = 195\ 699{\cdot}5$; $Y = 200\ 216{\cdot}6$; $\alpha' = 62°\ 16'\ 17''{\cdot}37$.

8. $\tan \frac{1}{2}(A + B) =$
$\cos\frac{1}{2}(\phi - \phi') \operatorname{cosec}\frac{1}{2}(\phi + \phi') \cot \frac{\Delta L}{2}$;
$\tan \frac{1}{2}(A - B) =$
$\sin \frac{1}{2}(\phi - \phi') \sec \frac{1}{2}(\phi + \phi') \cot \frac{\Delta L}{2}$;

9. $\sin \frac{s}{r} = \sin \Delta L \cos \phi' \operatorname{cosec} A$
$= \sin \Delta L \cos \phi \operatorname{cosec} B$;
$A = 74°\ 32'{\cdot}0$, $B = 264°\ 21'{\cdot}3$;
$s = 3194{\cdot}9$ miles.

10. 38° 04′ 19″·04.

11. $\log s = 4{\cdot}933\ 4005$; $A = 351°\ 43'\ 34''{\cdot}18$.

12. $X_R = +\ 826\ 176{\cdot}79$; $Y_R = -\ 669\ 424{\cdot}60$; $X_M = +\ 826\ 176{\cdot}79$; $Y_M = -\ 669\ 539{\cdot}48$.

13. $\gamma = 23'\ 55''{\cdot}38$; $X_B = +\ 704\ 012{\cdot}1$; $Y_B = -\ 633\ 741{\cdot}7$.

14. 11° 56′ 04″·12 N; 9° 46′ 22″·79 W.

15. $X_c = +\ 908\ 565{\cdot}18$; $Y_c = -\ 729\ 692{\cdot}22$.

16. $X = +\ 553\ 165{\cdot}9$; $Y = -\ 513\ 002{\cdot}5$; $\gamma = -\ 1°\ 23'\ 21''{\cdot}54$.

17. $\delta = 3'\ 26''{\cdot}48$; x 12·51 ft.; a — 50·05 ft.

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